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Support Set Invariancy Sensitivity Analysis in Bi-parametric Linear Optimization

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Abstract

In bi-parametric linear optimization, perturbation happens in both the right-hand-side and the objective function coefficient with different nonzero parameters. In this paper we are interested in identifying the region where the support set of a given optimal solution for primal and dual problem is invariant in bi-parametric linear optimization. In general, we denote that this region is a rectangle in $\mathbb{R}^2$ if its not a line or a singleton. It is proved that the boundaries of these regions can be identified in polynomial time.

Keywords: Parametric Optimization, Sensitivity Analysis, Linear Optimization, Interior Point Method, Optimal Partition, Support Set Invariancy.

1 Introduction

In practice, input data in optimization problems might perturb according to time or economic requirements. Sometimes, this variation cause to lose optimality or feasibility of optimal solutions. Investigating the behavior of the problem when input data changes is referred to as *sensitivity analysis* and *parametric programming.*

In Linear Optimization (LO) problem, variation usually happens in the Right-Hand-Side (RHS) or in the objective function coefficient data. If perturbation occurs either in the RHS or in objective function coefficient or in the both but with identical parameter, then the problem is called *uni-parametric* programming. However, when these parameters vary independently, the problem is referred to as *bi-parametric* LO problem.

Let a bi-parametric LO problem be defined as

$$LP(\epsilon,\lambda) \qquad\qquad \min \quad \{(c+\lambda\Delta c)^T x \mid Ax = b+\epsilon\Delta b,\ x\geq 0\},$$

where $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, and $c\in\mathbb{R}^n$, ϵ and λ are two real parameters not necessarily equal, $\Delta b\in\mathbb{R}^m$ and $\Delta c\in\mathbb{R}^n$ are perturbing directions, and $x\in\mathbb{R}^n$ is an unknown vector. Its dual is defined as

$$LD(\epsilon,\lambda) \qquad\qquad \max \quad \{(b+\epsilon\Delta b)^T y \mid A^T y + s = c+\lambda\Delta c,\ s\geq 0\},$$

where $y\in\mathbb{R}^m$ and $s\in\mathbb{R}^n$ are unknown vectors.

A vector $x(\epsilon)\geq 0$ is called a primal feasible solution of $LP(\epsilon,\lambda)$ if it satisfies the constraint $Ax=b+\epsilon\Delta b$, and any vector $(y(\lambda),s(\lambda))$ with $s(\lambda)\geq 0$ satisfying $A^Ty+s=c+\lambda\Delta c$ is referred to as a dual feasible solution of $LD(\epsilon,\lambda)$. Observe that the primal feasible solution dependents only on parameter ϵ and the dual feasible solution just varies depending on λ. According to

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weak duality property, any primal feasible solution $x(\epsilon)$ and any dual feasible solution $(y(\lambda), s(\lambda))$ satisfies $(c + \lambda\Delta c)^T x(\epsilon) \geq (b + \epsilon\Delta b)^T y(\lambda)$. The equality $x(\epsilon)^T s(\lambda) = 0$ holds if and only if these feasible solutions are optimal (*complementary slackness* property). In this case, according to strong duality property, objective functions values of primal and dual problems are equal, i.e., $(c + \lambda\Delta c)^T x(\epsilon) = (b + \epsilon\Delta b)^T y(\lambda)$. Let $\mathcal{LP}(\epsilon, \lambda)$ and $\mathcal{LD}(\epsilon, \lambda)$ denote feasible solution sets of problems $LP(\epsilon, \lambda)$ and $LD(\epsilon, \lambda)$, respectively. Further, let $\mathcal{LP}^*(\epsilon, \lambda)$ and $\mathcal{LD}^*(\epsilon, \lambda)$ denote their optimal solution sets, correspondingly. For the cases when there is no perturbation, we drop ϵ and λ from the notation. Analogously, for uni-parametric LO problem, the corresponding ϵ or λ are omitted from all notation.

We say $(x^*(\epsilon), y^*(\lambda), s^*(\lambda))$ is a primal-dual *strictly* complementary optimal solution if for all $i \in \{1, \ldots, n\}$, either $x_i^*(\epsilon)$ or $s_i^*(\lambda)$ is zero but not both. It means that $x^*(\epsilon)^T s^*(\lambda) = 0$ with $x^*(\epsilon) + s^*(\lambda) > 0$. If $\mathcal{LP}(\epsilon, \lambda)$ and $\mathcal{LD}(\epsilon, \lambda)$ are nonempty, then there is such primal-dual optimal solution (Goldman-Tucker Theorem [9]).

