A simple algorithm for computing all zeros of a nonlinear function of a variable in a given interval [a,b]

Neculai Andrei
Research Institute for Informatics,

April 16, 1975

In this technical report we present a simple algorithm for computing all zeros of a nonlinear function in a given interval, that is find *all points* $x^* \in [a,b]$, such that

$$f(x^*) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined in interval $[a,b]$ and $a < b$. The idea of the algorithm is as follows:

Algorithm

1) Divide the interval $[a,b]$ into a number of ni subintervals with step h
2) Compute the values: $fa = f(a)$
3) For $i = 1,2,\ldots,$ compute $f(a+ih)$ until $f(a+ih)f(a+(i-1)h) \le 0.$ In subinterval $[a+ih, a+(i-1)h]$ there is a zero of function f
4) In interval $[a+ih, a+(i-1)h]$ apply the algorithm for finding a zero of function f
5) Set $a = a+(i+1)h$ and go to step 3. ♦

The interval $[a,b]$ is divided in a number of subintervals. Every subinterval is analyzed to see whether there is a zero of function f in it. As soon as the sign condition $f(a+ih)f(a+(i-1)h) \le 0.$ is satisfied it follows that in subinterval $[a+ih, a+(i-1)h]$ there is a zero of function f. Therefore in this subinterval we may apply the algorithm for finding a zero of function f. In the following the next subinterval is analyzed, until all subintervals have been considered.

Example 1
Let us find the zeros of the function

$$f(x) = x^2 - 4.$$

in interval $[-10,+10]$. The plot of this function in this interval is presented in Figure 1.

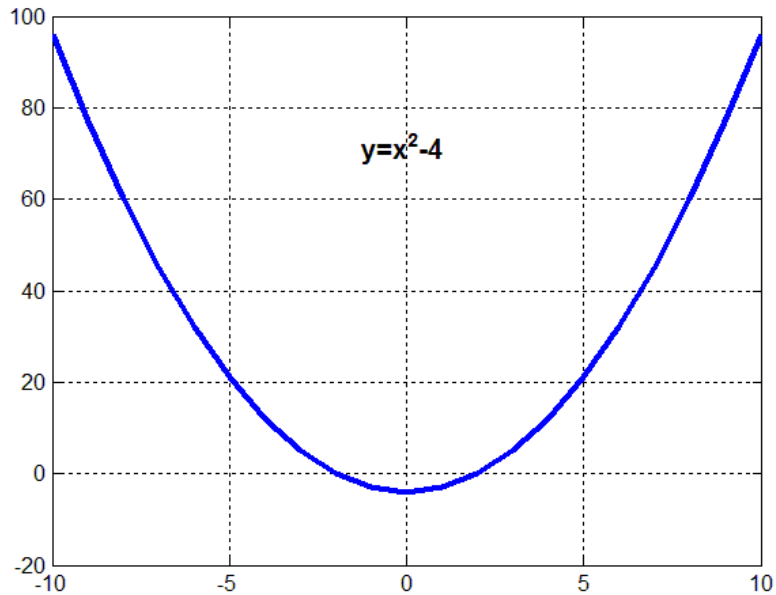

Fig. 1. Function $f(x) = x^2 - 4$.

The result of the algorithm is as follows:

```
   All zeros of a nonlinear function
   in interval [-0.1000000000000E+02, 0.1000000000000E+02 ]
   ==========================================================

Zero # 1
ZERO: -0.2000000000000E+01    Function value:  0.7105427357601E-14
Number of iterations:                41
Number of function evaluations:      42

Zero # 2
ZERO:  0.2000000000000E+01    Function value: -0.8881784197001E-14
Number of iterations:                20
Number of function evaluations:      21

========================================
TOTAL # OF iterations:                61
TOTAL # of function evaluations:      63
```

Example 2

Consider the function

$$f(x) = x^2 - 8x - 9$$

Its graphical representation for $x \in [-10, +10]$ is given in Figure 2.

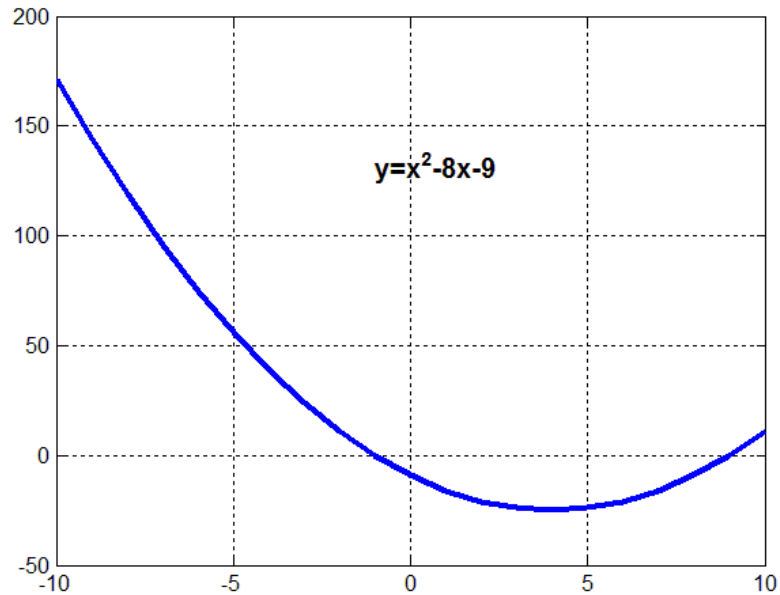

Fig. 2 Function $f(x) = x^2 - 8x - 9$

The zeros of this function in interval $[-10, +10]$ are as follows.

```
All zeros of a nonlinear function
in interval [-0.1000000000000E+02, 0.1000000000000E+02 ]
=========================================================

Zero # 1
ZERO: -0.1000000000000E+01    Function value:  0.1953992523340E-13
Number of iterations:                46
Number of function evaluations:      47

Zero # 2
ZERO:  0.9000000000000E+01    Function value: -0.1421085471520E-13
Number of iterations:                50
Number of function evaluations:      51

========================================
TOTAL # OF iterations:                96
TOTAL # of function evaluations:      98
```

Example 3

Let us find all zeros of function

$$f(x) = x^3 + 4x^2 - 4x - 16,$$

in interval $[-5, +5]$. Figure 3 presents the graphical representation of this function.

