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Self-tuning optimal control of periodic-review production inventory systems with deteriorating items

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Abstract

Control of inventory in production and operation systems is very important for better management and utilization of resources. In this paper, the optimal control of a periodic-review inventory system with items deterioration is considered. We are concerned with the problem of determining the production rate that minimizes a certain cost function. We deal with both cases where the deterioration coefficient is known and unknown. In the case where it is unknown, the recursive least squares algorithm is used to estimate this parameter. The main control objective is to maintain the inventory at a desired level in spite of the unknown model parameter and variable demand. The self-tuning optimal control is presented and the effectiveness of the proposed control algorithm is illustrated by simulation.

Keywords: Inventory systems, periodic-review, deterioration, optimal control, parameter estimation, self-tuning controller, goal.

1 Introduction

The inventory of a manufacturing system can be reviewed continuously or periodically. In a continuous-review model, the inventory is monitored continually and production can be started at any time. In contrast, in periodic-review models, there is some standard time when the inventory is reviewed and a decision is made whether to produce or not. The continuous-review policy is efficient in utilizing the system facilities and can certainly

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be implemented, given today's computerized access to inventory levels in real time. The continuous-review models can result in lower annual costs than periodic-review models. However, when a company stocks many products (hundreds or even thousands), it might be more convenient to produce periodically and a continuous-review policy may be expensive and unnecessary.

The following example of a periodic-review policy in a manufacturing system is mentioned in Bozer and Srinivisan (1991). Consider a flexible manufacturing system (FMS) that consists of a number of workstations connected by a single automated guided vehicle (AGV). Jobs arrive at the FMS and request service at a particular workstation, each workstation having a number of identical machines that can perform various operations. The AGV visits sequentially the workstations and loads/unloads the jobs. When a job arrives at the FMS, it has to wait in a receiving area for the AGV to come. Once the AGV comes and finds a machine is available in the desired workstation, the first job in the line is loaded and sent to that workstation for processing. After a job is finished at the workstation, it waits again for the AGV to come for departure. When the AGV comes, it loads the job to the finished area and the job leaves the FMS. If we consider a single station, since it is visited periodically by the AGV, a periodic-review policy is more appropriate than a continuous one.

We consider in this paper the problem of controlling the production rate of a periodic-review manufacturing system with unknown deteriorating rate of items. Items deterioration is of great importance in inventory theory, as shown by the surveys of Nahmias (1982), Raafat (1991), and Goyal and Giri (2001). The self-tuning optimal control approach is to be novel in this framework. There seems to be no literature on the optimal control of periodic-review manufacturing systems with unknown deteriorating items rate. The optimal control of continuous-review models with deteriorating items has been addressed by Bounkhel and Tadj (2005) and Tadj *et al.* (2005). Optimal control of inventory systems where items deterioration has not been taken into account is available though. We cite for example Salama (2000), Riddals and Bennett (2001), Zhang *et al.* (2001), Khemlnitsky and Gerchak (2002), Kiesmüller (2003), and Dobos (2003).

The optimal control approach allows us to determine the optimal strategy (choice of the production rate) so that the overall costs are minimized. We deal with both the finite and infinite planning horizon. Besides the time-varying optimal control, we also consider the self-tuning optimal control. Indeed, the deterioration parameters may be unknown in the real world. The proposed control algorithm estimates online these coefficients and feeds the controller to take the optimal production decision. Simulations and sensitivity analysis are conducted to evaluate the quality of the solutions obtained.

Following this introduction, the model is introduced in Section 2. Section 3 deals with the optimal control problem when the deterioration rate is known while Section 4 deals with the self-tuning counterpart. Our results are summarized in Section 5, where we also provide some directions for further related research.

2 Model Formulation

Let us consider a manufacturing firm producing a single product. To state the model we use the following notation:

$I(t)$	:	inventory level at time t,
$P(t)$	:	production rate at time t, $(P(t) \geq 0)$,
$D(t)$	:	demand rate at time t,
θ	:	deterioration coefficient, $(\theta > 0)$,
q	:	penalty cost for deviating from the inventory goal level $\hat{I}$, $(q > 0)$,
r	:	penalty cost for deviating from the control goal level $\hat{P}$, $(r > 0)$,
H	:	length of the planning horizon, $(H > 0)$,
I_0	:	initial inventory level,
$\hat{I}(t)$	:	inventory goal level at time t,
$\hat{P}(t)$	:	production goal rate at time t, $(\hat{P}(t) \geq 0)$.