The support set of nonnegative vector ν is defined as $\sigma(\nu) = \{i : \nu_i > 0\}$. We partition the index set $\{1, \ldots, n\}$ into two subsets

$$\begin{aligned}
\mathcal{B}(\epsilon, \lambda) &= \{i \mid x_i^*(\epsilon) > 0 \ \text{ for a primal optimal solution } x^*(\epsilon)\};\\
\mathcal{N}(\epsilon, \lambda) &= \{i \mid s_i^*(\lambda) > 0 \ \text{ for a dual optimal solution } (y^*(\lambda), s^*(\lambda))\}.
\end{aligned}$$

This partition is called *optimal partition* of the index set $\{1, \ldots, n\}$ for problems $LP(\epsilon, \lambda)$ and $LD(\epsilon, \lambda)$ and is denoted by $\pi(\epsilon, \lambda) = (\mathcal{B}(\epsilon, \lambda), \mathcal{N}(\epsilon, \lambda))$. Because optimal solution sets $\mathcal{LP}^*(\epsilon, \lambda)$ and $\mathcal{LD}^*(\epsilon, \lambda)$ are convex, the optimal partition is unique. After the landmark paper of Karmarkar [10], interior point methods are widely used to solve LO problems in polynomial time [11]. An interior point method terminates in a primal-dual strictly complementary optimal solution that is capable to identify the optimal partition as well.

Recently Koltai and Terlaky [8] categorized sensitivity analysis in LO to three different classes. The first type is referred to as *basis invariancy* sensitivity analysis, that the goal is finding the range of variation of parameters where the given optimal basic solution remains optimal. The second type is known as *support set invariancy* sensitivity analysis and in this point of view, identifying the range of variation of parameters is aimed where the support set of an optimal solution of the perturbed problem remains identical with the support set of the given optimal solution of the unperturbed problem. In the third point of view to sensitivity analysis, so-called *optimal partition invariancy* sensitivity analysis, we are interested in finding the region for the variation of parameters where for any parameters in this range, optimal partition is invariant.

Uni-parametric programming has been studied since the simplex method invented by Dantzig [1]. However, there are few published results in bi-parametric programming. Recently, Ghaffari et. al. [4] studied optimal partition invariancy sensitivity analysis in bi-parametric linear optimization. Let us review their result in a nutshell. The region where optimal partition remains invariant is referred to as *invariancy region*. It was proved that the invariancy region is a rectangle if it is not a line-segment or the singleton $\{(0,0)\}$. All invariancy regions altogether generate a mesh-like area in $\mathbb{R}^2$ constructed by vertical and horizontal (half-)line segments [4]. The region that contains the origin is referred to as *actual* invariancy region. The lines between two differen rectangular-type invariancy regions are called *transition lines* and the intersection of transition lines are referred to as *transition points*. It is worth mentioning that optimal partition on transition lines and transition points are different from the optimal partition in the invariancy region they enclosed. Thus, they are invariancy regions themselves, and referred to as *trivial* invariancy regions in contrast to the rectangular-type regions that are referred to as *non-trivial* invariancy regions.

Support set invariancy sensitivity analysis has been studied for Uni-parametric LO problem when it is in standard form [3, 7] and when it is in general form including free variables and inequalities in addition to nonnegative variables [2].

In this paper, we consider the problem $LP(\epsilon, \lambda)$ when both Δb and Δc are nonzero vectors and the parameters ϵ and λ are not necessarily equal. We answer the questions: "What is the range of the parameters, where there exists a primal optimal solution for perturbed problem with the same support set of a given primal optimal solution of the unperturbed problem?". We also answer analogous question for the dual problem considering the support set of the slack variable and for the primal and dual problems simultaneously. Computational methods are presented that enable us to identify these regions in polynomial time.

The paper is organized as follows. In section 2, the results obtained in uni-parametric LO problem are summarized. Section 3 is devoted to derive some fundamental properties. In the sequel, main theorems are proved that enable us to identify the desired regions by solving auxiliary LO problems. In section 4 a simple example is presented to illustrate the obtained results. The final section includes a summary of results and future research line.

2 preliminaries

In this section we first define some basic concepts and then review the results of support set invariancy sensitivity analysis obtained in [3, 7], because they have a major rule in identifying the corresponding regions in bi-parametric LO problem.

2.1 Basic concepts

Let $\pi = (\mathcal{B}, \mathcal{N})$ be the optimal partition of the index set $\{1, \dots, n\}$ for problems LP and LD. Moreover, let (x^*, y^*, s^*) be a primal-dual optimal solution for problems LP and LD with properties $\sigma(x^*) = P$ and $\sigma(s^*) = \widehat{P}$. It should be mentioned that x^* is not necessarily a basic nor a strictly complementary optimal solution.

In addition to the optimal partition, one can partition the index set $\{1, \dots, n\}$ to three other partitions. The first partition is (P, Z) where $P = \{i : x_i^* > 0\}$ and $Z = \{1, \dots, n\} \backslash P$. It is obvious that $P \subseteq \mathcal{B}$ and $\mathcal{N} \subseteq Z$ and the equality holds when the given primal-dual optimal solution (x^*, y^*, s^*) is strictly complementary. In support set invariancy sensitivity analysis, we want to find the range of variation of parameters ϵ and λ, where for any parameters value in this region, there is a primal optimal solution $x^*(\epsilon)$ with $\sigma(x^*(\epsilon)) = P$.