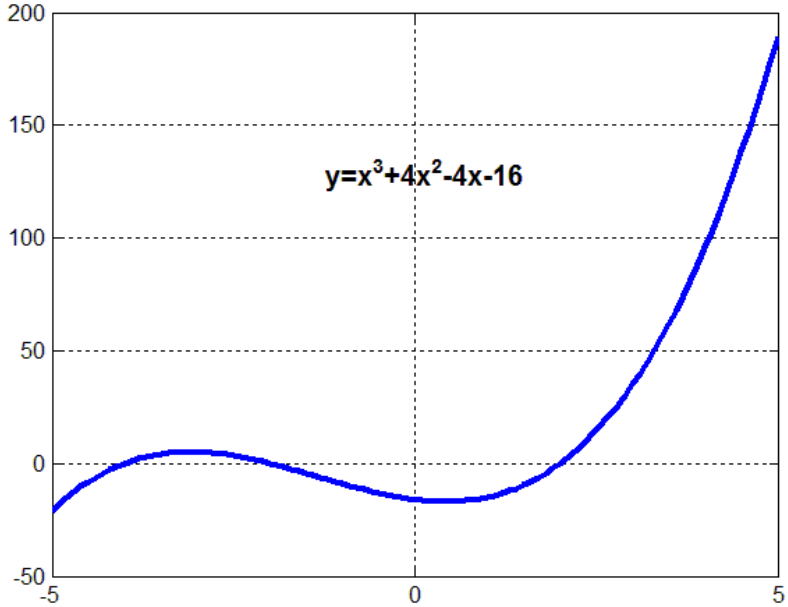

Fig. 3. Function $f(x) = x^3 + 4x^2 - 4x - 16,$

All zeros of this function in interval $[-5, +5]$ are as follows:

```
All zeros of a nonlinear function
in interval [-0.5000000000000E+01, 0.5000000000000E+01 ]
=========================================================

Zero # 1
ZERO: -0.4000000000000E+01    Function value: -0.4263256414561E-13
Number of iterations:                11
Number of function evaluations:      12

Zero # 2
ZERO: -0.2000000000000E+01    Function value:  0.1421085471520E-13
Number of iterations:                20
Number of function evaluations:      21

Zero # 3
ZERO:  0.2000000000000E+01    Function value: -0.7105427357601E-14
Number of iterations:                40
Number of function evaluations:      41

========================================
TOTAL # OF iterations:               71
TOTAL # of function evaluations:     74
```

Example 4
Let us find all the zeros of function

$$f(x) = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6$$

in interval $[-2, +2]$. Figure 4 shows the graphical representation of this function

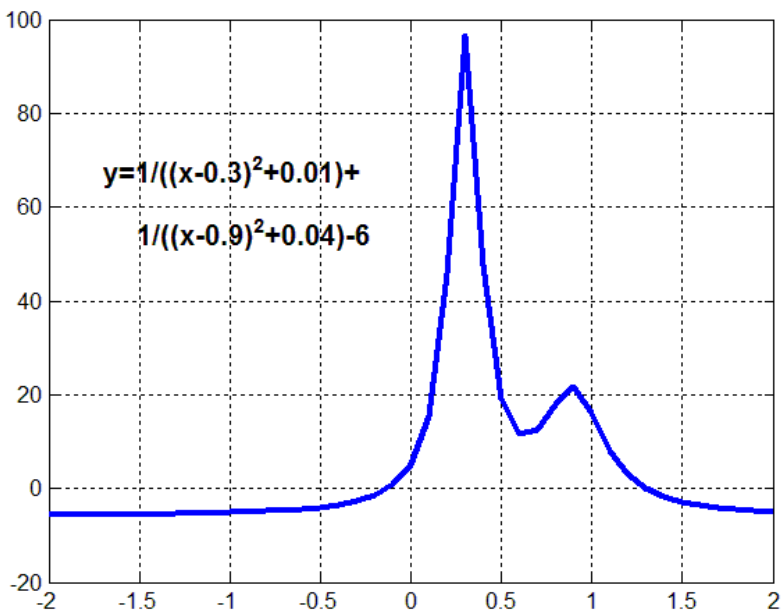

Fig. 4. Function $f(x) = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6$

The zeros of this function are as follows:

```
All zeros of a nonlinear function
in interval [-0.2000000000000E+01, 0.2000000000000E+01 ]
==========================================================

Zero # 1
ZERO: -0.1316180229187E+00    Function value: -0.1095845574284E-06
Number of iterations:                70
Number of function evaluations:      71

Zero # 2
ZERO:  0.1299549683332E+01    Function value:  0.6194646573476E-08
Number of iterations:                60
Number of function evaluations:      61

========================================
TOTAL # OF iterations:              130
TOTAL # of function evaluations:    132
```

Example 5
Find all zeros of the function

$$f(x) = x^4 - 5x^2 + 4$$

in interval $[-3, +3]$.
Figure 5 shows the graphic of this function.

