Applications solved by Levenberg-Marquardt Method

Neculai Andrei
Research Institute for Informatics,
Center for Advanced Modeling and Optimization,
8-10, Averescu Avenue, Bucharest 1, Romania
E-mail: nandrei@ici.ro

Abstract. *In this work we present the rezults of a crude implementation of Levenberg-Marquardt algorithm for solving some real applications considered in the book: "Critica Raþiunii Algoritmilor de Optimizare fãrã Restricþii", chapter 14.*

The algebraic formulation of the problem is as a least square one:

$$\min_{x \in R^n} f(x) = \frac{1}{2} r(x)^T r(x),$$

unde $r: R^n \to R^m$ is a continuous differentiable function, and $m \geq n$. This problem is a minimizing problem with special structure. Additionally, we can consider it as that of solving of a nonlinear algebraic system of equations:

$$r_i(x) = 0, \quad i = 1, \ldots, m,$$

where the functions $r_i: R^n \to R$ are known as residuals.

<u>Application Q1.</u> Circuit design problem.

The system of residuals is as follows:

$$\begin{aligned}
&(1 - x_1 x_2) x_3 \left\{ \exp\left[x_5 (g_{1k} - g_{3k} x_7 10^{-3} - g_{5k} x_8 10^{-3}) \right] - 1 \right\} \\
&\qquad + g_{4k} x_2 - g_{5k} = 0, \qquad k = 1, \ldots, 4, \\
&(1 - x_1 x_2) x_4 \left\{ \exp\left[x_6 (g_{1k} - g_{2k} - g_{3k} x_7 10^{-3} - g_{4k} x_9 10^{-3}) \right] - 1 \right\} \\
&\qquad + g_{4k} - g_{5k} x_1 = 0, \qquad k = 1, \ldots, 4, \\
&x_1 x_3 - x_2 x_4 = 0,
\end{aligned}$$

where:

$$g = \begin{bmatrix} 0.4850 & 0.7520 & 0.8690 & 0.9820 \\ 0.3690 & 1.2540 & 0.7030 & 1.4550 \\ 5.2095 & 10.0677 & 22.9274 & 20.2153 \\ 23.3037 & 101.7790 & 111.4610 & 191.2670 \\ 28.5132 & 111.8467 & 134.3884 & 211.4823 \end{bmatrix}.$$