The other partition can be defined as $(\widehat{P}, \widehat{Z})$ where $\widehat{P} = \{i : s_i^* > 0\}$ and $\widehat{Z} = \{1, \dots, n\} \backslash \widehat{P}$. It is obvious that $P \subseteq \mathcal{B} \subseteq \widehat{Z}$ and $\widehat{P} \subseteq \mathcal{N} \subseteq Z$ and equality holds when the given primal-dual optimal solution is strictly complementary. In active constraint set invariancy sensitivity analysis, we are interested to identify the range of the parameters ϵ and λ variation where for any parameters value in this region, the dual perturbed problem $LD(\epsilon, \lambda)$ has at least an optimal solution $(y^*(\lambda), s^*(\lambda))$ with $\sigma(s^*(\lambda)) = \sigma(s^*) = \widehat{P}$.

The last partition so-call characteristic invariancy partition can be defined as $(P, \widetilde{Z}, \widehat{P})$ where $\widetilde{Z} = \{1, \dots, n\} \backslash (P \cup \widehat{P})$. Considering this partition, we want to identify the range of variation of parameters where for any ϵ and λ in this region, there is a primal-dual optimal solution $(x^*(\epsilon), y^*(\lambda), s^*(\lambda))$ with properties $\sigma(x^*(\epsilon)) = \sigma(x^*) = P$ and $\sigma(s^*(\lambda)) = \sigma(s^*) = \widehat{P}$. It is easy to verify that the relations $Z = \widetilde{Z} \cup \widehat{P}$ and $\widehat{Z} = \widetilde{Z} \cup P$ hold and $\widetilde{Z} = \emptyset$ if and only if the given primal-dual optimal solution (x^*, y^*, s^*) is strictly complementary.

The optimal value function of problem $LP(\epsilon, \lambda)$ and $LD(\epsilon, \lambda)$ is defined as:

$$\phi(\epsilon, \lambda) = (c + \lambda \Delta c)^T x^*(\epsilon) = (b + \epsilon \Delta b)^T y^*(\lambda),$$

where Δb and Δc are perturbing directions and $(x^*(\epsilon), y^*(\lambda), s^*(\lambda))$ is a primal-dual optimal solution of problems $LP(\epsilon, \lambda)$ and $LD(\epsilon, \lambda)$. It is proved that the optimal value function is a bi-variate quadratic function on this domain [4]. It is univariate function, when either ϵ or λ is fixed (to zero)[4, 11]. In this case, the optimal value function is a piecewise continuous linear function [11]. When $\epsilon = \lambda$, the optimal value function is quadratic but uni-variate function[5]. In all cases, at the points (lines), where the representation of the optimal value function changes, optimal partition is changed as well. These lines and points are called transition lines and transition points respectively.

2.2 Uni-parametric LO case

Recall that for $\lambda = 0$ (or $\Delta c = 0$), problems $LP(\epsilon, \lambda)$ and $LD(\epsilon, \lambda)$ reduce to the following uni-parametric problems

$$
\begin{array}{ll}
LP(\epsilon) & \min \; \{c^T x \mid Ax = b + \epsilon\Delta b, x \geq 0\}, \\
LD(\epsilon) & \max \; \{(b + \epsilon\Delta b)^T y \mid A^T y + s = c, s \geq 0\}.
\end{array}
$$

On the other hand, for $\epsilon = 0$ (or $\Delta b = 0$), we have the following primal and dual LO problems:

$$
\begin{array}{ll}
LP(\lambda) & \min \; \{(c + \lambda\Delta c)^T x \mid Ax = b, x \geq 0\}, \\
LD(\lambda) & \max \; \{b^T y \mid A^T y + s = c + \lambda\Delta c, s \geq 0\}.
\end{array}
$$

For the case when both ϵ and λ are not zeros but $\epsilon = \lambda$, we have the following primal and dual LO problems.

$$
\begin{array}{ll}
LP(\epsilon, \lambda = \epsilon) & \min \; \{(c + \epsilon\Delta c)^T x \mid Ax = b + \epsilon\Delta b, x \geq 0\}, \\
LD(\epsilon, \lambda = \epsilon) & \max \; \{(b + \epsilon\Delta b)^T y \mid A^T y + s = c + \epsilon\Delta c, s \geq 0\}.
\end{array}
$$