The interpretation of the goal rates is as follows

- The inventory goal level $\hat{I}(t)$ is a safety stock that the company wants to keep on hand at time t. For example, $\hat{I}(1)$ could be 50 units of the finished product on day 1 and $\hat{I}(2)$ could be 70 units of the finished product on day 2, etc.
- The production goal rate $\hat{P}(t)$ is the most efficient rate desired by the firm at time t.

We therefore assume that the firm has set an inventory goal level $\hat{I}$ and a production goal rate $\hat{P}$ and is looking for the a pair (P, I) which converges to $(\hat{P}, \hat{I})$ and minimizes some cost function.

Since demand occurs at rate D, production occurs at the controllable rate P, and deterioration occurs at rate θ, it follows that the inventory level $I(t)$ evolves according to the state equation

$$\frac{d}{dt}I(t) = P(t) - D(t) - \theta I(t), \quad \forall t \in [0, H]. \tag{2.1}$$

Note that the goal pair $(\hat{P}, \hat{I})$ must satisfy the previous differential equation, to be feasible. The solution of (2.1) is given by

$$I(t) = I(0)e^{-\theta t} + \int_0^t [P(\tau) - D(\tau)]e^{-\theta(\tau - t)} d\tau, \quad \forall t \in [0, H]. \tag{2.2}$$

In order to state the periodic (discrete) optimal control problem associated with the production inventory system with deteriorating items governed by (2.1), we divide the planning horizon $[0, H]$ into $(N + 1)$ intervals $[t_k, t_{k+1}]$, with $t_k = kT$, $k = 0, 1, \cdots, N$ and we assume that the functions θ and $u = P - D$ are constant over each interval $[t_k, t_{k+1}]$. Let $I(k), P(k), D(k), \hat{I}(k), \hat{P}(k)$, and $u(k)$ denote respectively the value of the inventory level, the production rate, the demand rate, the inventory goal level, the production goal rate, and the control function over each interval $[t_k, t_{k+1}]$. Simple computations give the following simpler form of (2.2) at the end points t_k

$$I(t_{k+1}) = I(t_k)e^{-\theta T} + \int_{t_k}^{t_{k+1}} u(\tau)e^{-\theta(t_{k+1} - \tau)} d\tau, \quad \forall k = 0, 1, \cdots, N, \tag{2.3}$$

with $I(0) = I_0$. Using the fact that the function u is constant over each interval we can rewrite Equation (2.3) as follows

$$I((k + 1)T) = a\, I(kT) + b\, u(kT), \tag{2.4}$$

where $a = e^{-\theta T}$ and $b = \frac{1 - e^{-\theta T}}{\theta}$. Without loss of generality we may take $T = 1$. Introduce the shifted variables $\Delta I(k)$ and $\Delta u(k)$ as

$$\Delta I(k) = I(k) - \hat{I}(k) \quad \text{ and } \quad \Delta u(k) = u(k) - \hat{u}(k) = P(k) - \hat{P}(k),$$

where $\hat{u}(k) = \hat{P}(k) - D(k)$. Note that the inventory goal pair $(\hat{u}(k), \hat{I}(k))$ must verify the equation

$$\hat{I}(k + 1) = a\, \hat{I}(k) + b\, \hat{u}(k), \tag{2.5}$$

and therefore Equation (2.4) leads to

$$\Delta I(k+1) = a\ \Delta I(k) + b\ \Delta u(k). \tag{2.6}$$

Now we are in position to state our discrete model associated with the above production inventory system. It is a shifted input-output model that consists of minimizing the following objective function

$$J = \frac{1}{2} \sum_{k=0}^{N} \left[q \Delta I(k)^2 + r \Delta u(k)^2 \right], \tag{2.7}$$

subject to Equation (2.6).

3 Known Deterioration Coefficient

In order to determine the optimal control for this model, we will assume, in a first step, that the firm knows the value θ of the deterioration coefficient. The case where the deterioration coefficient is unknown is considered in the next section.

