AMO-Advanced Modeling and Optimization, Volume 9, Number 1, 2007, pp.37-51

Quadratic Pareto Optimal Control For Boundary Infinite Order Parabolic Equation With State-Control Constraints

G. M. BAHAA

Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, EGYPT

E-mail: Bahaa_GM@hotmail.com

Abstract: *A boundary Pareto optimal control problem for the parabolic operator with infinite order is considered. The performance index has an integral form. Constraints on controls and on states are imposed. To obtain optimality conditions for the Neumann problem, the generalization of the Dubovitskii-Milyutin Theorem given by Walczak in Refs.[33,34], was applied.*

AMS Subject Classification : 35K20, 49J20, 49K20, 93C20.

Keywords: *Pareto boundary control problems, Parabolic operators with infinite order, Neumann Problem, Dubovitskii-Milyutin Theorem, Conical approximations, Optimality conditions.*

1-Introduction.

In recent years, significant emphasis has been given to the study of optimal control of systems governed by parabolic partial differential equations (PPDE) with first boundary conditions or with Cauchy conditions. In these studies, the differential equations are either in general form or in divergence form. It is known that a general class of optimal control problems of systems governed by Ito stochastic differential equations with Markov (fixed) terminal time can be concerted into a class of optimal control problems of systems governed by linear second order (PPDE) with first boundary condition (Cauchy condition).

Questions concerning necessary conditions for optimality and existence of optimal controls for these problems have been investigated for example in [10-16,19-26]. Also, optimal control problems with systems governed by partial differential equations subject to control and state constraints have been extensively studied. We refer for instance to ([30],[32]), for necessary optimality conditions for especial cases of elliptic and parabolic problems. A typical approach to solve these problems is to discrete both the control and the state and use nonlinear programming to solve the resulting optimization problem.

In (Refs. [14,15,16,21,25]), the optimal control problems for systems described by parabolic and hyperbolic operators with infinite order and consist of one equation have been discussed. Also we extended the discussion in [10-13] to $n \times n$ coupled systems of elliptic, parabolic and hyperbolic types involving different types of operators. To obtain optimality conditions the arguments of (Ref.[30]) have been applied.

Making use of the Dubovitskii-Milyutin theorem from [17], following (Refs. [19-27]) Kotarski et. al. have obtained necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables and with Dirichlet and Neumann boundary conditions. The interest in the study of this class of operators is stimulated by problems in quantum field theory.

In Ref.[20,21], Kotarski considered Pareto optimization problem for a parabolic system and obtained necessary and sufficient conditions for optimality by applying the classical Dubovitskii-Milyutin Theorem (Ref.[17]). The performance index was more general than the quadratic one and had an integral form. The set representing the constraints on the controls was assumed to have a nonempty interior. This assumption can be easily removed if we apply the generalized version of the Dubovitskii-Milyutin Theorem (Ref.[28]), instead of the classical one (Ref.[17]) (as the approximation of the set of controls, the regular tangent cone is used instead of the regular admissible cone).

In [1] a time optimal control problem for parabolic equations involving second order operator with an infinite number of variables is considered. Also in [2] a time optimal control problem for parabolic equations involving infinite order operator with finite number of variables is considered. In [3,4,5] a distributed and boundary control problems for cooperative parabolic and elliptic systems governed by Schrödinger operator is investigated.

In [22] a distributed control problem for a hyperbolic system with mixed control state constraints involving operator of infinite order is studied. In [24] a distributed control problem for Neumann parabolic problem with time delay is considered. Also in [25], a distributed control problem for a hyperbolic system involving operator of infinite order with Dirichlet conditions is given.

In this paper the application of the generalized Dubovitskii-Milyutin Theorem will be demonstrated on an Pareto optimization problem for a system described by a parabolic operator of infinite order with Neumann boundary conditions. A necessary and sufficient conditions for Pareto optimality of boundary control Neumann problem are given.

This paper is organized as follows. In section 2, we introduce some functional spaces with infinite order. In section 3, we define a parabolic equation with infinite order. In section 4, we formulate the Pareto optimal control problem and we introduce the main results of this paper.

2-Some Functional Spaces (Refs.[6,7]).