2.2.1 Support set invariancy sensitivity analysis in Uni-parametric LO problem

Assume that we are given an optimal solution x^*, with the support set $\sigma(x^*) = P$. In this way the partition (P, Z) is defined for the index set $\{1, \ldots, n\}$, where $Z = \{1, \ldots, n\} \backslash P$. Considering this partition, the *Invariant Support Set* (ISS) interval of problem $LP(\epsilon)$ is denoted by $\Upsilon(\epsilon)$, that is the interval where there is an optimal solution $x^*(\epsilon)$ with the property $\sigma(x^*(\epsilon)) = P$ for each λ in this interval. It is proved that the ISS interval $\Upsilon(\epsilon)$ can be identified in polynomial time by solving the following auxiliary LO problems [3]:

$$\epsilon_l = \min \quad \{\epsilon : A_P x_P - \epsilon\Delta b = b, x_P \geq 0\}, \tag{1}$$

$$\epsilon_u = \max \quad \{\epsilon : A_P x_P - \epsilon\Delta b = b, x_P \geq 0\}, \tag{2}$$

where A_P and x_P are the parts of A and x corresponding to P, respectively. It is proved that if $\epsilon = 0$ is not a transition point of the optimal value function of $LP(\epsilon)$, then the ISS interval $\Upsilon(\epsilon)$ is an open interval, otherwise this is a half-closed interval if it is not the singleton $\{0\}$. Observe that this interval is the singleton $\{0\}$ when $\epsilon_l = \epsilon_u = 0$ [3].

Let $\Upsilon(\lambda)$ denote the ISS interval of problem $LP(\lambda)$. It is proved that when $\lambda = 0$ is not a transition point of the optimal value function of this problem, then the ISS interval $\Upsilon(\lambda)$ coincides the closure of the actual invariancy interval, the invariancy interval that contains the origin (See Theorem 2.10 in [3]). However, when $\lambda = 0$ is a transition point of the optimal value function then one of the following propositions holds (See Theorem 2.11 in [3]):

1. If $\lambda \in [\lambda_-, 0)$, then $\Upsilon(\lambda) = [\lambda_-, 0]$,
2. If $\lambda \in (0, \lambda_+]$, then $\Upsilon(\lambda) = [0, \lambda_+]$,
3. $\Upsilon(\lambda) = \{0\}$,

where λ_+ (λ_-) is the immediate transition point adjacent to the right (left) of transition point $\lambda = 0$. Observe that λ_+ (λ_-) might be $+\infty$ ($-\infty$).

For the case, when both the RHS of the constraint and the coefficient of the objective function are perturbed but with the same parameter, we have $\Upsilon(\epsilon, \lambda = \epsilon) = \Upsilon(\epsilon) \cap \Upsilon(\lambda = \epsilon)$.

2.2.2 Invariant active constraint set invariancy sensitivity analysis in Uni-parametric LO problem

Observe that invariancy of the support set for the slack variable of a dual optimal solution is equivalent to invariancy of the corresponding support set of constraints in dual problem. For this reason, the ISS interval for dual problem is referred to as *Invariant Active Constraint Set* (IACS) interval for problem $LD(\epsilon)$ that is denoted by $\Gamma(\epsilon)$.

Let (y^*, s^*) be a dual optimal solution of LD with the property $\sigma(s^*) = \widehat{P}$. In this way, the IACS partition $(\widehat{P}, \widehat{Z})$ of the index set $\{1, \ldots, n\}$ is defined. Let γ_l and γ_u be the extreme points of $\overline{\Gamma}(\epsilon)$. It is proved that these values can be identified by solving the following two auxiliary LO problems (See Theorems 3.5 and 3.10 in [7]):

$$\gamma_l = \min\{\epsilon : A_{\widehat{Z}} x_{\widehat{Z}} - \epsilon \Delta b = b, x_{\widehat{Z}} \geq 0\}, \tag{3}$$

$$\gamma_u = \max\{\epsilon : A_{\widehat{Z}} x_{\widehat{Z}} - \epsilon \Delta b = b, x_{\widehat{Z}} \geq 0\}. \tag{4}$$

It is also proved that the IACS interval $\Gamma(\epsilon)$ is always a closed interval.

On the other hand, if γ_l and γ_u denote the end points of $\overline{\Gamma}(\lambda)$, they can be determined by solving the following two auxiliary LO problems (See Theorems 3.5 and 3.13 in [7]):

$$\gamma_l = \min\{\lambda : A^T_{\widehat{Z}} y - \lambda \Delta c_{\widehat{Z}} = c_{\widehat{Z}}, A^T_{\widehat{P}} y + s_{\widehat{P}} - \lambda \Delta c_{\widehat{P}} = c_{\widehat{P}}, s_{\widehat{P}} \geq 0\}, \tag{5}$$

$$\gamma_u = \max\{\lambda : A^T_{\widehat{Z}} y - \lambda \Delta c_{\widehat{Z}} = c_{\widehat{Z}}, A^T_{\widehat{P}} y + s_{\widehat{P}} - \lambda \Delta c_{\widehat{P}} = c_{\widehat{P}}, s_{\widehat{P}} \geq 0\} \ . \tag{6}$$

We remind that if $\lambda = 0$ is not a transition point of the optimal value function, then the IACS interval $\Gamma(\lambda)$ is an open interval. Otherwise, it is a half-closed interval. Similar to the ISS interval in the case when $\epsilon = \lambda$, we have $\Gamma(\epsilon, \lambda = \epsilon) = \Gamma(\epsilon) \cap \Gamma(\lambda = \epsilon)$ [7].

