AMO - Advanced Modeling and Optimization, Volume 9, Number 2, 2007

New efficiency using undesirable factors of Data Envelopment Analysis

J. Saffar Ardabili [a], N. Aghayi [b] and A. Monzali [c]
[a] Department of Mathematics, Payame Noor University , Ardabil Center,
P. O. Box. 55135-469, Ardabil, Iran
E-mail: saffar@pnu.ac.ir
[b] Department of Mathematics, Center Tehran Branch, Islamic Azad University, Tehran, Iran
[c] Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran

Abstract

In this paper, the possibility of suitable production is presented, and then a new method is suggested taking into account the existence of some undesirable components is the outputs and inputs of the Decision Making Units (DMUs) in the set.
AMS Subject Classification(2000): 90B30, 90C31
Key words: Data Envelopment Analysis, Undesirable Inputs And Outputs , Efficiency.

1 Introduction

When there are no undesirable input and output in the performance of DMUs, models of Data Envelopment Analysis (DEA) to increase efficiency are based on the output increase or input decrease. But many applied problems may consist of inputs whose increase and decrease results in efficiency increase and decrease, respectively, As Koopman (1951) represented. Such reclamation operation needs to increase undesirable inputs in order to increase efficiency or increase and decrease of undesirable outputs decrease and increase efficiency, respectively. Suppose undesirable outputs be factory wastes that should decrease in order to increase efficiency (e.g. Allen, (1999), Smith, (1991)).

There are direct and indirect methods for consideration and using undesirable outputs in DEA. In indirect methods, undesirable inputs and outputs in every single DMU change into desirable inputs and outputs with a decreasing monotonous function. And then DMUs efficiency is evaluated using standard models of DEA. Koopmans (1951), Golany and Roll (1989) introduced [ADD] and [MLT] methods, respectively, for measuring efficiency with undesirable inputs and outputs. In direct methods, there are some suppositions to Production Possibility Set (PPS), so in evaluation will obtain suitable input and output.

This paper is structured as follows: section 2 gives definitions of proportionate PPS to undesirable inputs and outputs. The method for measuring efficiency with undesirable inputs and outputs is shown in section 3. Finally, an example with undesirable inputs and outputs, and then the conclusion will be given.

2 Production Possibility Set

Suppose we have n observations on n DMUs with input and output vectors $(\mathbf{x}_j, \mathbf{y}_j)$ for $j = 1, 2, \ldots, n$. Let $\mathbf{x}_j = (x_{1j}, \ldots, x_{mj})^T$ and $\mathbf{y}_j = (y_{1j}, \ldots, y_{sj})^T$. All $\mathbf{x}_j \in R^m$ and $\mathbf{y}_j \in R^s$ and $\mathbf{x}_j > \mathbf{0}, \mathbf{y}_j > \mathbf{0}$ for $j = 1, 2, \ldots, n$. The input matrix $\mathbf{X}$ and output matrix $\mathbf{Y}$ can be represented as

$$\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n], \quad \mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_j, \ldots, \mathbf{y}_n]$$

where $\mathbf{X}$ is an $(m \times n)$ matrix and $\mathbf{Y}$ an $(s \times n)$ matrix.

The production possibility set T is generally defined as

$$T = \{(\mathbf{x}, \mathbf{y}) |\ \mathbf{x}\ \textit{can produce}\ \mathbf{y}\}. \tag{1}$$

In DEA, the production possibility set under a Variable Return to Scale (VRS) technology is constructed form the observed data $(\mathbf{x}_j, \mathbf{y}_j)$ for $j = 1, 2, \ldots, n$ as follows:

$$T = \left\{ (\mathbf{x}, \mathbf{y}) | \mathbf{x} \geq \sum_{j=1}^{n} \lambda_j \mathbf{x}_j, \mathbf{y} \leq \sum_{j=1}^{n} \lambda_j \mathbf{y}_j, \lambda_j \geq \mathbf{0}, \sum_{j=1}^{n} \lambda_j = 1, j = 1, \ldots, n \right\}. \tag{2}$$