The object of this section is to give the definition of some functional spaces of infinite order, and the chains of the constructed spaces which will be used later. We define the Sobolev space $W^{\infty}\{a_\alpha,2\}(\mathbb{R}^n)$ (which we shall denote by $W^{\infty}\{a_\alpha,2\}$) of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of $\mathbb{R}^n, n \geq 1$, as follows,

$$W^{\infty}\{a_\alpha,2\} = \left\{\phi(x) \in C^{\infty}(\mathbb{R}^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha ||\mathbb{D}^\alpha \phi||_2^2 < \infty\right\},$$

where $a_\alpha \geq 0$ is a numerical sequence and $||.||_2$ is the canonical norm in the space $L^2(\mathbb{R}^n)$(all functions are assumed to be real valued), and

$$\mathbb{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}....(\partial x_n)^{\alpha_n}},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $W^{-\infty}\{a_\alpha,2\}$ is defined as the formal conjugate space to the space $W^{\infty}\{a_\alpha,2\}$, namely:

$$W^{-\infty}\{a_\alpha,2\} = \{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha \mathbb{D}^\alpha \psi_\alpha(x)\},$$

where $\psi_\alpha \in L^2(\mathbb{R}^n)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha ||\psi_\alpha||_2^2 < \infty$.

The duality pairing of the spaces $W^{\infty}\{a_\alpha,2\}$ and $W^{-\infty}\{a_\alpha,2\}$ is postulated by the formula

$$(\phi,\psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{\mathbb{R}^n} \psi_\alpha(x) \mathbb{D}^\alpha \phi(x) dx,$$

where

$$\phi \in W^{\infty}\{a_\alpha,2\}, \qquad \psi \in W^{-\infty}\{a_\alpha,2\}.$$

From above, $W^{\infty}\{a_\alpha,2\}$ is everywhere dense in $L^2(\mathbb{R}^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha,2\}$ denotes the topological dual space with respect to $L^2(\mathbb{R}^n)$, so we have the following chain:

$$W^{\infty}\{a_\alpha,2\} \subseteq L^2(\mathbb{R}^n) \subseteq W^{-\infty}\{a_\alpha,2\}.$$

We now introduce $L^2(0,T;L^2(\mathbb{R}^n))$ which we shall denote by $L^2(Q)$, where $Q = \mathbb{R}^n \times]0,T[$, denotes the space of measurable functions $t \to \phi(t)$ such that

$$||\phi||_{L^2(Q)} = (\int_0^T ||\phi(t)||_2^2 dt)^{\frac{1}{2}} < \infty,$$

endowed with the scalar product $(f,g) = \int_0^T (f(t),g(t))_{L^2(\mathbb{R}^n)} dt$, $L^2(Q)$ is a Hilbert space. In the same manner we define the spaces $L^2(0,T;W^{\infty}\{a_\alpha,2\})$, and $L^2(0,T;W^{-\infty}\{a_\alpha,2\})$, as its formal conjugate.

Finally we have the following chains:

$$L^2(0,T;W^{\infty}\{a_\alpha,2\}) \subseteq L^2(Q) \subseteq L^2(0,T;W^{-\infty}\{a_\alpha,2\}),$$

Finally, let us introduce the space

$$W(0,T) := \left\{ y; \quad y \in L^2(0,T;W^{\infty}\{a_\alpha,2\}), \frac{\partial y}{\partial t} \in L^2(0,T;W^{-\infty}\{a_\alpha,2\}) \right\},$$

in which a solution of a parabolic equation with infinite order will be contained.

3. Parabolic Equation (see Refs.[19]-[26]).

In Ref [14], which is a review article for previous results that earlier obtained by I. M. Gali, H. A. El-Saify and S. A. El-Zahaby. The results obtained there are for the case of operators with an infinite number of variables which are elliptic, parabolic, hyperbolic or well-posed in the sense of Petrowsky.