2.2.3 Characteristics Invariancy sensitivity analysis in Uni-parametric LO problem

Considering the partition $(P, \widetilde{Z}, \widehat{P})$, the Invariant Characteristics (IC) interval can be defined. Let us denote this interval for the perturbation of the RHS data and the objective function data with $\Theta(\epsilon)$ and $\theta(\lambda)$, respectively. It is proved [7] that $\Theta(\epsilon) = \Upsilon(\epsilon)$ and $\Theta(\lambda) = \Gamma(\lambda)$ and consequently, $\Theta(\epsilon, \lambda = \epsilon) = \Upsilon(\epsilon) \cap \Gamma(\lambda = \epsilon)$.

2.2.4 Completing remarks

Note that the auxiliary LO problems (1-6) can be solved in polynomial time by an interior point method. It is also proved that in uni-parametric LO problem, when perturbation occurs either in the RHS of the constraints or in the coefficient of the objective function, the optimal value function is linear on the ISS intervals $\Upsilon(\epsilon)$ or $\Upsilon(\lambda)$ [3]. However, for the case $\epsilon = \lambda$, the optimal value function is a univariate quadratic function on these intervals [3, 7]. These properties of the optimal value function implies that non of the ISS, IACS and IC intervals do not cover more that an invariancy interval with the exemption that they might include the end points of the corresponding invariancy interval.

3 Bi-parametric LO problem

For the case of bi-parametric LO problem, we prefer to follow the notation used in uni-parametric case. Let $\Upsilon(\epsilon, \lambda)$, $\Gamma(\epsilon, \lambda)$ and $\Theta(\epsilon, \lambda)$ denote the ISS, IACS and IC regions in bi-parametric LO problem, respectively.

3.1 Fundamental properties

The following lemma states that the ISS, IACS and IC regions are convex sets and consequently, to identify these regions, it is enough to identify their boundaries.

Lemma 3.1 *The ISS, IACS and IC regions are convex sets.*

Proof: We only prove this property for the ISS region. The proofs of the convexity for IACS and IC regions go analogously. The proof is obvious when the ISS region is the singleton $\{(0, 0)\}$. Let (ϵ_1, λ_1) and (ϵ_2, λ_2) be two arbitrary points in $\Upsilon(\epsilon, \lambda)$. Further, let $(x^{(1)}, y^{(1)}, s^{(1)})$ and

$(x^{(2)}, y^{(2)}, s^{(2)})$ be primal-dual optimal solutions of problems $LP(\epsilon,\lambda)$ and $LD(\epsilon,\lambda)$ at these points, respectively. We know that $\sigma(x^{(1)}) = \sigma(x^{(2)}) = P$, $\sigma(s^{(1)}) \subseteq Z$ and $\sigma(s^{(2)}) \subseteq Z$. For an arbitrary point $(\overline{\epsilon},\overline{\lambda})$ on the line segment between two points (ϵ_1,λ_1) and (ϵ_2,λ_2), there is a $\theta \in (0,1)$ such that:

$$\overline{\epsilon} = \theta\epsilon_1 + (1-\theta)\epsilon_2,$$
$$\overline{\lambda} = \theta\lambda_1 + (1-\theta)\lambda_2.$$

We define

$$\begin{aligned} x(\overline{\epsilon}) &= \theta x^{(1)} + (1-\theta)x^{(2)}, \\ y(\overline{\lambda}) &= \theta y^{(1)} + (1-\theta)y^{(2)}, \\ s(\overline{\lambda}) &= \theta s^{(1)} + (1-\theta)s^{(2)}. \end{aligned}$$

It is easy to verify that $x(\overline{\epsilon}) \in \mathcal{LP}(\overline{\epsilon},\overline{\lambda})$ and $(y(\overline{\lambda}), s(\overline{\lambda})) \in \mathcal{LD}(\overline{\epsilon},\overline{\lambda})$. Moreover, $\sigma(x(\overline{\epsilon})) = \sigma(x^{(1)}) \cup \sigma(x^{(2)}) = P$, and $\sigma(s(\overline{\lambda})) = \sigma(s^{(1)}) \cup \sigma(s^{(2)}) \subseteq Z$, that prove the optimality of $(x(\overline{\epsilon}), y(\overline{\lambda}), s(\overline{\lambda}))$ for problems $LP(\overline{\epsilon},\overline{\lambda})$ and $LD(\overline{\epsilon},\overline{\lambda})$, as well as the invariancy of the support sets of the primal optimal solution $x^*(\epsilon)$. The proof is complete. □

The following theorem talks about the fact that when both Δb and Δc are nonzero vectors and $\epsilon \neq \lambda$, the optimal value function $\phi(\epsilon,\lambda)$ is a bivariate quadratic function on the ISS, IACS and IC regions. The proof is similar to the proof of Theorem 2.13 in [4] and is omitted.