In the absence of undesirable factors when a $DMU_o, o \in \{1, 2, \ldots, n\}$, is under evaluation, we can use the following BCC model:

$$\begin{array}{ll} \min & \theta \\ s.t & \theta \mathbf{x}_o - \mathbf{X}\lambda \geq \mathbf{0}, \\ & \mathbf{Y}\lambda \geq \mathbf{y}_o, \\ & \mathbf{1}^T \lambda = 1, \\ & \lambda \geq \mathbf{0}. \end{array} \tag{3}$$

Corresponding to each output $\mathbf{y}$, $L(\mathbf{y})$ is defined as the following:

$$L(\mathbf{y}_j) = \{\mathbf{x} | (\mathbf{x}, \mathbf{y}_j) \in T\}. \tag{4}$$

In fact, $L(\mathbf{y}_j)$ is a function that $\mathbf{y}_j$ portrays to a subset of inputs so that inputs can produce $\mathbf{y}_j$.

Now suppose that some inputs are undesirable so input matrix $\mathbf{X}$ can be represented as $\mathbf{X} = (\mathbf{X}^d, \mathbf{X}^u)^T$, where $\mathbf{X}^d = (x^d_{1j}, \ldots, x^d_{m_1 j}), j = 1, \ldots, n$ and $\mathbf{X}^u = (x^u_{1j}, \ldots, x^u_{m_2 j}), j = 1, \ldots, n$ are $(m_1 \times n)$ and $(m_2 \times n)$ matrixes that represent desirable (good) and undesirable (bad) inputs, respectively. And similarly, suppose that some outputs are undesirable so output matrix $\mathbf{Y}$ can be represented as $\mathbf{Y} = (\mathbf{Y}^g, \mathbf{Y}^b)^T$, where $\mathbf{Y}^g = (y^g_{1j}, \ldots, y^g_{s_1 j}), j = 1, \ldots, n$ and $\mathbf{Y}^b = (y^b_{1j}, \ldots, y^b_{s_2 j}), j = 1, \ldots, n$ are $(s_1 \times n)$ and $(s_2 \times n)$ matrixes that represent desirable (good) and undesirable (bad) inputs, respectively.

Definition 1: Let DMU of $(\mathbf{x}^d_1, \mathbf{x}^u_1, \mathbf{y}^g_1, \mathbf{y}^b_1)$ is dominant to DMU of $(\mathbf{x}^d_2, \mathbf{x}^u_2, \mathbf{y}^g_2, \mathbf{y}^b_2)$ if $\mathbf{x}^d_1 \leq \mathbf{x}^d_2$, $\mathbf{x}^u_1 \geq \mathbf{x}^u_2$, $\mathbf{y}^g_1 \geq \mathbf{y}^g_2$, and $\mathbf{y}^b_1 \leq \mathbf{y}^b_2$ and the unequal be strict at least in a component. So that,

$$\begin{bmatrix} -\mathbf{x}^d_1 \\ \mathbf{x}^u_1 \\ \mathbf{y}^g_1 \\ -\mathbf{y}^b_1 \end{bmatrix} \geq \begin{bmatrix} -\mathbf{x}^d_2 \\ \mathbf{x}^u_2 \\ \mathbf{y}^g_2 \\ -\mathbf{y}^b_2 \end{bmatrix}.$$

Definition 2: DMU_o is efficient if in T there is no DMU to be dominant over it.

We consider the properties of the Production Possibility Set as the following:

(1) T is convex.

(2) T is closed.