Finite Horizon. The main tool in the analysis of the above model is the method of Lagrange multipliers which consists of combining Equation (2.7) and Equation (2.6) as the Lagrangian

$$L = \frac{1}{2} \sum_{k=0}^{N} \left[q \Delta I(k)^2 + r \Delta u(k)^2 \right] + \lambda(k+1) \left[-\Delta I(k+1) + a \Delta I(k) + b \Delta u(k) \right], \tag{3.1}$$

where $\lambda(k+1)$ is the Lagrange multiplier. Proceeding with the minimization leads to the control equation

$$\frac{\partial L}{\partial \Delta u(k)} = r \Delta u(k) + \lambda(k+1) b = 0,$$

and the adjoint equation

$$\frac{\partial L}{\partial \Delta I(k)} = q \Delta I(k) - \lambda(k) + a \lambda(k+1) = 0.$$

The control equation leads to

$$\Delta u(k) = -r^{-1} b \lambda(k+1), \tag{3.2}$$

and the adjoint equation to

$$\lambda(k) = q\Delta I(k) + a\lambda(k+1). \tag{3.3}$$

The sweep method (by Bryson and Ho (1975)) assumes that

$$\lambda(k) = s(k)\Delta I(k), \tag{3.4}$$

with $s(k) > 0$, for $k = 0, 1, \cdots, N$. Substituting Equations (3.4) and (2.6) into (3.2) yields

$$\begin{aligned} \Delta u(k) &= -r^{-1}\, b\, s(k+1)\Delta I(k+1) \\ &= -r^{-1} b\, s(k+1)\left[a\Delta I(k) + b\Delta u(k)\right], \end{aligned}$$

which gives

$$\Delta u(k) = -\alpha(k+1)\Delta I(k), \tag{3.5}$$

where

$$\alpha(k+1) = \frac{a\, b\, s(k+1)}{r + b^2\, s(k+1)}.$$

If we substitute Equation (3.4) into Equation (3.3), then we obtain

$$s(k)\Delta I(k) = as(k+1)\Delta I(k+1) + q\Delta I(k). \tag{3.6}$$

Combining (3.5) and (3.6) with (2.6) leads to

$$s(k) = a^2\, s(k+1) + a\, b\, s(k+1)\left\{-\left[r + b^2\, s(k+1)\right]^{-1} a\, b\, s(k+1)\right\} + q,$$

or

$$s(k) = a^2\left[s(k+1) - \frac{b^2\, s(k+1)^2}{r + b\, s(k+1)}\right] + q. \tag{3.7}$$

Equation (3.7) is called discrete Riccatti equation. Note that $\Delta u(N)$ has no effect on the minimum of the objective function, so we may take $\Delta u(N) = 0$. This ensures, along with Equations (3.2) and (3.4), that

$$s(N) = q. \tag{3.8}$$

The recursion equation (3.7) for $s(k)$ is solved backwards starting from the end point N. Therefore, the desired optimal solutions $(I(k), P(k))$ are given by the following recursive formulas

$$I(k+1) = \hat{I}(k+1) + [a - b\alpha(k+1)]\left[I(k) - \hat{I}(k)\right],$$

and

$$P(k) = \hat{P}(k) - \alpha(k+1)\left[I(k) - \hat{I}(k)\right].$$

Infinite Horizon. In this case, we look for the steady state solution of the Riccatti equation (3.7). We will denote s_∞ the limit of the sequence $s(N)$ when $N \to \infty$. From (3.7) we have

$$s_\infty = a^2\left[s_\infty - \frac{b^2\, s_\infty^2}{r + b^2\, s_\infty}\right] + q,$$

which has the following positive solution

$$s_\infty \;=\; \frac{-\left[(1-a^2)\, r - q\, b^2\right] + \sqrt{\left[r(1-a^2) - q\, b^2\right]^2 + 4\, b^2 q\, r}}{2\, b^2}. \tag{3.9}$$

In this case the desired optimal solutions $(I(k), P(k))$ are given by

$$I(k+1) = \hat{I}(k+1) + [a - b\alpha_\infty]\left[I(k) - \hat{I}(k)\right],$$

and

$$P(k) = \hat{P}(k) - \alpha_\infty\left[I(k) - \hat{I}(k)\right],$$

where

$$\alpha_\infty = \frac{a\, b\, s_\infty}{r + b^2\, s_\infty}.$$

Simulation. To illustrate the results obtained, we perform a simulation in the infinite horizon case. We let $\theta = 0.01$, $T = 1$, $q = r = 1$, and

$$\hat{I}(k) = \begin{cases} 1, & 0 \le k \le 15, \\ 2, & k > 15, \end{cases}, \qquad D(k) = \begin{cases} k, & 0 \le k \le 15, \\ 10, & k > 15. \end{cases}$$

Then we find $a = 0.99$, $b = 0.995$, and

$$\hat{P}(k) = \begin{cases} k + 0.01, & 0 \le k \le 15, \\ 10.02, & k > 15, \end{cases}, \qquad \hat{u}(k) = \begin{cases} 0.01, & 0 \le k \le 15, \\ 0.02, & k > 15. \end{cases}$$

Two simulations were conducted in this case, the first one for $I(0) = 0$, i.e., $I_0 < \hat{I}(0)$. The simulation results are shown in Figure (1-a) and Figure (1-b).