Theorem 3.2 *The optimal value function $\phi(\epsilon,\lambda)$ is a bi-variate quadratic function on the closure of regions $\Upsilon(\epsilon,\lambda)$, $\Gamma(\epsilon,\lambda)$ and $\Theta(\epsilon,\lambda)$.*

Corollary 3.3 *The regions $\Upsilon(\epsilon,\lambda)$, $\Gamma(\epsilon,\lambda)$ and $\Theta(\epsilon,\lambda)$ can not cover more than an (actual) invariancy region with the exemption that they might include some parts of the borders of the corresponding invariancy region.*

3.2 Identifying the ISS regions

Let use establish a theorem that is about the relationship between the ISS region and the two corresponding intervals in uni-parametric case. This relationship plays a major role in identifying this region and speaks of the fact that this identification can be carried out in polynomial time.

Theorem 3.4 *Consider the bi-parametric LO problem $LP(\epsilon,\lambda)$. Let $\Upsilon(\epsilon)$ be the ISS interval of problems $LP(\epsilon)$ and $LD(\epsilon)$. Further, let $\Upsilon(\lambda)$ be the ISS interval of problems $LP(\lambda)$ and $LD(\lambda)$. then*

$$\Upsilon(\epsilon,\lambda) = \Upsilon(\epsilon) \times \Upsilon(\lambda).$$

Proof: Let (x^*, y^*, s^*) be a primal-dual optimal solution of problems LP and LD with $\sigma(x^*) = P$ and $\sigma(s^*) \subseteq Z$. First we prove inclusion

$$\Upsilon(\epsilon) \times \Upsilon(\lambda) \subseteq \Upsilon(\epsilon,\lambda). \tag{7}$$

Let $\overline{\epsilon} \in \Upsilon(\epsilon)$ be an arbitrary fixed parameter. According to lemma 2.2 in [3], the dual optimal solution set $\mathcal{LD}^*(\epsilon)$ is invariant for all $\epsilon \in \Upsilon(\epsilon)$. Therefore, $(x^*(\overline{\epsilon}), y^*, s^*)$ is a primal-dual optimal solution of problems $LP(\overline{\epsilon})$ and $LD(\overline{\epsilon})$, with the property $\sigma(x^*(\overline{\epsilon})) = P$. Similarly, let $\overline{\lambda} \in \Upsilon(\lambda)$ be an arbitrary fixed parameter. According to lemma 2.8 in [3], the primal optimal solution set $\mathcal{LP}^*(\lambda)$ is invariant for all $\lambda \in \Upsilon(\lambda)$. Therefore, one might consider $(x^*, y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ as a primal-dual optimal solution of problems $LP(\overline{\lambda})$ and $LD(\overline{\lambda})$. It is obvious that $x^*(\overline{\epsilon})$ and $(y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ are feasible solution of problems $LP(\overline{\epsilon},\overline{\lambda})$ and $LD(\overline{\epsilon},\overline{\lambda})$, respectively. Moreover, equality $x^*(\overline{\epsilon})^T s^*(\overline{\lambda}) = 0$ implies the optimality of these solutions. It means that $(\overline{\epsilon},\overline{\lambda}) \in \Upsilon(\epsilon,\lambda)$ that proves the inclusion (7).

Now, we prove inclusion

$$\Upsilon(\epsilon,\lambda) \subseteq \Upsilon(\epsilon) \times \Upsilon(\lambda). \tag{8}$$

Let $(\overline{\epsilon},\overline{\lambda}) \in \Upsilon(\epsilon,\lambda)$ be given and $(x^*(\overline{\epsilon}), y^*(\overline{\lambda}), s^*(\overline{\epsilon}))$ be a primal-dual optimal solution of problems $LP(\overline{\epsilon},\overline{\lambda})$ and $LD(\overline{\epsilon},\overline{\lambda})$. Thus, $\sigma(x^*(\overline{\epsilon})) = P$ and $\sigma(s^*(\overline{\lambda})) \subseteq Z$. Therefore, $(x^*(\overline{\epsilon}), y^*, s^*)$ is a primal-dual optimal solution of problems $LP(\overline{\epsilon})$ and $LD(\overline{\epsilon})$. Analogously, $(x^*, y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ is a primal-dual optimal solution of problems $LP(\overline{\lambda})$ and $LD(\overline{\lambda})$. It means that $\overline{\epsilon} \in \Upsilon(\epsilon)$ and $\overline{\lambda} \in \Upsilon(\lambda)$ and the inclusion (8) is proved. The proof is complete. $\square$