(3) The monotony property of desirable inputs and outputs. So that,

$$\forall \mathbf{u} \in R^{m_1}_+, \mathbf{v} \in R^{s_1}_+, (\mathbf{x}^d, \mathbf{x}^u, \mathbf{y}^g, \mathbf{y}^b) \in T \Longrightarrow (\mathbf{x}^d + \mathbf{u}, \mathbf{x}^u, \mathbf{y}^g - \mathbf{v}, \mathbf{y}^b) \in T.$$

This property is not necessarily established for undesirable factors, because in this case, T has no efficient DMU.

we can define the Production Possibility Set T satisfying (1) through (3) by

$$T = \{(\mathbf{x}^d, \mathbf{x}^u, \mathbf{y}^g, \mathbf{y}^b) | \mathbf{x}^d \geq \textstyle\sum_{j=1}^n \lambda_j \mathbf{x}^d_j, \mathbf{x}^u = \sum_{j=1}^n \lambda_j \mathbf{x}^u_j, \mathbf{y}^g \leq \sum_{j=1}^n \lambda_j \mathbf{y}^g_j,$$

$$\mathbf{y}^b = \sum_{j=1}^n \lambda_j \mathbf{y}^b_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq \mathbf{0}, j = 1, \ldots, n\}. \quad (5)$$

3 Measures Of Efficiency Using Undesirable Factors

In input oriented data, the efficiency of the DMU under evaluation is obtained by decreasing and increasing the desirable and undesirable input, respectively. And similarly, in output oriented data, we increase desirable output and decrease the undesirable output. Farell (1989) introduced a model to increase and decrease desirable and undesirable output, respectively. But there is a problem with his model and it is its nonlinear form. [TRβ] method introduced by Ali and Seiford (1990) simultaneously increase desirable outputs and decrease undesirable outputs, but measures of efficiency is dependent on the β value.

There are some other methods such as [WD] and [MLT] that were introduced by Far (1989) and Galony and Roll (1989) respectively, that decrease undesirable outputs only with decreasing desirable outputs. We, however, believe that in order to improve efficiency, desirable and undesirable outputs need to be increased and decreased respectively. suppose $\mathrm{DMU}_o = (\mathbf{x}_o^d, \mathbf{x}_o^u, \mathbf{y}_o^g, \mathbf{y}_o^b)$ be unit under evaluation, corresponding to the output $\mathbf{y}_o = (\mathbf{y}_o^g, \mathbf{y}_o^b)$, and using (2) $L(\mathbf{y}_o^g, \mathbf{y}_o^b)$ in defined as follows

$$L(\mathbf{y}_o^g, \mathbf{y}_o^b) = \{(\mathbf{x}^d, \mathbf{x}^u) | (\mathbf{x}^d, \mathbf{x}^u, \mathbf{y}_o^g, \mathbf{y}_o^b) \in T\}, \tag{6}$$

and we consider the subset of $L(\mathbf{y}_o^g, \mathbf{y}_o^b)$ as

$$\partial^s L(\mathbf{y}_o^g, \mathbf{y}_o^b) = \{(\mathbf{x}^d, \mathbf{x}^u) | \forall (\mathbf{u}, \mathbf{v}) \geq \mathbf{0}, (\mathbf{u}, \mathbf{v}) \neq \ \mathbf{0} \Rightarrow (\mathbf{x}^d - \mathbf{u}, \mathbf{x}^u + \mathbf{v}) \notin L(\mathbf{y}_o^g, \mathbf{y}_o^b)\}. \tag{7}$$

That $\partial^s L(\mathbf{y}_o^g, \mathbf{y}_o^b)$ includes all inputs of the efficient DMUs which can produce $(\mathbf{y}_o^g, \mathbf{y}_o^b)$.

The model to evaluate the efficiency of DMU_o with the most decrease of $\mathbf{x}_o^d$ and the most increase of $\mathbf{x}_o^u$ is as follows :

$$\begin{array}{lll} \gamma_o = \max & \beta - \alpha & \\ s.t & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j^d + \mathbf{s}^- = \alpha_o \mathbf{x}_o^d, & \\ & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j^u = \beta \mathbf{x}_o^u, & \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j^g - \mathbf{s}^+ = \mathbf{y}_o^g, & \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j^b = \mathbf{y}_o^b, & \\ & \sum_{j=1}^{n} \lambda_j = 1, & \\ & \beta_o \geq 1, & \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n. & \end{array} \tag{8}$$

The constraint $\beta \geq 1$ restricts the decrease of both α and β .