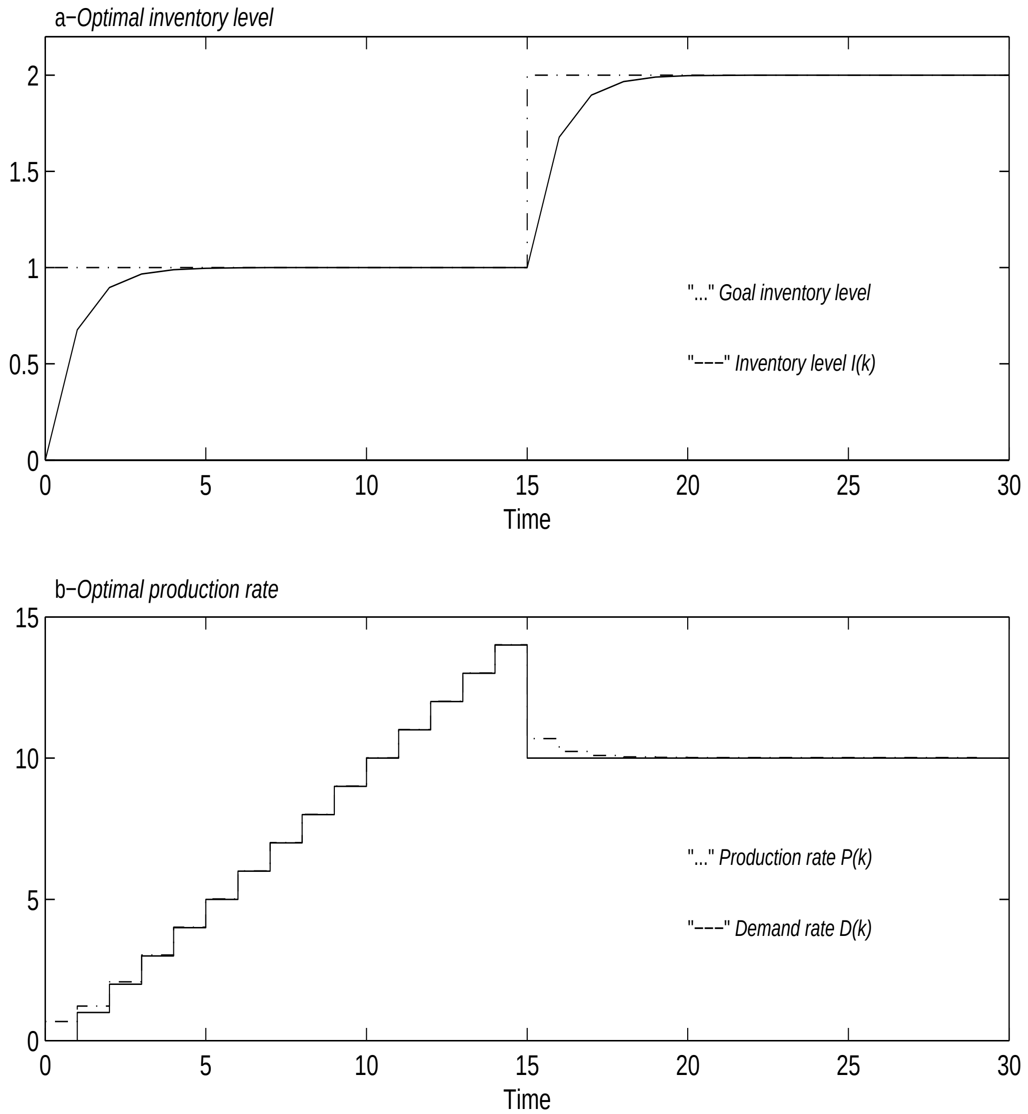

Figure 1. Simulation results, θ known, infinite horizon case, $I_0 < \hat{I}(0)$.

The simulation has been conducted for 30 units of time only. We observe from the figure that the inventory level I and the production rate P converge to the desired goal inventory level $\hat{I}$ and the desired goal production rate $\hat{P}$.