The solution given by LMS1.FOR is:

```
h =   .1000000000000E+00     Iter:      85    fxnorm=  .8384754695116E-12
```

```
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999559344E+00     .9938272427235E-08
       2     .4499879970448E+00    -.4444011381111E-08
       3     .1000007034318E+01    -.2774882545964E-09
       4     .2000067324432E+01     .1161680529549E-08
       5     .7999970340548E+01     .2145045812085E-06
       6     .7999702107041E+01     .8626667238332E-06
       7     .5000029802673E+01    -.2248754071843E-08
       8     .9999877560642E+00    -.4836714992962E-06
       9     .2000050728785E+01    -.2455853853611E-08
 ------------------------------------------------------------
h =   .1000000000000E-01     Iter:       7    fxnorm=  .6138217763747E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999473471E+00    -.1380992564748E-07
       2     .4499866396205E+00     .6150919773518E-08
       3     .1000005608341E+01     .3976552420681E-09
       4     .2000070481081E+01    -.1700030338725E-08
       5     .7999973182774E+01    -.3415567988441E-06
       6     .7999677710567E+01    -.1370612125129E-05
       7     .5000033611104E+01     .3549132543412E-08
       8     .9999876716382E+00     .7685177081385E-06
       9     .2000055271083E+01     .6762468363064E-10
 ------------------------------------------------------------
h =   .1000000000000E-02     Iter:       5    fxnorm=  .1427293121687E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999383347E+00    -.4427152333619E-07
       2     .4499852193690E+00    -.2498694584574E-08
       3     .1000004116256E+01    -.3823191008223E-07
       4     .2000073791786E+01    -.4332902392434E-07
       5     .7999976156117E+01    -.9317445446300E-06
       6     .7999652140351E+01    -.3707203944714E-05
       7     .5000037597680E+01    -.3551379279543E-07
       8     .9999875830828E+00     .2084896863153E-05
       9     .2000060028636E+01    -.9860063876488E-09
 ------------------------------------------------------------
h =   .1000000000000E-03     Iter:       4    fxnorm=  .6558840185056E-15
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999425795E+00     .5391352502215E-06
       2     .4499861771758E+00     .3946345309203E-05
       3     .1000005157730E+01     .1095941355089E-04
       4     .2000072047380E+01     .1585009354699E-04
       5     .7999974061335E+01    -.8897681489373E-06
       6     .7999667112955E+01    -.2137460086260E-05
       7     .5000034245025E+01    -.2566746218235E-05
       8     .9999876179594E+00     .1504306823108E-05
       9     .2000056843242E+01    -.1901407357119E-06
 ------------------------------------------------------------
h =   .1000000000000E-04     Iter:       4    fxnorm=  .7801734441277E-19
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999505220E+00     .4394033439326E-06
       2     .4499873309575E+00     .3564212612162E-05
       3     .1000006347522E+01     .1021667258527E-04
       4     .2000068921822E+01     .1460680769583E-04
       5     .7999971688090E+01    -.2264429745935E-06
       6     .7999689778780E+01    -.1307599006850E-06
       7     .5000031039605E+01    -.2027470344501E-05
       8     .9999876960072E+00     .2964226268887E-06
       9     .2000052682565E+01    -.1257026993606E-07
 ------------------------------------------------------------
```

Gauss-Newton method

```
h =   .0000000000000E+00     Iter:       4    fxnorm=  .3440037629328E-23
     === Solution ===
       n            x(i)                  f(x)
       1     .8999999512723E+00     .4272501126223E-06
```

```
       2     .4499874384527E+00     .3517968622191E-05
       3     .1000006457980E+01     .1012692840874E-04
       4     .2000068623126E+01     .1445813695966E-04
       5     .7999971467658E+01    -.1616614575539E-06
       6     .7999691922614E+01     .5812107417569E-07
       7     .5000030741761E+01    -.1969393537138E-05
       8     .9999877033896E+00     .1816298436097E-06
       9     .2000052289922E+01     .7004126278964E-08
 ------------------------------------------------------------
```

<u>Application Q2.</u> Propan combustion in aer

The system of residuals is:

$$x_1x_2 + x_1 - 3x_5 = 0,$$

$$2x_1x_2 + x_1 + 2R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + R_9x_2x_4 + R_8x_2 - Rx_5 = 0,$$

$$2x_2x_3^2 + R_7x_2x_3 + 2R_5x_3^2 + R_6x_3 - 8x_5 = 0,$$

$$R_9x_2x_4 + 2x_4^2 - 4Rx_5 = 0,$$

$$x_1x_2 + x_1 + R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + R_9x_2x_4 + R_8x_2 + R_5x_3^2 + R_6x_3 + x_4^2 - 1 = 0,$$

where:

$$R_5 = 0{,}193 \qquad R_6 = 0.4106217541E-3 \qquad R_7 = 0.5451766686E-3$$

$$R_8 = 0.44975E-6 \qquad R_9 = 0.3407354178E-4 \qquad R_{10} = 0.9615E-6$$

$$R = 10.$$

The solution given by LMS2.FOR is:

```
h =   .1000000000000E-04     Iter:    4719    fxnorm=  .9981537537787E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .3114827660508E-02     .8708039096472E-06
       2     .3458990467533E+02    -.4409059379817E-06
       3     .6504930784612E-01     .2203724754346E-06
       4     .8593780077147E+00     .8416787267151E-09
       5     .3695184957122E-01    -.1380675018758E-08
 ------------------------------------------------------------
h =   .1000000000000E-05     Iter:     477    fxnorm=  .9779276862365E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .3114830318720E-02     .8739929981544E-06
       2     .3458987529275E+02    -.4425255882290E-06
       3     .6504933543535E-01     .2211770430138E-06
       4     .8593780075575E+00     .8447664789912E-09
       5     .3695184953612E-01    -.1388563708460E-08
 ------------------------------------------------------------
h =   .1000000000000E-06     Iter:      54    fxnorm=  .8725222441331E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .3114884851937E-02     .9392861086005E-06
       2     .3458927247837E+02    -.4761553264454E-06
       3     .6504990143503E-01     .2374118648341E-06
       4     .8593780043335E+00     .9080820539964E-09
       5     .3695184881608E-01    -.1818957207078E-08
 ------------------------------------------------------------
h =   .1000000000000E-07     Iter:      13    fxnorm=  .2488880475000E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .3115025953045E-02     .1086573651332E-05
       2     .3458770588733E+02    -.6302696353155E-06
       3     .6505136744268E-01     .2343456701470E-06
       4     .8593779961106E+00     .1072952393599E-08
       5     .3695184695824E-01    -.4764511241095E-07
```

```
 ------------------------------------------------------------
h =   .1000000000000E-08     Iter:       9    fxnorm=  .4030944932120E-13
     === Solution ===
       n            x(i)                  f(x)
       1     .3115229058723E-02    -.1156008102771E-05
       2     .3458466261322E+02    -.8225482029223E-05
       3     .6505363137659E-01    -.5072569901732E-05
       4     .8596595100161E+00     .9682409468272E-03
       5     .3695184367521E-01     .4789729941601E-03
 ------------------------------------------------------------
```

Gauss-Newton method

```
h =   .0000000000000E+00     Iter:       9    fxnorm=  .3133111967483E-13
     === Solution ===
       n            x(i)                  f(x)
       1     .3114207641762E-02    -.9739969896172E-06
       2     .3459640509381E+02    -.3140451874994E-05
       3     .6504295171116E-01    -.2252082174248E-05
       4     .8596595857510E+00     .9683156343674E-03
       5     .3695185691980E-01     .4820239019967E-03
 ------------------------------------------------------------
```

<u>Application Q3.</u> Stationar solution of a chemical reactor.

The residuals are as follows:

$$
\begin{aligned}
&1 - x_1 - k_1 x_1 x_6 + r_1 x_4 = 0,\\
&1 - x_2 - k_2 x_2 x_6 + r_2 x_5 = 0,\\
&-x_3 + 2k_3 x_4 x_5 = 0,\\
&k_1 x_1 x_6 - r_1 x_4 - k_3 x_4 x_5 = 0,\\
&1{,}5\left(k_2 x_2 x_6 - r_2 x_5\right) - k_3 x_4 x_5 = 0,\\
&1 - x_4 - x_5 - x_6 = 0,
\end{aligned}
$$

where: $k_1 = 31{,}24 \quad k_2 = 0{,}272 \quad k_3 = 303{,}03 \quad r_1 = 2{,}062 \quad r_2 = 0{,}02$.