Subsequently, J. L. Lions [30] suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimension in the form

$$A(t)\Phi(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathbb{D}^{2\alpha} \Phi(x,t), \tag{3.1}$$

For this operator the bilinear form $\pi(t;\Phi,\Psi) := (A\Phi,\Psi)_{L^2(R^n)}$ is coercive on $W^{\infty}\{a_\alpha,2\}$. The operator $A(t)$ is a bounded self-adjoint elliptic operator with infinite order mapping $W^{\infty}\{a_\alpha,2\}$ onto $W^{-\infty}\{a_\alpha,2\}$.

We consider the following evaluation equation:

$$\frac{\partial y}{\partial t} + A(t)y = 0, \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{3.2}$$

$$y(x,0) = y_p(x), \quad x \in \mathbb{R}^n, \tag{3.3}$$

$$\frac{\partial^{\omega} y(x,t)}{\partial \nu_A^{\omega}} = f, \quad x \in \Gamma, \quad t \in (0,T), \tag{3.4}$$

where

$$f \in L^2(0,T;W^{-\infty}\{a_\alpha,2\}(\Gamma)), \quad y_p \in L^2(\mathbb{R}^n),$$

are given functions and $\frac{\partial^\omega}{\partial\nu_A^\omega}$ is the co-normal derivatives with respect to $A(t)$, i.e. $\frac{\partial^\omega}{\partial\nu_A^\omega} = \frac{\partial^\omega}{\partial\nu^\omega} cos(\nu;x_k)$; $cos(\nu;x_k) = k$ - th direction cosine of ν; ν being the normal to the boundary Γ of R^n for $|\omega| = 0,1,2,..,$ $|\omega| \leq \alpha - 1$.

$A(t)$ is a bounded self-adjoint elliptic partial differential operator with infinite order mapping $W^\infty\{a_\alpha,2\}$ onto $W^{-\infty}\{a_\alpha,2\}$, which takes the above form (3.1).

For each $t \in]0,T[$, we define the following bilinear form on $W^\infty\{a_\alpha,2\}$:

$$\pi(t;\phi,\psi) = (A(t)\phi,\psi)_{L^2(\mathbb{R}^n)}, \quad \phi,\psi \in W^\infty\{a_\alpha,2\}.$$

Then

$$\begin{aligned}
\pi(t;\phi,\psi) &= \Big(A(t)\phi,\psi\Big)_{L^2(\mathbb{R}^n)} \\
&= \Big(A(t)\phi(x),\psi(x)\Big)_{L^2(\mathbb{R}^n)} \\
&= \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha \mathbb{D}^{2\alpha}\phi(x,t),\psi(x)\Big)_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n}\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}\mathbb{D}^{\alpha}\phi(x)\mathbb{D}^{\alpha}\psi(x)\,dx.
\end{aligned} \tag{3.5}$$

The bilinear form (3.5) is coercive on $W^\infty\{a_\alpha,2\}$ that is, there exists $\eta \in \mathbb{R}$, such that:

$$\pi(t;\phi,\phi) = \eta\,||\phi||^2_{W^\infty\{a_\alpha,2\}}, \quad \eta > 0. \tag{3.6}$$

It is well known that the ellipticity of $A(t)$ is sufficient for the coerciveness of $\pi(t;\phi,\psi)$ on $W^\infty\{a_\alpha,2\}$. In fact,

$$\begin{aligned}
\pi(t;\phi,\phi) =& \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha\mathbb{D}^{2\alpha}\phi(x,t),\phi(x,t)\Big) \\
&\geq \Big(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha||\mathbb{D}^{\alpha}\phi(x)||^2_{L^2(\mathbb{R}^n)}\Big) \\
&= \eta||\phi(x)||^2_{W^\infty\{a_\alpha,2\}}.
\end{aligned}$$

$$\begin{cases} \forall\phi,\psi \in W^\infty\{a_\alpha,2\} \text{ the function } t \to \pi(t;\phi,\psi) \text{ is continuously} \\ \text{differentiable in }]0,T[; \text{ and} \end{cases}$$

$$\pi(t;\phi,\psi) = \pi(t;\psi,\phi). \tag{3.7}$$

Note. The operator $\frac{\partial}{\partial t} + A(t)$ is parabolic operator with an infinite order which maps $L^2(0,T;W^\infty\{a_\alpha,2\})$ onto $L^2(0,T;W^{-\infty}\{a_\alpha,2\})$.