In the case when $I_0 > \hat{I}(0)$, (we took $I_0 = 1.5$ and $\hat{I}(0) = 1$), the simulation results are given in Figure 2.

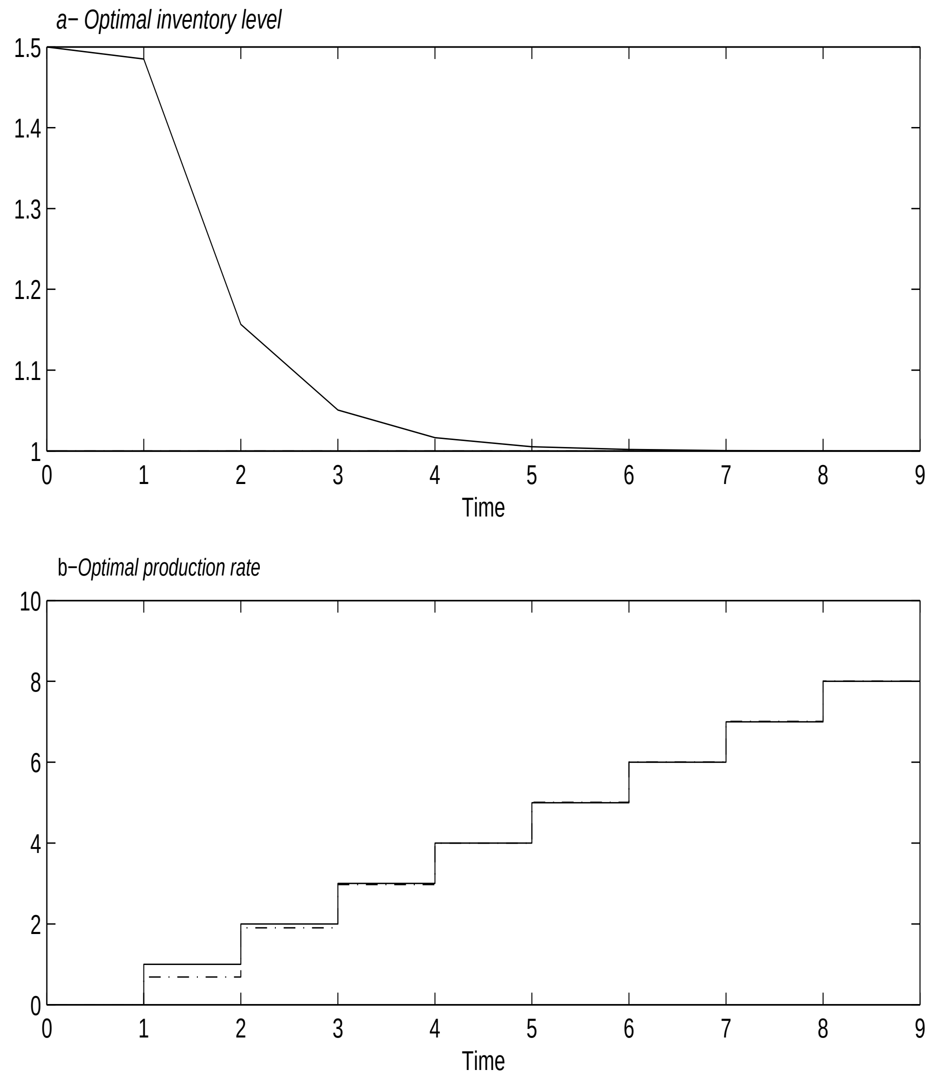

Figure 2. Simulation results, θ known, infinite horizon case, $I_0 > \hat{I}(0)$.

From Figure 2 we observe that the optimal production rate P is equal to zero in the first step. This is due to the control algorithm which assigns the value zero as an optimal production whenever the computed production rate P is negative. The simulation has been conducted for 9 units of time only. Note that I converges to $\hat{I}$ and P converges to $\hat{P}$ in this case too.

4 Self-Tuning Optimal Control

In reality, the rate of items deterioration may be unknown and can vary with time. In this section, the deterioration coefficients a and b of the periodic-review model are estimated online and the previous control algorithm will be used with the estimated parameters. The dynamic relation between the input and the output of the inventory can be written as

$$I(k) = aI(k-1) + bu(k-1) = [I(k-1) \quad u(k-1)] \begin{bmatrix} a \\ b \end{bmatrix} = \phi(k)\beta, \tag{4.1}$$

where $\phi(k) = [I(k-1) \quad u(k-1)]$ and $\beta = [a \quad b]^\top$.