The solution given by LMS3.FOR is:

```
h =   .1000000000000E-01     Iter:       6    fxnorm=  .1098400295305E-13
     === Solution ===
       n            x(i)                  f(x)
       1     .9742459420319E+00    -.3810706958873E-06
       2     .9828291020383E+00     .3103860655447E-07
       3     .5150916779263E-01     .5248391618390E-06
       4     .9356712743612E+00    -.4072771266628E-06
       5     .9083433483791E-04     .1454068687901E-05
       6     .6423788844000E-01     .2864003353920E-08
 ------------------------------------------------------------
h =   .1000000000000E-02     Iter:       5    fxnorm=  .4400326046312E-18
     === Solution ===
       n            x(i)                  f(x)
       1     .9742435089340E+00     .1943647420788E-07
       2     .9828290656387E+00     .6461607386695E-08
       3     .5151298955793E-01    -.2503807229937E-04
       4     .9356712801626E+00     .1249588668226E-04
       5     .9079599503335E-04     .1241610668555E-04
       6     .6423811604606E-01    -.1922037037722E-06
 ------------------------------------------------------------
h =   .1000000000000E-03     Iter:       5    fxnorm=  .4213151023470E-21
     === Solution ===
       n            x(i)                  f(x)
       1     .9742436038490E+00     .1641891844706E-07
       2     .9828290729074E+00     .1224157514565E-08
       3     .5151279663935E-01    -.6934472075282E-04
       4     .9356712112919E+00     .3465377283390E-04
       5     .9071752932342E-04     .3466284330811E-04
```

```
       6     .6423810210228E-01    -.3092354962375E-07
 ------------------------------------------------------------
h =   .1000000000000E-04     Iter:       5    fxnorm=  .1672948839244E-21
     === Solution ===
       n            x(i)                  f(x)
       1     .9742436105458E+00     .1753614720101E-07
       2     .9828290740800E+00     .2196811052991E-09
       3     .5151277935719E-01    -.7632248730467E-04
       4     .9356711975686E+00     .3814348314417E-04
       5     .9070519529305E-04     .3816011557214E-04
       6     .6423810047422E-01    -.3238112264436E-08
 ------------------------------------------------------------
h =   .1000000000000E-05     Iter:       5    fxnorm=  .1650235746191E-21
     === Solution ===
       n            x(i)                  f(x)
       1     .9742436112470E+00     .1765599266790E-07
       2     .9828290742032E+00     .1143234524843E-09
       3     .5151277755098E-01    -.7705358074660E-04
       4     .9356711961195E+00     .3850911188354E-04
       5     .9070390300769E-04     .3852653866587E-04
       6     .6423810030283E-01    -.3253020630645E-09
 ------------------------------------------------------------
```

Gauss-Newton method

```
h =   .0000000000000E+00     Iter:       5    fxnorm=  .1649174591340E-21
     === Solution ===
       n            x(i)                  f(x)
       1     .9742436113254E+00     .1766938861891E-07
       2     .9828290742169E+00     .1025623971039E-09
       3     .5151277734927E-01    -.7713520362498E-04
       4     .9356711959576E+00     .3854993242372E-04
       5     .9070375873050E-04     .3856744796885E-04
       6     .6423810028368E-01     .0000000000000E+00
 ------------------------------------------------------------
```

<u>Application Q4.</u> Robot kinematics problem

The rsiduals are as follows:

$$
\begin{aligned}
&4{,}731(10^{-3})x_1x_3 - 0{,}3578x_2x_3 - \\
&0{,}1238x_1 + x_7 - 1{,}637(10^{-3})x_2 - 0{,}9338x_4 - 0{,}3571 = 0\,, \\
&0{,}2238x_1x_3 + 0{,}7623x_2x_3 + \\
&0{,}2638x_1 - x_7 - 0{,}07745x_2 - 0{,}6734x_4 - 0{,}6022 = 0\,, \\
&x_6x_8 + 0{,}3578x_1 + 4{,}731(10^{-3})x_2 = 0\,, \\
&-0{,}7623x_1 + 0{,}2238x_2 + 0{,}3461 = 0\,, \\
&x_1^2 + x_2^2 - 1 = 0\,, \\
&x_3^2 + x_4^2 - 1 = 0\,, \\
&x_5^2 + x_6^2 - 1 = 0\,, \\
&x_7^2 + x_8^2 - 1 = 0\,.
\end{aligned}
$$