Theorem 1: The DMU_o in model (8) is efficient if and only if

1) $\alpha^* = \beta^* = 1$

2) All slacks be zero for all optimal solutions.

Theorem 2: If an optimal solution of model (8) be $(\alpha^*, \beta^*, \lambda^*, \mathbf{s}^-, \mathbf{s}^+)$, then

$$(\alpha^* \mathbf{x}^d - \mathbf{s}^{-*}, \beta^* \mathbf{x}^u) \in \partial^s L(\mathbf{y}_o^g, \mathbf{y}_o^b).$$

Considering Theorem 1 it is clear that $\beta^* = 1$ is the efficiency value and $\beta^* > 1$ is the inefficiency value of undesirable inputs. So $0 < \frac{1}{\beta^*} \leq 1$ shows the efficiency value of undesirable inputs. And $\alpha^* \leq 1$ shows the efficiency value of desirable inputs. Therefore, the efficiency value for the DMU_o is weight (geometrical) average α^* and β^*. So that,

$$\gamma_o = \sqrt[m_1+m_2]{\frac{(\alpha^*)^{m_1}}{(\beta^*)^{m_2}}} \tag{9}$$

where $0 < \gamma_o \leq 1$.

Remark 1: The DMU$_o$ in model (8) is efficient if and only if $\gamma_o = 1$.

Theorem 3: If DMU$_k$ be dominant over DMU$_j$ and γ_k, γ_j be the efficiency value in model (8) then, $\gamma_k \leq \gamma_j$.

Proof: Let optimal solutions of model (8) in evaluation of DMU$_k$ and DMU$_j$ be $(\alpha^*, \beta^*, \lambda^*, \mathbf{s}^-, \mathbf{s}^+)$ and $(\bar{\alpha}, \bar{\beta}, \bar{\lambda}, \bar{\mathbf{s}}, \bar{\mathbf{s}})$, respectively. We have:

$$\begin{bmatrix} -\mathbf{x}_j^d \\ \mathbf{x}_j^u \\ \mathbf{y}_j^g \\ -\mathbf{y}_j^b \end{bmatrix} \leq \begin{bmatrix} -\mathbf{x}_k^d \\ \mathbf{x}_k^u \\ \mathbf{y}_k^g \\ -\mathbf{y}_k^b \end{bmatrix} \Rightarrow \begin{bmatrix} -\alpha^*\mathbf{x}_j^d \\ \beta^*\mathbf{x}_j^u \\ \mathbf{y}_j^g \\ -\mathbf{y}_j^b \end{bmatrix} \leq \begin{bmatrix} -\alpha^*\mathbf{x}_k^d \\ \beta^*\mathbf{x}_k^u \\ \mathbf{y}_k^g \\ -\mathbf{y}_k^b \end{bmatrix},$$

since

$$(\alpha^*\mathbf{x}_k^d, \beta^*\mathbf{x}_k^u, \mathbf{y}_k^g, \mathbf{y}_k^b) \in T,$$

so,

$$(\alpha^*\mathbf{x}_j^d, \beta^*\mathbf{x}_j^u, \mathbf{y}_j^g, \mathbf{y}_j^b) \in T.$$

Then $(\alpha^*, \beta^*, \lambda^*, \mathbf{s}^-, \mathbf{s}^+)$ is a feasible solution for model (8) in evaluating DMU$_j$ Thus, $\bar{\beta} - \bar{\alpha} \geq \alpha^* - \beta^*$ so that, $\gamma_j \geq \gamma_k$.

4 Numerical example

As an example, consider seven DMUs with one desirable input, one undesirable input and one desirable outputs.

Regarding Table 1 and Figure 1, it can be seen that DMUs D, E, and F are efficient and they are on the $\partial^s L(\mathbf{y}_G^g)$. On the other hand, efficiency of other DMUs have been examined through their image on $\partial^s L(\mathbf{y}_G^g)$. (Efficient Frontiers)

Table 1. The inputs and outputs data for 5 DMUs.