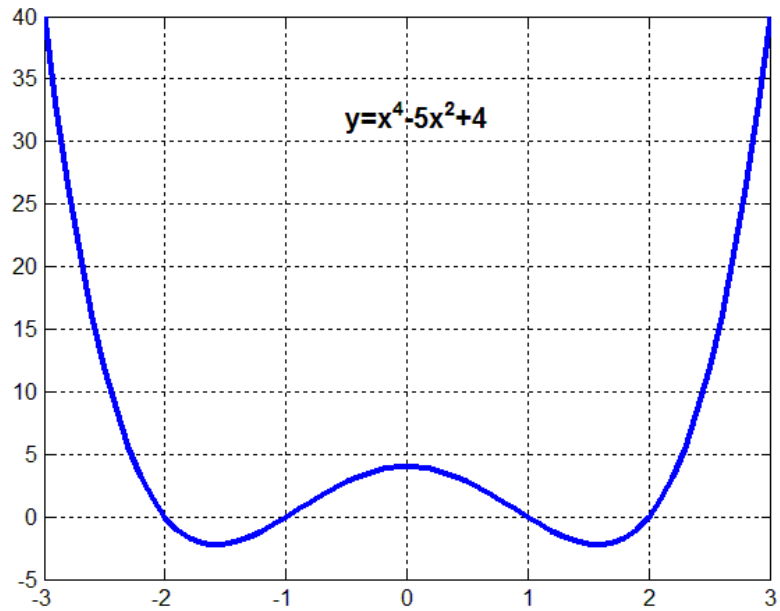

Fig. 5. Function $f(x) = x^4 - 5x^2 + 4$

The zeros of this function in interval $[-3, +3]$ are as follows:

```
All zeros of a nonlinear function
in interval [-0.3000000000000E+01, 0.3000000000000E+01 ]
==========================================================

Zero # 1
ZERO: -0.2000000000001E+01    Function value:  0.6977529665164E-11
Number of iterations:                52
Number of function evaluations:      53

Zero # 2
ZERO: -0.1000000000003E+01    Function value: -0.1745270594711E-10
Number of iterations:                50
Number of function evaluations:      51

Zero # 3
ZERO:  0.9999999999994E+00    Function value:  0.3477662602336E-11
Number of iterations:                66
Number of function evaluations:      67
```

```
Zero # 4
ZERO:  0.1999999999997E+01    Function value: -0.3488764832582E-10
Number of iterations:                50
Number of function evaluations:      51

=========================================
TOTAL # OF iterations:              218
TOTAL # of function evaluations:    222
```

Example 6
Consider the function

$$f(x) = x^6 - 14x^4 + 49x^2 - 36$$

for which the graphical representation in interval $[-4,+4]$ is as in Figure 6.

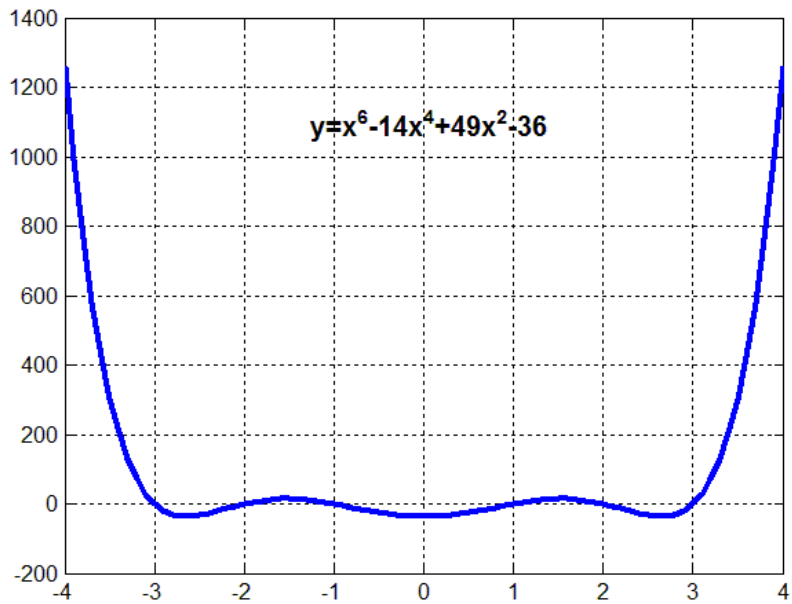

Fig. 6. Function $f(x) = x^6 - 14x^4 + 49x^2 - 36$

```
  All zeros of a nonlinear function
  in interval [-0.5000000000000E+01, 0.5000000000000E+01 ]
  ==========================================================

Zero # 1
ZERO: -0.3000000000000E+01    Function value:  0.1762145984685E-11
Number of iterations:                41
Number of function evaluations:      42

Zero # 2
ZERO: -0.2000000000000E+01    Function value: -0.6252776074689E-12
```

```
Number of iterations:                20
Number of function evaluations:      21

Zero # 3
ZERO: -0.1000000000000E+01    Function value:  0.4689582056017E-12
Number of iterations:                20
Number of function evaluations:      21

Zero # 4
ZERO:  0.1000000000000E+01    Function value: -0.4263256414561E-12
Number of iterations:                40
Number of function evaluations:      41

Zero # 5
ZERO:  0.2000000000000E+01    Function value:  0.4831690603169E-12
Number of iterations:                20
Number of function evaluations:      21

Zero # 6
ZERO:  0.3000000000000E+01    Function value: -0.2728484105319E-11
Number of iterations:                20
Number of function evaluations:      21

========================================
TOTAL # OF iterations:               161
TOTAL # of function evaluations:     167
```