The solution given by LMS4.FOR is:

```
h =   .1000000000000E-01     Iter:       7    fxnorm=  .5915886376069E-12
     === Solution ===
       n            x(i)                  f(x)
       1     .6715407682379E+00    -.9103012442724E-06
       2     .7409659480087E+00     .2729223913533E-05
       3     .9518912953669E+00     .2758624526834E-04
       4    -.3064339179757E+00     .1265153657554E-04
```

```
       5     .9638664122755E+00    -.2460485947764E-05
       6     .2665510553002E+00    -.1215718749670E-05
       7     .4046397414012E+00     .8792579453343E-04
       8    -.9144785048995E+00     .4256244496403E-05
 ----------------------------------------------------------
h =   .1000000000000E-02     Iter:       6    fxnorm=  .4302573775894E-14
     === Solution ===
       n            x(i)                  f(x)
       1     .6715527686521E+00    -.2692397836346E-06
       2     .7409565686515E+00    -.6987915945889E-07
       3     .9518930722387E+00     .3871444749719E-05
       4    -.3064311900992E+00     .1404520715009E-05
       5     .9639008615295E+00    -.2422879495922E-06
       6     .2665824548926E+00     .4952416730042E-06
       7     .4046416169527E+00     .1710761137776E-03
       8    -.9144756912210E+00     .6280042184059E-06
 ----------------------------------------------------------
h =   .1000000000000E-03     Iter:       6    fxnorm=  .1459868868863E-15
     === Solution ===
       n            x(i)                  f(x)
       1     .6715541172345E+00    -.2569210139081E-07
       2     .7409554927638E+00    -.4408150733326E-08
       3     .9518928980838E+00     .3819150107814E-06
       4    -.3064313362833E+00     .1357126497292E-06
       5     .9638962530280E+00    -.2536848286994E-07
       6     .2665868558520E+00     .2532788290210E-06
       7     .4046414710624E+00     .1645383144655E-03
       8    -.9144754751267E+00     .1147117394407E-06
 ----------------------------------------------------------
h =   .1000000000000E-04     Iter:       6    fxnorm=  .5891253549596E-16
     === Solution ===
       n            x(i)                  f(x)
       1     .6715542474078E+00    -.2556955203126E-08
       2     .7409553901825E+00    -.4151002874764E-09
       3     .9518928825003E+00     .3815377503606E-07
       4    -.3064313481262E+00     .1352390271991E-07
       5     .9638957952293E+00    -.2548168454020E-08
       6     .2665872887687E+00     .2308691529063E-06
       7     .4046414594802E+00     .1638865938012E-03
       8    -.9144754524705E+00     .6390146012336E-07
 ----------------------------------------------------------
h =   .1000000000000E-05     Iter:       6    fxnorm=  .5279143608685E-16
     === Solution ===
       n            x(i)                  f(x)
       1     .6715542603782E+00    -.2555914369040E-09
       2     .7409553799743E+00    -.4121025742876E-10
       3     .9518928809602E+00     .3829665152274E-08
       4    -.3064313492828E+00     .1351916634196E-08
       5     .9638957494805E+00    -.2549298550036E-09
       6     .2665873319885E+00     .2286459699352E-06
       7     .4046414583519E+00     .1638214433362E-03
       8    -.9144754501945E+00     .5882564080295E-07
 ----------------------------------------------------------
     Gauss-Newton method
h =   .0000000000000E+00     Iter:       6    fxnorm=  .5213964419254E-16
     === Solution ===
       n            x(i)                  f(x)
       1     .6715542618189E+00    -.2065014825803E-13
       2     .7409553788406E+00     .4807265696627E-13
       3     .9518928807893E+00     .1645896917021E-10
       4    -.3064313494110E+00     .5551115123126E-16
       5     .9638957443977E+00    -.1110223024625E-15
       6     .2665873367899E+00     .2283991491492E-06
       7     .4046414582269E+00     .1638142046612E-03
       8    -.9144754499415E+00     .5826171944889E-07
 ----------------------------------------------------------
```