DMU_j	x^d	x^u	y^g	β^*	α^*	γ_j
A	3	1	1	7	1	0.37
B	2	2	1	2.5	0.5	0.45
C	1	3	1	1.66	1	0.78
D	1	5	1	1	1	1
E	2	6	1	1	1	1
F	3	7	1	1	1	1
G	4	4	1	1.75	0.75	0.65

Similar discussion can be presented for the output oriented.

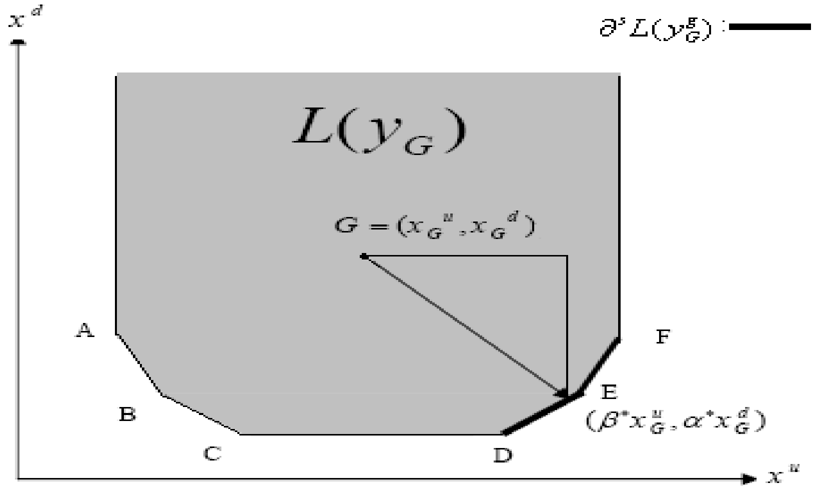

Figure 1: The graph of the $L(\mathbf{y}_G)$

5 Conclusion

Throughout this paper, a new model is defined for the evaluation of efficiency where some inputs and outputs may be undesirable. Also, this model assures that the DMUs under evaluation will be compared with a corresponding unit of $\partial^s L(\mathbf{y}_G^g)$. (Efficient Frontiers)

References

[1] Allen, K., 1999. DEA in the ecological context - an overview. In Wassermann. G. (Ed.), Data Envelopment Analysis in the service sector. Galber, Wiesbaden. 203-235.

[2] Ali, A.I., Seiford, L.M., 1990. Translation invariance in Data Envelopment Analysis. Operational Research Letters 9, 403-405.

[3] Banker, R.D, Charnes, A., Cooper, W.W., 1984. Some models for estimating technical and scale inefficiencies in Data Envelopment Analysis. Management Science 30, 1078-1092.

[4] Fare, R., Grosskopf, S., Lovel, C.A.K., Pasurka, C., 1989. Multilateral productivity comparisons when some outputs are undesirable: A nonparametric approach. The Review Economics and Statistics 71, 90-98.

[5] Golany, B., Roll, Y., 1989. An application procedure for DEA. Omega: The International of Management Science 17,237-250.

[6] Koopmans, T.C., 1951. Analysis of production as an efficient combination of activities. In: Koopmans, T.C. (Ed), Activity Analysis of production and allocation. Cowles Commission, Wiely, New York, 33-97.

[7] Lawrence, M. seiford, Zhu, J., 2002. Modeling undesirable factor in efficiency evaluation. European Journal of Operational Research 142, 16-20.

[8] Lewis, H.F., Sexton, T.r., 1999. Data Envelopment Analysis with reverse inputs. Paper presented at North America Productivity Workshop, Union College, Schenectady, NY,July 1999.

[9] Lovell, C.A.K., Pastor, J.T., 1995. Units invariant and translation invariant DEA models. Operational Research Letters 18, 147-151.

[10] Pastor, T., 1996. Translation invariance in DEA: Generalization Annals of Operational Research 66, 93-102.

[11] Smith, P., 1990. Data Envelopment Analysis applied to financial statements. Omega: The International of Management Science 18, 131-138.