Example 7

For function

$$f(x) = x^7 + 7x^6 - 14x^5 - 98x^4 + 49x^3 + 343x^2 - 36x - 252$$

the graphical representation in interval $[-8,4]$ is given in Figure 7. All the zeros of this function are as follows:

```
All zeros of a nonlinear function
in interval [-0.8000000000000E+01, 0.4000000000000E+01 ]
=========================================================

Zero # 1
ZERO: -0.7000000000000E+01    Function value: -0.3771020828935E-08
Number of iterations:                57
Number of function evaluations:      58

Zero # 2
ZERO: -0.3000000000001E+01    Function value:  0.6180016498547E-09
Number of iterations:               101
Number of function evaluations:     102

Zero # 3
ZERO: -0.2000000000003E+01    Function value: -0.8911911208997E-09
Number of iterations:                49
Number of function evaluations:      50
```

```
Zero # 4
ZERO: -0.1000000000012E+01    Function value:  0.3537707016221E-08
Number of iterations:                48
Number of function evaluations:      49

Zero # 5
ZERO:  0.9999999999970E+00    Function value: -0.1140392669186E-08
Number of iterations:                67
Number of function evaluations:      68

Zero # 6
ZERO:  0.1999999999988E+01    Function value:  0.6632433269260E-08
Number of iterations:                48
Number of function evaluations:      49

Zero # 7
ZERO:  0.2999999999999E+01    Function value: -0.1535823912491E-08
Number of iterations:                52
Number of function evaluations:      53

========================================
TOTAL # OF iterations:               422
TOTAL # of function evaluations:     429
```

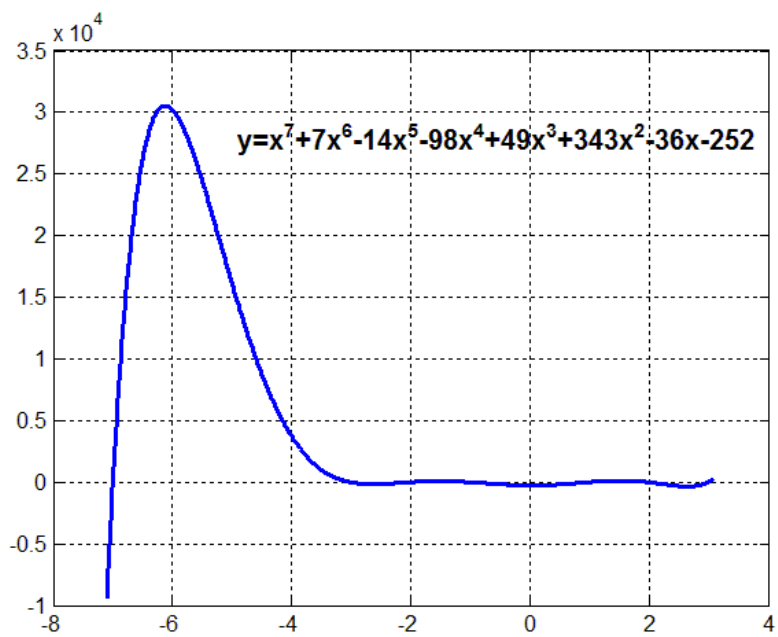

Fig. 7. Function $f(x) = x^7 + 7x^6 - 14x^5 - 98x^4 + 49x^3 + 343x^2 - 36x - 252$

Example 8
Let us determine al the zeros of the function

$$f(x) = x^3 - 2x^2 + 1$$

in interval $[-1,2]$. The graphical representation of this function is given in Figure 8.

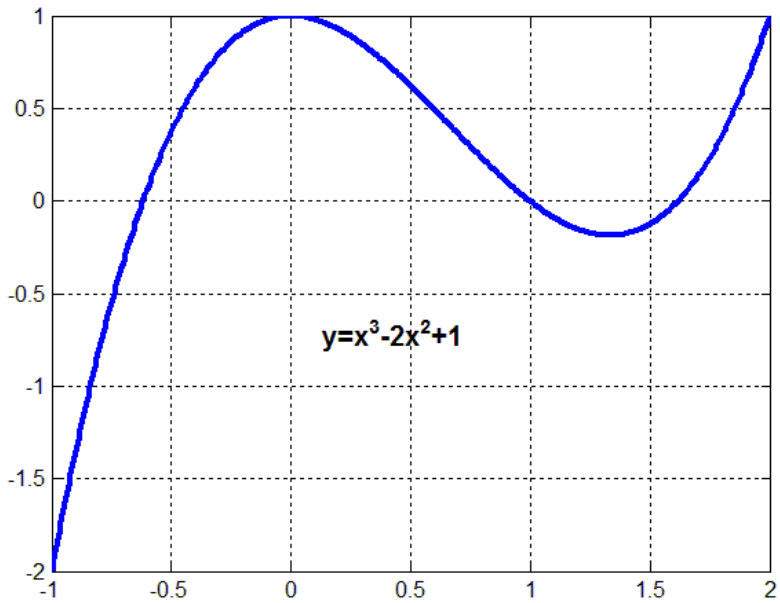

Fig. 8. Function $f(x) = x^3 - 2x^2 + 1$

```
  All zeros of a nonlinear function
  in interval [-0.1000000000000E+01, 0.2000000000000E+01 ]
  ==========================================================

Zero # 1
ZERO: -0.6180339956284E+00    Function value: -0.2488650863164E-07
Number of iterations:                47
Number of function evaluations:      48

Zero # 2
ZERO:  0.9999999988079E+00    Function value:  0.1192091425573E-08
Number of iterations:               130
Number of function evaluations:     131

Zero # 3
ZERO:  0.1618033988476E+01    Function value: -0.3787938851474E-09
Number of iterations:                63
Number of function evaluations:      64

========================================
TOTAL # OF iterations:              240
TOTAL # of function evaluations:    243
```

Example 9
Find all the zeros of the function

$$f(x) = xe^x - 7$$

in interval $[-2, +2]$.
The graphical representation of this function is as in Figure 9.