<u>Application Q5</u>. Solid Fuel Ignition Problem

The solution given by LMS5.FOR is:

```
h =    .1000000000000E-02      Iter:      65     fxnorm=   .7624256186521E-12

h =    .1000000000000E-03      Iter:      10     fxnorm=   .7490028859566E-12

h =    .1000000000000E-04      Iter:       4     fxnorm=   .8641278511537E-14

h =    .1000000000000E-05      Iter:       3     fxnorm=   .5543097490450E-16

h =    .1000000000000E-06      Iter:       2     fxnorm=   .4891086164035E-13

      Gauss-Newton method
h =    .0000000000000E+00      Iter:       2     fxnorm=   .4793697162660E-14
```

The graphical representation of the solution is given in Figure Q5.

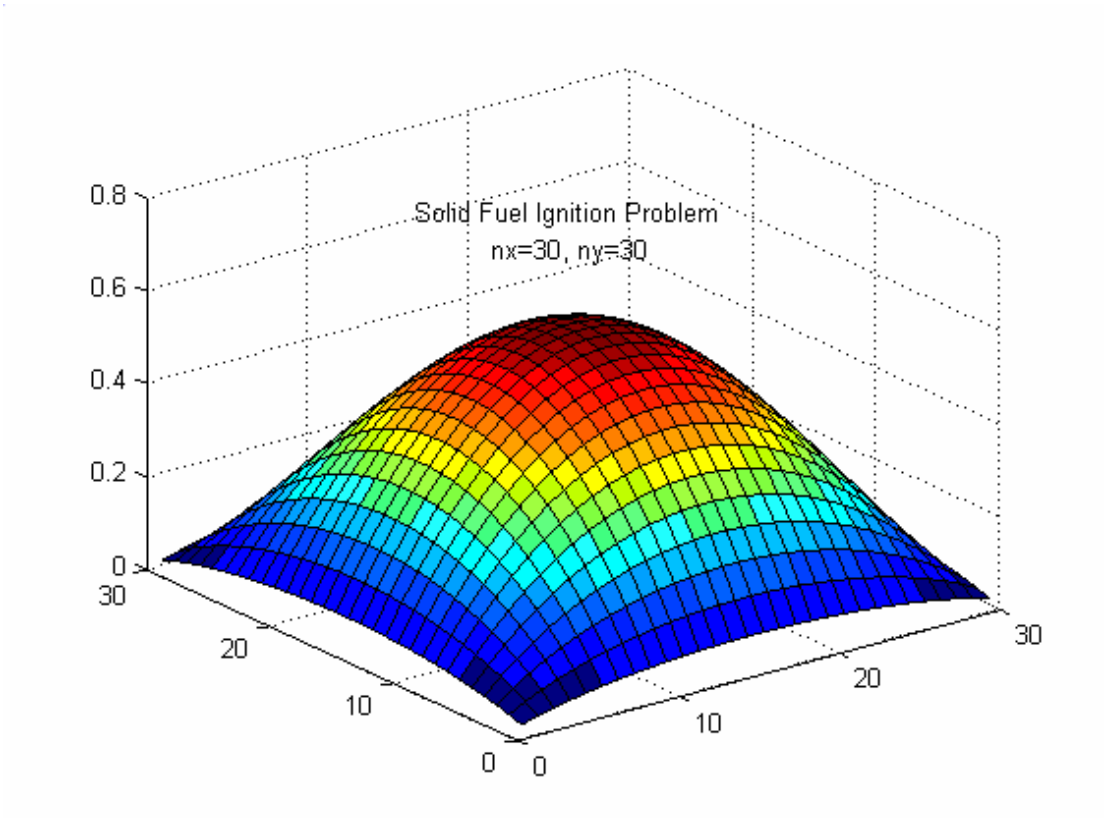

Fig. Q5. Solution of the Bratu Problem.