Using the observations $u(0), u(1), \cdots, u(N)$ and $I(0), I(1), \cdots, I(N)$, we wish to compute the values of a and b, which will best fit the observed data.

Let $e(k, \hat{\beta})$ be the equation error defined as $e(k, \hat{\beta}) = I(k) - \phi(k)\hat{\beta}$, where $\hat{\beta}$ is the estimate of β. The principle of least squares says that the estimate $\hat{\beta}$ is the point of minimum of the performance measure

$$J_2(\hat{\beta}) = \sum_{k=0}^{N} e^2(k, \hat{\beta}) = E^\top(N, \hat{\beta})E(N, \hat{\beta}), \tag{4.2}$$

where

$$E(N, \hat{\beta}) = [e(0, \hat{\beta}) \quad e(1, \hat{\beta}) \cdots e(N, \hat{\beta})]^\top = Y(N) - \Phi(N)\hat{\beta},$$

$$Y(N) = [I(0) \quad I(1), \cdots, I(N)]^\top, \quad \text{and} \quad \Phi(N) = [\phi(0) \quad \phi(1), \cdots, \phi(N)]^\top.$$

Taking the derivative of J_2 with respect to $\hat{\beta}$, we obtain

$$\frac{\partial J_2(\hat{\beta})}{\partial \hat{\beta}} = 0 \Longleftrightarrow \hat{\beta} = \left[\Phi(N)^\top \Phi(N)\right]^{-1} \Phi(N)Y(N). \tag{4.3}$$

If $\Delta u(k)$ is persistently exciting, then (see for instance Lyung (1987)) the matrix $\Phi(N)^\top \Phi(N)$ is non singular and so the solution is well defined.

Recursive Least Squares Algorithm (Lyung (1987)). The least squares calculation for $\hat{\beta}$ given by Equation (4.3) is a batch calculation since one has a batch of data from which the matrix Φ and vector Y are composed. However, the observations of the inventory are obtained sequentially. If the least squares problem has been solved for k observations, it seems to be a waste of computational resources to start from scratch when a new observation is obtained. Moreover, the decision of the optimal production should be taken at each period. Hence it is desirable to arrange the computations in such a way that the results obtained for k observations can be used in order to get the estimates for $(k+1)$ observations.

Let $\hat{\beta}(k)$ denote the least squares estimate based on k measurements. Then, from Equation (4.3) we have

$$\hat{\beta}(k) = \left[\Phi^\top(k)\Phi(k)\right]^{-1} \Phi(k)Y(k).$$

It is assumed that the matrix $\left[\Phi(k)^\top\Phi(k)\right]$ is nonsingular for all k. When an additional measurement is obtained a row is added to the matrix Φ and an element is added to the vector Y. Hence

$$\Phi(k+1) = \begin{bmatrix} \Phi(k) \\ \phi(k+1) \end{bmatrix}; \qquad Y(k+1) = \begin{bmatrix} Y(k) \\ I(k+1) \end{bmatrix}.$$

The estimated $\hat{\beta}(k+1)$ based on $(k+1)$ measurements can be written as

$$\begin{aligned} \hat{\beta}(k+1) &= \left[\Phi^\top(k+1)\Phi(k+1)\right]^{-1} \Phi(k+1)Y(k+1) \\ &= \left[\Phi^\top(k)\Phi(k) + \phi^\top(k+1)\phi(k+1)\right]^{-1} \left[\Phi^\top(k)Y(k) + \phi^\top(k+1)I(k+1)\right]. \end{aligned} \tag{4.4}$$

We define the (2×2) matrix M as

$$M(k+1) = \left[\Phi^\top(k+1)\Phi(k+1)\right]^{-1} = \left[M^{-1}(k) + \phi^\top(k+1)\phi(k+1)\right]^{-1}.$$

By using the well-known matrix inversion Lemma, the following result can be established

$$M(k+1) = M(k) - M(k)\phi^\top(k+1)\left[1 + \phi(k+1)M(k)\Phi^\top(k+1)\right]^{-1} \phi(k+1)M(k).$$

Substituting the expression for $M(k+1)$ into Equation (4.4), we obtain

$$\begin{aligned} \hat{\beta}(k+1) &= \hat{\beta}(k) + G(k)\left[I(k+1) - \phi(k+1)\hat{\beta}(k)\right], \\ G(k) &= M(k)\phi^\top(k+1)\left[1 + \phi(k+1)M(k)\phi^\top(k+1)\right]^{-1}, \\ M(k+1) &= \left[I_2 - G(k)\phi(k+1)\right]M(k), \end{aligned}$$

where I_2 is the 2×2 identity matrix. We note that any recursive algorithm requires some initial value to be started up and it is common to start the recursion at $k=0$ with some invertible matrix $M(0)$ and a vector $\hat{\beta}(0)$.