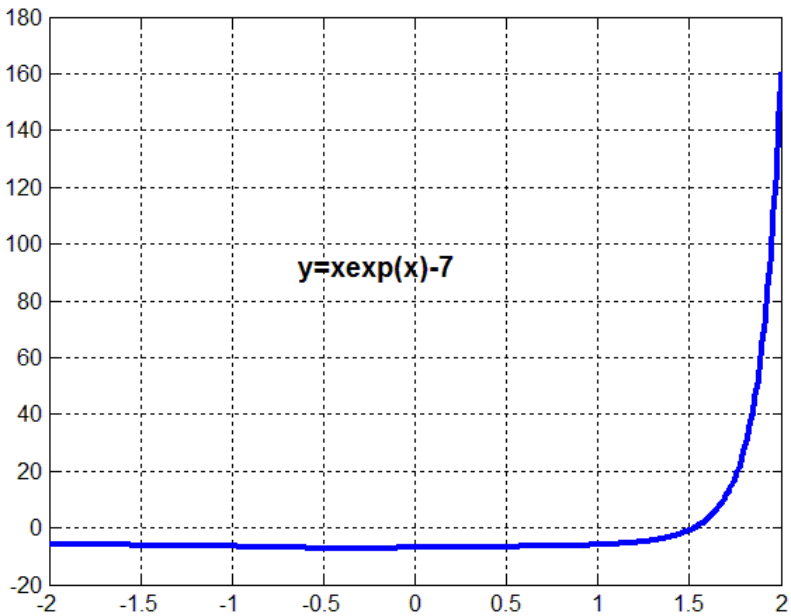

Fig. 9. Function $f(x) = xe^x - 7$

The function has one single zero as illustrated by

```
All zeros of a nonlinear function
in interval [-0.2000000000000E+01, 0.2000000000000E+01 ]
==========================================================

Zero # 1
ZERO:  0.1524345204979E+01    Function value: -0.5749978271297E-10
Number of iterations:              204
Number of function evaluations:    205

========================================
TOTAL # OF iterations:             204
TOTAL # of function evaluations:   205
```

Example 10
Consider the function

$$f(x) = x - \cos(x).$$

The graphical representation of this function in interval $[-2,+2]$ is as in Figure 10.

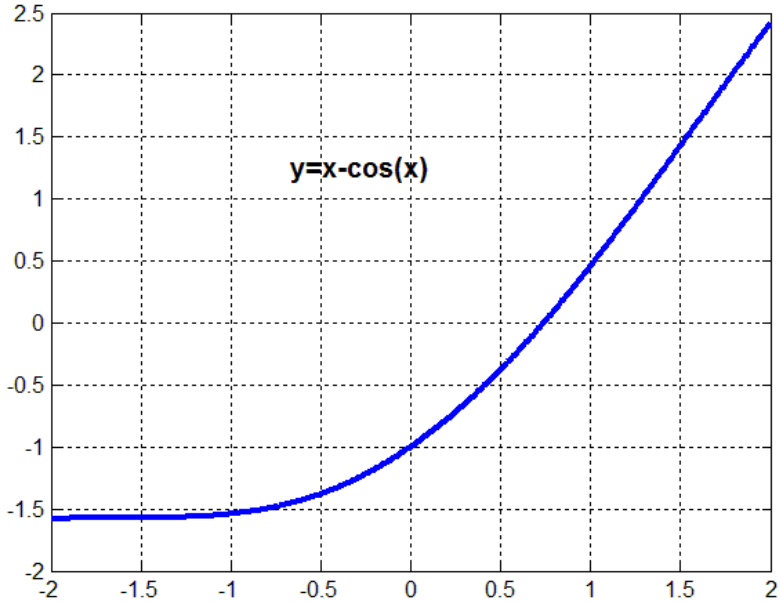

Fig 10. Function $f(x) = x - \cos(x)$.

```
All zeros of a nonlinear function
in interval [-0.2000000000000E+01, 0.2000000000000E+01 ]
==========================================================

Zero # 1
ZERO:  0.7390851330757E+00    Function value: -0.2333762392863E-09
Number of iterations:                160
Number of function evaluations:      161

=========================================
TOTAL # OF iterations:               160
TOTAL # of function evaluations:     161
```

Example 11
For function

$$f(x) = \cos(x)$$

its graphical representation is as in Figure 11.