<u>Application Q6</u>. Flow in a Driven Cavity Problem

The solution given by LMS6.FOR is:

```
h =    .1000000000000E-04      Iter:      56     fxnorm=   .9851913736209E-12

h =    .1000000000000E-05      Iter:      11     fxnorm=   .3451363846708E-12

h =    .1000000000000E-06      Iter:       7     fxnorm=   .9548401270013E-14

h =    .1000000000000E-07      Iter:       7     fxnorm=   .5349712048329E-14

h =    .1000000000000E-08      Iter:       7     fxnorm=   .2155391695484E-13
```

```
      Gauss-Newton method
h =    .0000000000000E+00      Iter:        7    fxnorm=  .2555944142404E-13
```

The graphical representation of the solution is given in Figure Q6.

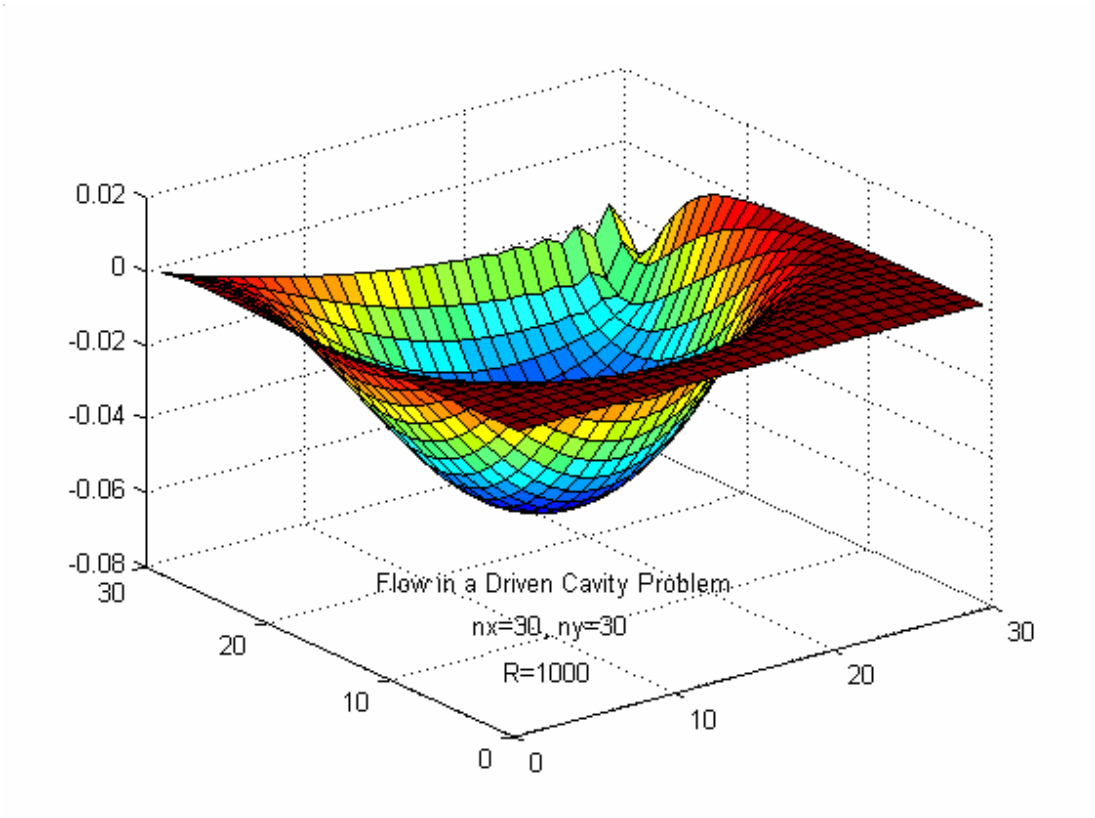

Fig. Q6. Flow in a Driven Cavity Problem.

January 10, 2007