4-Statement of the Pareto Optimal Control Problem and Optimization Theorem (see Refs.[20],[21]).

The Pareto optimal control problem is stated as follows:

$$\frac{\partial y}{\partial t} + A(t)y = 0, \qquad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{4.1}$$

$$y(x,0) = y_p(x), \qquad x \in \mathbb{R}^n, \tag{4.2}$$

$$\frac{\partial^\omega y(x,t)}{\partial \nu_A^\omega} = u, \qquad x \in \Gamma, \quad t \in (0,T). \tag{4.3}$$

$$I(y,u) = \begin{bmatrix} I_1(u) \\ I_2(y) \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^T \int_{\mathbb{R}^n} u^2 dx dt \\ \displaystyle\int_{\mathbb{R}^n} (y(T,x) - z_d(x))^2 dx \end{bmatrix} \quad \to \text{Pareto min.} \tag{4.4}$$

Control constraints:
We assume the following constraints on controls:

$$u \in \mathcal{U}_{ad} \subset \mathcal{U} := L^2(0,T; W^\infty\{a_\alpha, 2\}(\Gamma)), \quad \mathcal{U}_{ad} \text{ is closed and convex.} \tag{4.5}$$

State constraints:
We assume the following constraints on states:

$$y \in Y_{ad} \subset Y := L^2(0,T; W^\infty\{a_\alpha, 2\}), \quad Y_{ad} \text{ is closed convex with} \tag{4.6}$$

a non-empty interior in Y.

Where $y_p, z_d \in L^2(\mathbb{R}^n)$ are given. $A(t)$ is the same operator defined in section 3. The control time T is assume to be fixed in our problem.

We also assume that there exists $(\tilde{y}, \tilde{u})$ such as $\tilde{u} \in \mathcal{U}_{ad}, \tilde{y} \in \text{int}Y_{ad}$ and $(\tilde{y}, \tilde{u})$ satisfy equations (4.1)-(4.3) (Slater's condition).

The solution of the stated Pareto optimal control problem (4.1)-(4.6) is equivalent to seeking of a pair $(y^0, u^0) \in E := Y \times \mathcal{U}$, which satisfies equations (4.1)-(4.3) and minimizes in the Pareto sense the vector functional (4.4) under constraints (4.5)-(4.6). We formulate necessary and sufficient conditions of optimality for the problem (4.1)-(4.6) in the following optimization theorem.

Theorem (4.1). For every $\lambda_1, \lambda_2 > 0$ such as $\lambda_1 + \lambda_2 = 1$ there is the unique solution (y^0, u^0) to the Pareto optimal control problem (4.1)-(4.6). Moreover, there are two adjoint states p and ω such as $p \in W(0,T)$, and $\xi \in L^2(0,T; W^{-\infty}\{a_\alpha, 2\}))$. Besides, p and u^0 satisfy (in the weak sense) the adjoint equations given below.

The necessary and sufficient conditions of optimality are characterized by the the following system of partial differential equations and inequalities:

State equation:

$$\frac{\partial y^0}{\partial t} + A(t)y^0 = 0, \qquad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{4.7}$$

$$y^0(x,0) = y_p(x), \qquad x \in \mathbb{R}^n. \tag{4.8}$$

$$\frac{\partial^\omega y^0(x,t)}{\partial \nu_A^\omega} = u^0, \qquad x \in \Gamma, \quad t \in (0,T), \tag{4.9}$$

Adjoint equations:

$$-\frac{\partial p}{\partial t} + A^*(t)p = 0, \qquad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{4.10}$$

$$p(x,T) = \lambda_2[y^0(x,T) - z_d], \qquad x \in \mathbb{R}^n, \tag{4.11}$$

$$\frac{\partial^\omega p(x,t)}{\partial \nu_A^\omega} = 0, \qquad x \in \Gamma, \quad t \in (0,T). \tag{4.12}$$

$$-\frac{\partial u^0}{\partial t} + A^*(t)u^0 = \xi, \qquad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{4.13}$$

$$u^0(x,T) = -\frac{1}{\lambda_1}p(x,T), \qquad x \in \mathbb{R}^n, \tag{4.14}$$

$$\frac{\partial^\omega u^0(x,t)}{\partial \nu_A^\omega} = 0, \qquad x \in \Gamma, \quad t \in (0,T). \tag{4.15}$$