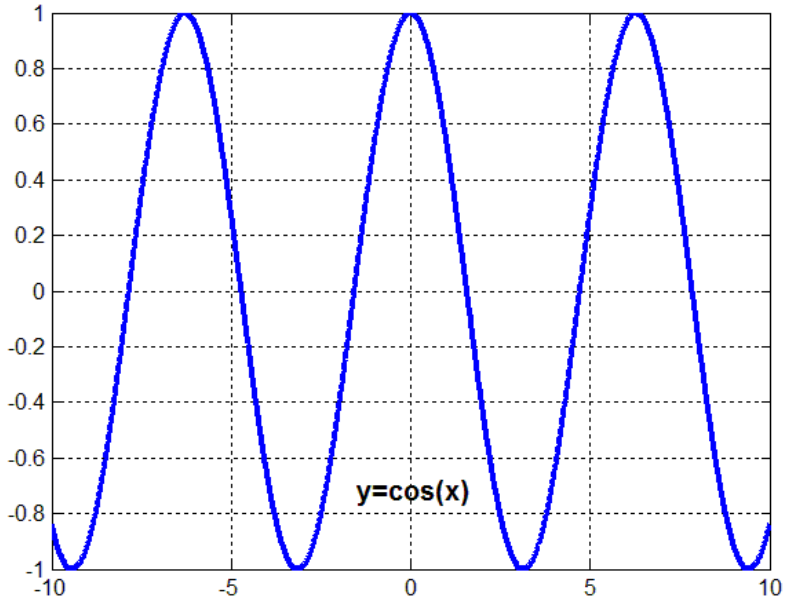

Fig. 11. Function $f(x) = \cos(x)$

All the zeros in interval $[-10, +10]$ are as follows:

```
All zeros of a nonlinear function
in interval [-0.1000000000000E+02, 0.1000000000000E+02 ]
=========================================================

Zero # 1
ZERO: -0.7853981634974E+01    Function value: -0.1000004217481E-08
Number of iterations:                48
Number of function evaluations:      49

Zero # 2
ZERO: -0.4712388980389E+01    Function value:  0.3969973845106E-11
Number of iterations:                56
Number of function evaluations:      57

Zero # 3
ZERO: -0.1570796328783E+01    Function value: -0.1988157577291E-08
Number of iterations:                56
Number of function evaluations:      57

Zero # 4
ZERO:  0.1570796325803E+01    Function value:  0.9921117873361E-09
Number of iterations:                56
Number of function evaluations:      57

Zero # 5
ZERO:  0.4712388977408E+01    Function value: -0.2976298502604E-08
Number of iterations:                56
Number of function evaluations:      57
```

```
Zero # 6
ZERO:  0.7853981623054E+01    Function value:  0.1092096133055E-07
Number of iterations:                54
Number of function evaluations:      55

=========================================
TOTAL # OF iterations:               326
TOTAL # of function evaluations:     332
```

Example 12

In interval $[-10,+10]$, the function

$$f(x) = \sin(x)$$

has the graphical representation as in Figure 12.

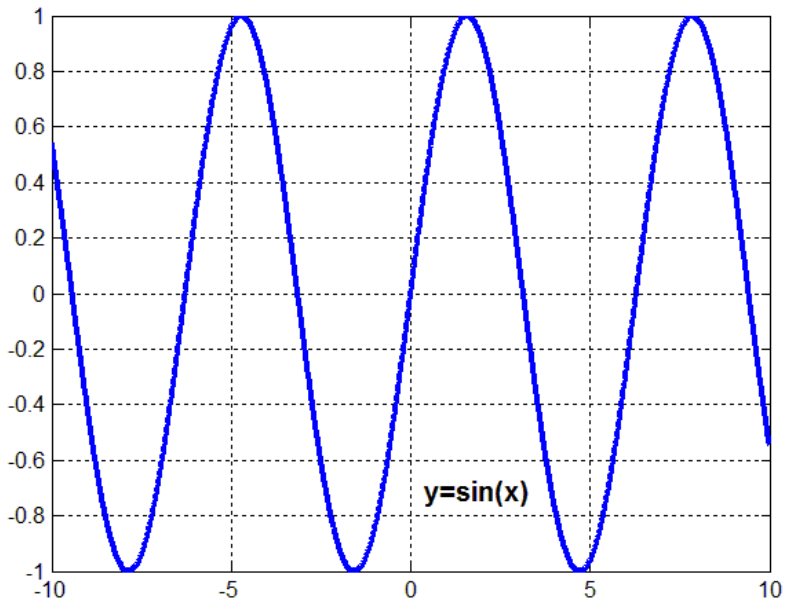

Fig. 12. Function $f(x) = \sin(x)$

```
All zeros of a nonlinear function
in interval [-0.1000000000000E+02, 0.1000000000000E+02 ]
=========================================================

Zero # 1
ZERO: -0.9424777960777E+01    Function value:  0.7906196910264E-11
Number of iterations:                28
Number of function evaluations:      29

Zero # 2
ZERO: -0.6283185309172E+01    Function value: -0.1992104014408E-08
Number of iterations:                56
Number of function evaluations:      57
```

```
Zero # 3
ZERO: -0.3141592654586E+01    Function value:  0.9960653298805E-09
Number of iterations:               56
Number of function evaluations:     57

Zero # 4
ZERO: -0.1852176230973E-13    Function value: -0.1852176230973E-13
Number of iterations:               56
Number of function evaluations:     57

Zero # 5
ZERO:  0.3141592651606E+01    Function value:  0.1984203590658E-08
Number of iterations:               56
Number of function evaluations:     57

Zero # 6
ZERO:  0.6283185306191E+01    Function value: -0.9881653502191E-09
Number of iterations:               56
Number of function evaluations:     57

Zero # 7
ZERO:  0.9424777957797E+01    Function value:  0.2972362279539E-08
Number of iterations:               56
Number of function evaluations:     57

========================================
TOTAL # OF iterations:             364
TOTAL # of function evaluations:   371
```

Example 13

Consider the function

$$f(x) = \cos(x^2).$$

Its graphical representation is given in Figure 13. All the zeros given by our algorithm are as follows:

```
All zeros of a nonlinear function
in interval [-0.5000000000000E+01, 0.4900000000000E+01 ]
=========================================================

Zero # 1
ZERO: -0.4854064781435E+01    Function value:  0.4425552954429E-09
Number of iterations:               31
Number of function evaluations:     32

Zero # 2
ZERO: -0.4518888386514E+01    Function value: -0.1437784206173E-08
Number of iterations:               40
Number of function evaluations:     41

Zero # 3
```

```
ZERO: -0.4156772737928E+01    Function value:  0.3536126206845E-10
Number of iterations:               40
Number of function evaluations:     41

Zero # 4
ZERO: -0.3759942412082E+01    Function value: -0.1019722421655E-08
Number of iterations:               43
Number of function evaluations:     44

Zero # 5
ZERO: -0.3315957522016E+01    Function value:  0.2503220008392E-09
Number of iterations:               42
Number of function evaluations:     43

Zero # 6
ZERO: -0.2802495619874E+01    Function value: -0.6544096434107E-07
Number of iterations:               41
Number of function evaluations:     42

Zero # 7
ZERO: -0.2170803763732E+01    Function value:  0.2496712109789E-09
Number of iterations:               51
Number of function evaluations:     52

Zero # 8
ZERO: -0.1253314143047E+01    Function value: -0.1436606316668E-07
Number of iterations:               59
Number of function evaluations:     60

Zero # 9
ZERO:  0.1253314136684E+01    Function value:  0.1583061742973E-08
Number of iterations:              126
Number of function evaluations:    127

Zero #10
ZERO:  0.2170803763639E+01    Function value: -0.1546045130297E-09
Number of iterations:               63
Number of function evaluations:     64

Zero #11
ZERO:  0.2802495607611E+01    Function value:  0.3296629053914E-08
Number of iterations:               50
Number of function evaluations:     51

Zero #12
ZERO:  0.3315957521923E+01    Function value: -0.3672195725941E-09
Number of iterations:               46
Number of function evaluations:     47

Zero #13
ZERO:  0.3759942411251E+01    Function value:  0.5227230755314E-08
Number of iterations:               42
Number of function evaluations:     43

Zero #14
ZERO:  0.4156772737835E+01    Function value: -0.7387537322601E-09
Number of iterations:               42
```

```
Number of function evaluations:     43

Zero #15
ZERO:  0.4518888386052E+01    Function value:  0.2736935030798E-08
Number of iterations:               40
Number of function evaluations:     41

Zero #16
ZERO:  0.4854064781342E+01    Function value: -0.4614149365595E-09
Number of iterations:               38
Number of function evaluations:     39

=========================================
TOTAL # OF iterations:              794
TOTAL # of function evaluations:    810
```