Remark 3.5 *The ISS region $\Upsilon(\epsilon,\lambda)$ is a (half-)line segment containing the origin, whenever either $\Upsilon(\epsilon)$ or $\Upsilon(\lambda)$ is the singleton $\{0\}$. If both $\Upsilon(\epsilon)$ and $\Upsilon(\lambda)$ are $\{0\}$, then $\Upsilon(\epsilon,\lambda) = \{(0,0)\}$. Otherwise, the ISS region $\Upsilon(\epsilon,\lambda)$ is a* half-open rectangle *in $\mathbb{R}^2$. By the half-open rectangle, we mean that horizontal line borders always are included in $\Upsilon(\epsilon,\lambda)$. Because the ISS interval $\Upsilon(\lambda)$ is always a closed interval [3]. However, its vertical line borders might not belong to $\Upsilon(\epsilon,\lambda)$. Because, $\Upsilon(\epsilon)$ is an open interval if $\epsilon = 0$ is not a transition point of the optimal value function $\phi(\epsilon)$ and if $\epsilon = 0$ is one of the end points of $\Upsilon(\epsilon)$ (equivalently, $\epsilon = 0$ is a transition point of the optimal value $\phi(\epsilon)$) then the ISS region $\Upsilon(\epsilon,\lambda)$ includes one of its vertical borders as well.*

Remark 3.6 *According to Theorem 3.4, to identify the ISS region $\Upsilon(\epsilon,\lambda)$, it is enough to identify the ISS intervals $\Upsilon(\epsilon)$ and $\Upsilon(\lambda)$. Since these intervals can be identified in polynomial time, thus the ISS region $\Upsilon(\epsilon,\lambda)$ can be identified in polynomial time as well.*

3.3 Identifying the IACS region

Let us present the method of identifying the IACS region $\Gamma(\epsilon,\lambda)$. The proof of the following theorem is similar to the proof of Theorem 3.4 and is omitted.

Theorem 3.7 *Let $\Gamma(\epsilon)$ be the IACS interval of problems $LP(\epsilon)$ and $LD(\epsilon)$. Further, let $\Gamma(\lambda)$ be the IACS interval of problems $LP(\lambda)$ and $LD(\lambda)$. Then*

$$\Gamma(\epsilon,\lambda) = \Gamma(\epsilon) \times \Gamma(\lambda).$$

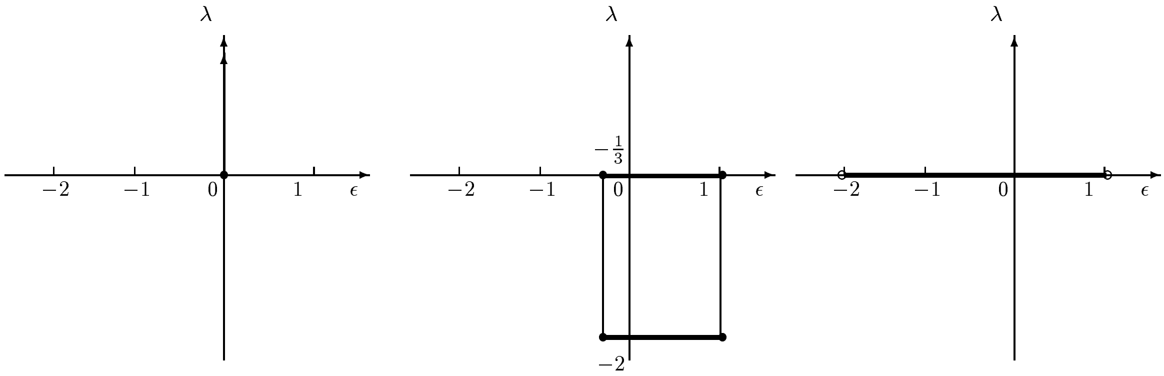
Remark 3.8 *The IACS region $\Gamma(\epsilon,\lambda)$ is a (half-)line segment containing the origin, whenever either $\Gamma(\epsilon)$ or $\Gamma(\lambda)$ is the singleton $\{0\}$. If both $\Gamma(\epsilon)$ and $\Gamma(\lambda)$ are singleton $\{0\}$, then $\Gamma(\epsilon,\lambda) = \{(0,0)\}$. Otherwise, the IACS region $\Gamma(\epsilon,\lambda)$ is a half-open rectangle in $\mathbb{R}^2$. By the half-open rectangle here, we mean that its vertical line borders always are included in $\Gamma(\epsilon,\lambda)$. Because the IACS interval $\Gamma(\epsilon)$ is always a closed interval [7]. However, its horizontal line borders might not belong to $\Gamma(\epsilon,\lambda)$, because $\Gamma(\lambda)$ is an open interval if $\lambda = 0$ is not a transition point of the optimal value function $\phi(\lambda)$. If $\lambda = 0$ is one of the end points of $\Gamma(\lambda)$ (equivalently, $\lambda = 0$ is a transition points of the optimal value function $\phi(\lambda)$), then the IACS region $\Gamma(\epsilon,\lambda)$ includes one of its horizontal borders as well.*

3.4 Identifying the IC region

Recall that the IC region is the intersection of the ISS region and the IACS region in uni-parametric LO problem [7]. It can be easily concluded that this property holds in bi-parametric LO case as well.