Maximum conditions:

$$\int_0^T \int_{\mathbb{R}^n} (p + \lambda_1 u^0)(u - u^0)\,dxdt \geq 0 \quad \forall u \in U_{ad}, \tag{4.16}$$

$$\int_0^T \int_{\mathbb{R}^n} \xi(y - y^0)\,dxdt \geq 0 \quad \forall y \in Y_{ad}. \tag{4.17}$$

Proof. Note that the conditions $\inf_{(y,u)} I_i(y,u) < I_i(y^0,u^0), i = 1,2$ hold, I_1, I_2 are strictly convex, hence they are Ponstein convex (strict convexity implies the Ponstein convexity). I_1, I_2 are also Frèchet differentiable. Therefore all assumptions of Theorem 1.6.1 in [21] are met. The stated Pareto optimal control problem (4.1)-(4.6) is equivalent to the one with the scalar performance functional $I = \lambda_1 I_1 + \lambda_2 I_2, \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$. To this scalar problem we apply Theorem 1.8.1 in [21]. We approximate the set $\mathcal{U}_{ad}$ by the admissible cone, the set Y_{ad} and the constraints given by equations (4.1)-(4.3) by the tangent cones and the scalar functional by the cone of decrease.

(a.) Analysis of constraints on controls.

The set $Q_1 = Y \times \mathcal{U}_{ad} \subset E$ represents equality constraints. Using Theorem 10.5 [17] we find the functional belonging to the adjoint tangent cone i.e.

$$f_1(\overline{y}, \overline{u}) \in [RTC(Q_1, (y^0, u^0))]^*.$$

The functional $f_1(\overline{u}, \overline{u})$ can be expressed as follows

$$f_1(\overline{u}, \overline{u}) = f_1^1(\overline{y}) + f_1^2(\overline{u})$$

where $f_1^1(\overline{y}) = 0 \quad \forall \overline{y} \in Y$ (Theorem 10.1 [15]) and $f_1^2(\overline{u})$ is the support functional to the set U_{ad} at the point u^0 (Theorem 10.5 [17]).

(b.) Analysis of constraints on states.

The set $Q_2 = Y_{ad} \times Y \subset E$ represents inequality constraints. Using Theorem 10.5 [17] we find the functional belonging to the adjoint regular admissible cone i.e.

$$f_2(\overline{y}, \overline{u}) \in [RAC(Q_2, (y^0, u^0))]^*.$$

Similarly as above we have that $f_2(\overline{y}, \overline{u}) = f_2^1(\overline{y})$ is equal to the support functional to the set Y_{ad} at the point y^0.

(c.) Analysis of equations (4.1)-(4.3).

The set

$$Q_3 := \left\{ (y,u) \in E; \quad \begin{array}{lll} \dfrac{\partial y}{\partial t} + A(t)y = 0, & x \in \mathbb{R}^n, & t \in (0,T), \\ y(x,0) = y_p(x), & x \in \mathbb{R}^n, & \\ \dfrac{\partial^\omega y(x,t)}{\partial \nu_A^\omega} = u, & x \in \Gamma, & t \in (0,T) \end{array} \right\}$$

represents the equality constraints. On the basis of Lusternik's theorem (Theorem 9.1 [17]) the regular tangent cone has the form

$$\begin{aligned} RTC(Q_3, (y^0, u^0)) &= \left\{ (\overline{y}, \overline{u}) \in E;\ P'(y^0, u^0)(\overline{y}, \overline{u}) = 0 \right\} \\ &= \left\{ (\overline{y}, \overline{u}) \in E; \quad \begin{array}{lll} \dfrac{\partial \overline{y}}{\partial t} + A(t)\overline{y} = 0, & x \in \mathbb{R}^n, & t \in (0,T) \\ \overline{y}(x,0) = 0, & x \in \mathbb{R}^n, & \\ \dfrac{\partial^\omega \overline{y}(x,t)}{\partial \nu_A^\omega} = \overline{u}, & x \in \Gamma, & t \in (0,T) \end{array} \right\} \end{aligned}$$