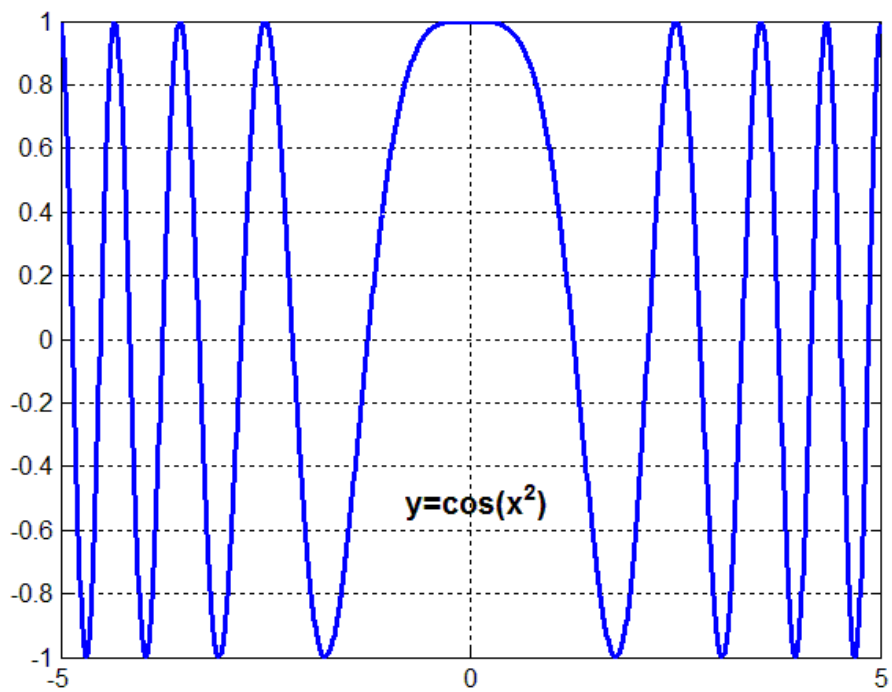

Fig. 13. Function $f(x) = \cos(x^2)$.

Example 14

Let us determine all the zeros of the function

$$f(x) = \cos(x) - x^2$$

in interval $[-5, +5]$. The function is represented as in Figure 14.

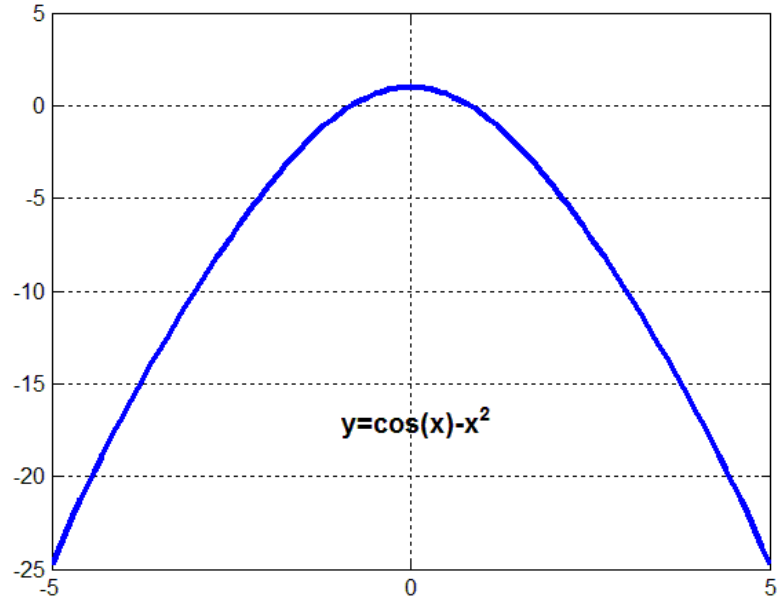

Fig. 14. Function $f(x) = \cos(x) - x^2$

```
   All zeros of a nonlinear function
   in interval [-0.5000000000000E+01, 0.5000000000000E+01 ]
   ==========================================================

 Zero # 1
 ZERO: -0.8241323128343E+00    Function value: -0.1266746809607E-08
 Number of iterations:              110
 Number of function evaluations:    111

 Zero # 2
 ZERO:  0.8241323120892E+00    Function value:  0.5081914391256E-09
 Number of iterations:               58
 Number of function evaluations:     59

 ========================================
 TOTAL # OF iterations:             168
 TOTAL # of function evaluations:   170
```