Theorem 3.9 *Let $\Upsilon(\epsilon)$ be the ISS interval of problem LP and $\Gamma(\lambda)$ be the IACS interval for problem LD. Then*

$$\Theta(\epsilon,\lambda) = \Theta(\epsilon) \times \Theta(\lambda) = \Upsilon(\epsilon) \times \Gamma(\lambda).$$



Case 3. Optimal degenerate basic solution. **Case 2. Optimal non-degenerate basic solution.** **Case 1. Strictly complementary optimal solution.**

Figure 1: Different ISS regions corresponding to different kinds of optimal solutions in Example 1.

4 Illustrate Example

In this section, a simple example is presented to illustrate the obtained results in identifying the ISS region for the case when the primal optimal solution is not unique. Analogous examples can be designed to illustrate obtained results in identifying the IACS and the IC regions.

Example 1: Consider the primal problem as:

$$\begin{array}{lrrrrrl} \min & -x_1 & -x_2 & & & & \\ s.t. & x_1 & +x_2 & +x_3 & & & =2 \\ & -x_1 & +x_2 & & +x_4 & & =1 \\ & 2x_1 & -x_2 & & & +x_5 & =4 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & \geq 0. \end{array} \tag{9}$$

The optimal partition of the index set $\{1,2,3,4,5\}$ is $\pi = (\mathcal{B},\mathcal{N}) = (\{1,2,4,5\},\{3\})$. Let $\Delta b = (1,-2,1)^T$ and $\Delta c = (-2,1,0,0,0)^T$ be perturbing directions. The actual invariancy region is the transition line $\{(\epsilon,0) \mid -2 < \epsilon < 1\}$. One can categorizes optimal solutions of problem (9) in three cases.

- **Case 1. Strictly complementary optimal solution**, such as $x^{(1)} = (1,1,0,1,3)^T$. For this kind of optimal solutions, we have $P = \sigma(x^{(1)}) = \{1,2,4,5\}$. In this case, the ISS region coincides the actual invariancy region and consequently, $\Upsilon(\epsilon,\lambda) = (-2,1) \times 0$ (See Figure 1).
- **Case 2. Primal optimal non-degenerate basic solution**, such as $x^{(2)} = (\frac{1}{2},\frac{3}{2},0,0,\frac{9}{2})^T$. For this optimal solution, the ISS interval $\Upsilon(\epsilon)$ is $(-\frac{1}{3},1)$ and because $\lambda = 0$ is a transition point, $\Upsilon(\lambda) = [-2,0]$ the closure of the invariancy interval to the left of the transition point $\lambda = 0$. Thus, $\Upsilon(\epsilon,\lambda) = (-\frac{1}{3},1) \times [-2,0]$, Observe that in this case, the region is a rectangle that contains its horizontal borders (See Figure 1).
- **Case 3. Primal optimal degenerate basic solution**, such as $x^{(3)} = (2,0,0,3,0)^T$. In this case, $\epsilon = 0$ is a transition point of the optimal value function $\phi(\epsilon)$ and consequently, the ISS interval $\Upsilon(\epsilon)$ is the singleton $\{0\}$. On the other hand, $\lambda = 0$ is a transition point of the optimal value function $\phi(\lambda)$ and we have $\Upsilon(\lambda) = [0,+\infty)$. Thus, $\Upsilon(\epsilon,\lambda) = 0 \times [0,+\infty)$ (See Figure 1).

5 Conclusion

We investigated support set, active constraint set and characteristic invariancy sensitivity analysis for primal and dual LO problem in bi-parametric case. It was proved that the corresponding regions are rectangles in general and in special cases they might be line-segments or even a singleton. All these regions includes the origin but they do not contain more than an invariancy region with the exemption that they might include some parts of their vertical or horizontal borders.

Support set invariancy sensitivity analysis has been studied for the LO problem in general case, when the LO problem has free variables and inequalities in addition to nonnegative variables and equalities in uni-parametric case [2]. Moreover, these point of views to sensitivity analysis have been studied in convex quadratic optimization problem [7]. Our approach can be generalized to cover these cases too. Furthermore, support set expansion sensitivity analysis has been studied in both uni-parametric LO and convex quadratic problems [6]. One my interested in studying support set expansion sensitivity analysis in bi-parametric case for these problems too.

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