Simulation. Let all the parameters I_0, θ, T, q, r, and $\hat{I}(k)$ be kept as in the first simulation and let $\hat{\beta}(0) = [0.5 \quad 0.4]^\top$ and $M(0) = 100I_2$. Implementation of the recursive least squares algorithm gave the estimates of the unknown parameters a and b shown in Figure 3.

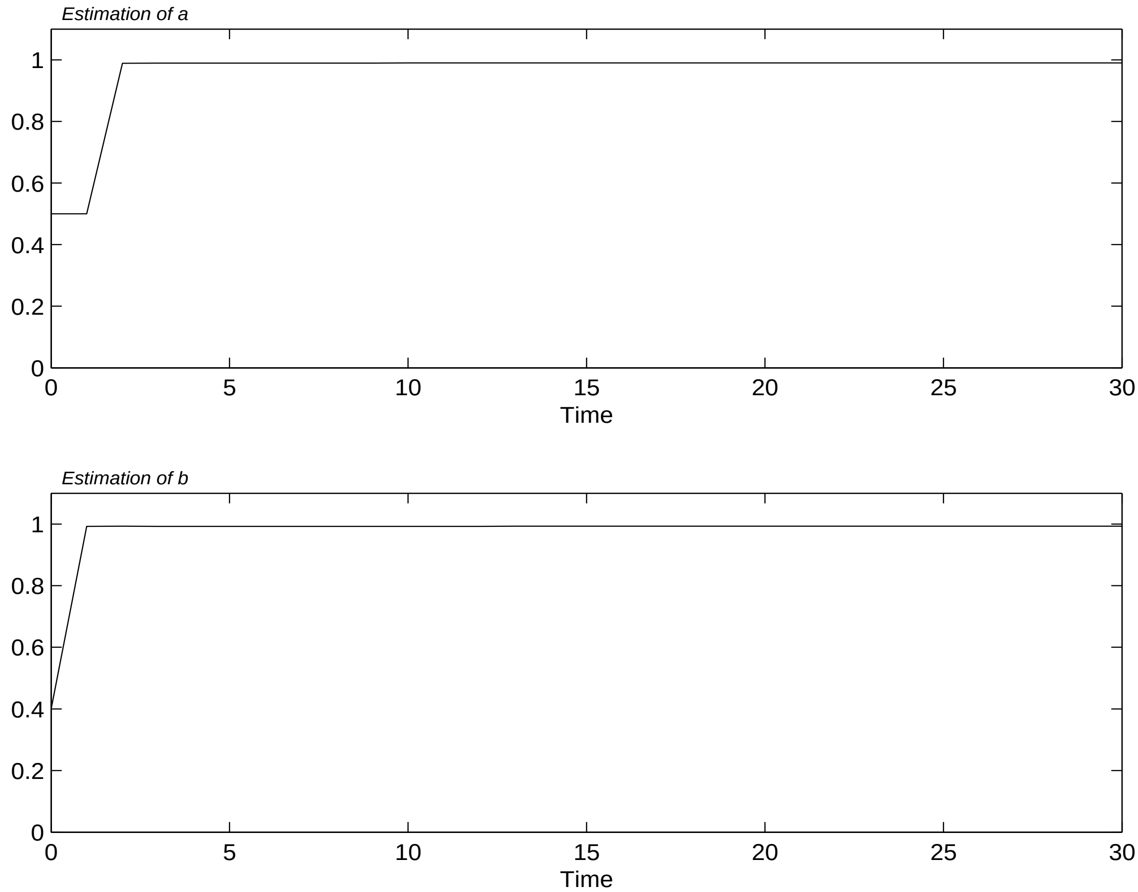

Figure 3. Parameters Estimation.

Observe that the estimated parameter vector $\hat{\beta}$ reaches the vector β after 2 steps. Figure 4 shows the convergence of I and P to $\hat{I}$ and $\hat{P}$, respectively. Note the pick in the inventory level curve in the initial phase of the simulation. It is due to the fact that the vector $\hat{\beta}$ had not reached the value β yet.