The Fortran program of the algorithm is as follows:

```
C==============================================================
C A simple algorithm for computing all the zeros of a nonlinear
C function of a variable in a given interval [a,b]. b>a.
C
C The algorithms tries to determine all the zeros of a nonlinear
C function in a given interval by discretizing of this interval
C and by find the intervals where the function changes its sign.
C
C Every subinterval is reduced by its halving for finding the
C zero of the function under accuracy epsm.
C
```

```
C Neculai Andrei
C April 16, 1975
C==============================================================


        subroutine func(x,f)
        real*8 x,f

        f = x**2 - 4.d0

c       f = x**2 - 8.d0*x - 9.d0
c       f = 4.d0*x**3 - 3.d0*x - 1.d0
c       f = x**3 + 4.d0*x**2 - 4.d0*x - 16.d0
c       f = 1.d0/((x-0.3d0)**2+0.01) + 1.d0/((x-0.9d0)**2+0.04) -6.d0
c       f = x**4 - 5.d0*x**2 + 4.d0
c       f = x**6 - 14.d0*x**4 + 49.d0*x**2 - 36.d0
c       f = x**7 + 7.d0*x**6 - 14.d0*x**5 - 98.d0*x**4 +
c     +    49.d0*x**3 +343.d0*x**2 - 36.d0*x - 252.d0
c       f = x**3 - 2.d0*x**2 + 1.d0
c       f = cos(x) - x**2
c       f = x*dexp(x) - 7.d0
c       f = x - cos(x)
c       f = cos(x)
c       f = sin(x)
c       f = cos(x**2)

        return
        end


c============================================ Main program
c                                            ==============
        real*8  a, b, c, x
        real*8 fa,fb,fc,fx
        real*8 h
        real*8 epsm
        integer iprint
        integer i, ni, iint
        integer iter, itertot
        integer fgcnt, fgcntot

        epsm=0.000000001d0
        iprint = 0

C Searching interval
c ==================
        a = -5.d0
        b =  5.d0

      open(unit=4,file='zeros.out',status='unknown')

        itertot = 0
        fgcntot = 0

        write(4,1)
1       format(4x,'All zeros of a nonlinear function')
        write(4,2) a,b
2       format(4x,'in interval [',e20.13,','e20.13,' ]')
        write(4,3)
3       format(4x,56('='),/)

C-------------------------------------------------------
        call func(a, fa)
        fgcnt = fgcnt + 1


C Take a discretization of the interval [a,b]
C ============================================

        ni = 200
        h = (b-a)/float(ni)
```

```
c Start iteratuions
c =================
        iint= 0
        i   = 1

400     continue
        iter  = 0
        fgcnt = 0

        if(i .gt. ni) go to 998

30        b = a + h
          iter = iter+1
          call func(b, fb)
          fgcnt = fgcnt + 1

          if(fa*fb .gt. 0.d0) then
            a  = b
            fa = fb
            i = i+1
            if(i .gt. ni) go to 998
            go to 30
          else
c                       write(4,104)
c104                    format(4x,'fa*fb le 0.d0: Change of sign------')

                iint = iint+1
            if(fa .le. 0.d0) then
10                continue
              if(dabs(fa) .le. epsm .or. dabs(fb) .le. epsm) go to 999
              iter=iter+1
              c = a + (b-a)/2.d0
              call func(c,fc)
              fgcnt = fgcnt + 1
              if(fc .ge. 0.d0) then
                b = c
                fb = fc
                if(iprint .eq. 1) then
                  write(4,99) iter
99                format(4x,'iter='i5)
                  write(4,110) a,b,c
110               format(4x,' a='e20.13,4x,' b=',e20.13,4x,' c=',e20.13)
                  write(4,111) fa,fb,fc
111               format(4x,'fa='e20.13,4x,'fb=',e20.13,4x,'fc=',e20.13)
                  write(4,112) b-a
112                format(4x,'fa<0. b-a=',e20.13)
               end if
                go to 10
              else
                a = c
                fa = fc
                if(iprint .eq. 1) then
                  write(4,99) iter
                  write(4,110) a,b,c
                  write(4,111) fa,fb,fc
                  write(4,112) b-a
                 end if
                  go to 10
              end if
            end if

            if(fa. ge. 0.d0) then
20                continue
              if(dabs(fa) .le. epsm .or. dabs(fb) .le. epsm) go to 999
              iter=iter+1
              c = a + (b-a)/2.d0
              call func(c,fc)
              fgcnt = fgcnt + 1
              if(fc .ge. 0.d0) then
```

```
              a = c
              fa = fc
              if(iprint .eq. 1) then
                write(4,99) iter
                 write(4,110) a,b,c
                 write(4,111) fa,fb,fc
                 write(4,112) b-a
               end if
              go to 20
            else
              b  = c
              fb = fc
              if(iprint .eq. 1) then
                write(4,99) iter
                 write(4,110) a,b,c
                 write(4,111) fa,fb,fc
                 write(4,112) b-a
               end if
              go to 20
            end if
          end if
        end if
c       end do


c-------------------------------------------------------------


999     continue

        if(dabs(fa) .le. epsm .or. dabs(fb) .le. epsm) x=a
        call func(x, fx)
        fgcnt = fgcnt + 1

        write(4,119) iint
119     format(2x,'Zero #',i2)
        write(4,120) x, fx
120     format(2x,'ZERO: ',e20.13,4x,'Function value: ',e20.13)
        write(4,121) iter
121     format(2x,'Number of iterations:          ',i7)
        write(4,122) fgcnt
122     format(2x,'Number of function evaluations:',i7,/)

        a  = x + h
        call func(a,fa)
c       fgcnt = fgcnt + 1

        i = i+1

        itertot = itertot + iter
        fgcntot = fgcntot + fgcnt

        go to 400

998     continue

        write(4,130)
130     format(2x,39('='))
        write(4,131) itertot
131     FORMAT(2X,'TOTAL # OF iterations:          ',i6)
        write(4,132) fgcntot
132     format(2x,'TOTAL # of function evaluations:',i6)
        stop
        end
c=====================================================================
c Last line
```

-----ooooOoooooo-----