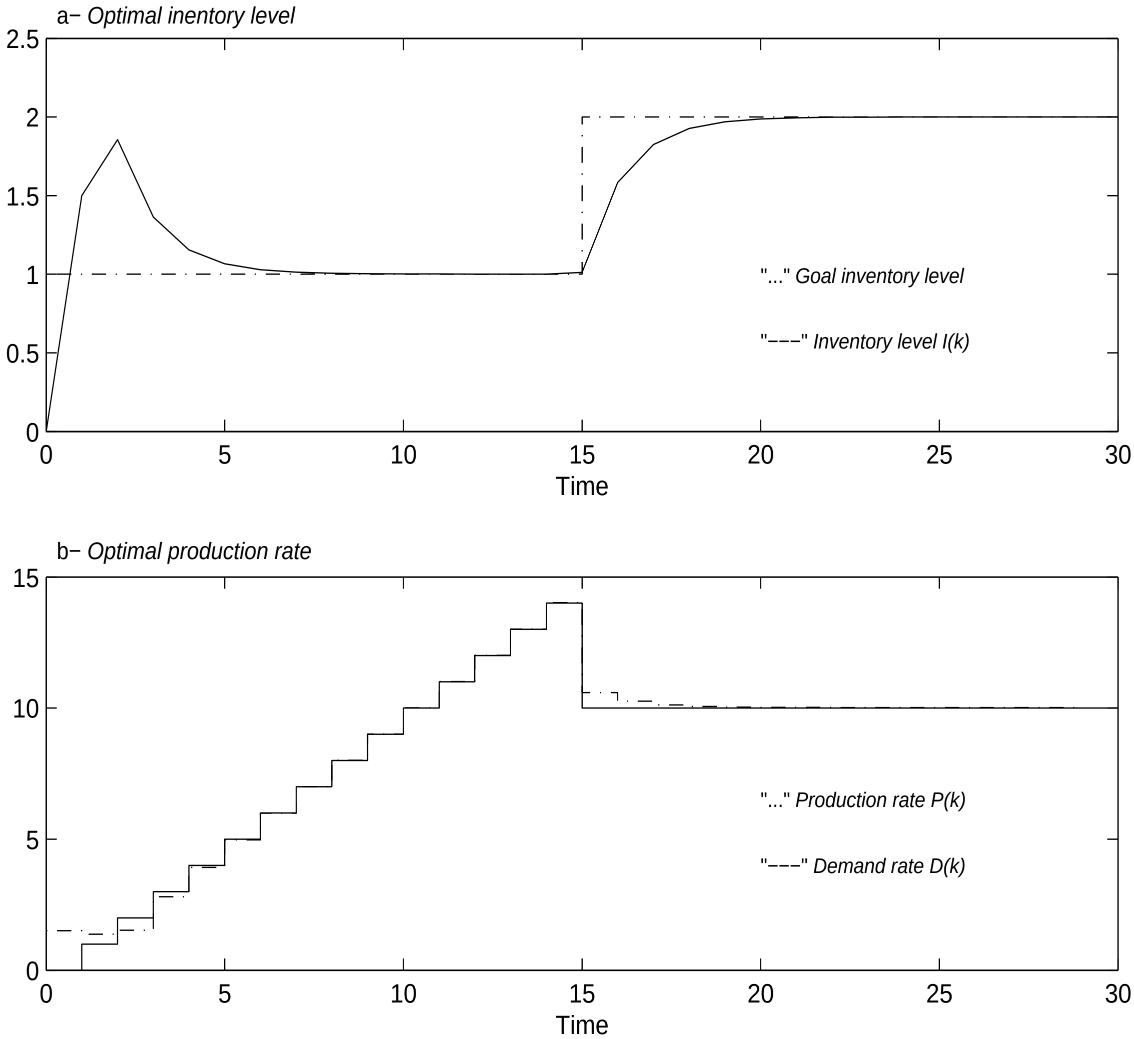

Figure 4. Simulation results, θ unknown.

5 Conclusion

We presented in this paper the optimal control of a periodic-review production inventory system with deteriorating items. The system is modelled by a recurrent difference equation where parameters, linked to the item deterioration rate, are unknown. An online algorithm (recursive least squares) has been used to estimate these parameters which allow the controller to take an optimal production decision. Numerical simulation shows the effectiveness of the proposed algorithm. The multi-variable and stochastic cases are being investigated by the authors.

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