where $P'(y^0,u^0)(\overline{y},\overline{u})$ is the Frèchet differential of the operator

$$P(y,u) := \left(\frac{\partial y}{\partial t} + A(t)y - u, y(x,0) - y_p(x)\right)$$

mapping from the space

$$\Im := L^2(0,T;W^\infty\{a_\alpha,2\}) \times L^2(0,T;W^\infty\{a_\alpha,2\})$$

into the space

$$\mathcal{Z} := L^2(0,T;W^{-\infty}\{a_\alpha,2\}) \times L^2(\mathbb{R}^n).$$

Knowing that there exists a unique solution to the equation (4.1)-(4.3) for every u and y_p it is easy to prove that $P'(y^0,u^0)$ is the mapping from the space $\Im$ onto $\mathcal{Z}$ as required in the Lusternik theorem.

(d.) Analysis of the performance functional.

Applying Theorem 7.5 [17] we find the cone

$$RFC(I,(y^0,u^0)) = \left\{(\overline{y},\overline{u}) \in E; \sum_{i=1}^{2} \lambda_i I_i'(y^0,u^0)(\overline{y},\overline{u}) < 0\right\},$$

where I_i' denotes the Frèchet differential of I_i.

It is easily seen that

$$I_1'(\overline{y},\overline{u}) = 2\int_0^T \int_{R^n} u^0 \overline{u}\,dxdt,$$
$$I_2'(\overline{y},\overline{u}) = 2\int_{\mathbb{R}^n} (y^0(T) - z_d)\overline{y}(T)dx.$$

From Theorem 10.2 [17] we find the functional belonging to the adjoint cone. It has the form

$$f_4(\overline{y},\overline{u}) = -\mu\lambda_1 \int_0^T \int_{\mathbb{R}^n} u^0\overline{u}\,dxdt - \mu\lambda_2 \int_{\mathbb{R}^n} (y^0(T) - z_d)\overline{y}(T)dx,$$

where $\mu \geq 0$. From Remark 1.5.1 [21] it follows that $\mu \neq 0$.

To write down the Euler-Lagrange Equation, we need to check the assumption (v) of Theorem 1.8.1 [21].

It is known that the tangent cones are closed [28]. Following the idea of [33], we shall show that:-

$$RTC(Q_1 \cap Q_3,(y^0,u^0)) = RTC(Q_1,(y^0,u^0)) \bigcap RTC(Q_3,(y^0,u^0)).$$

We only need to show the inclusion " $\subset$ ", because we always have " $\supset$ " [28].

It can be easily checked that in the neighborhood V_0 of the point (y^0, u^0) the operator P satisfies the assumptions of the implicit function theorem [33]. Consequently, the set Q_3 can be represented in the neighborhood V_0 in the form

$$\left\{(y,u) \in E; \quad y = \varphi(u)\right\}, \tag{4.18}$$

where $\varphi : U \to Y$ is an operator of the class C^1 satisfying the condition $P(\varphi(u), u) = 0$ for u such as $(\varphi(u), u) \in V_0$. From this we know that

$$RTC(Q_3, (y^0, u^0)) = \left\{(\overline{y}, \overline{u}) \in E; \overline{y} = \varphi_u(u^0)\overline{u}\right\}. \tag{4.19}$$

Let $(\overline{y}, \overline{u})$ be any element of the set

$$RTC(Q_1, (y^0, u^0)) \bigcap RTC(Q_3, (y^0, u^0)).$$

From the definition of the tangent cone we can see that there exists the operator $r_u^1 := \mathbb{R}^1 \to U$ such as $\frac{r_u^1(\epsilon)}{\epsilon} \to 0$ with $\epsilon \to 0^+$ and

$$(y^0, u^0) + \epsilon(\overline{y}, \overline{u}) + (r_y^1, r_u^1) \in Q_1 \tag{4.20}$$

for a sufficiently small ϵ and with any $r_y^1(\epsilon)$.

From (4.18) follows that for sufficiently small ϵ, we have

$$\left(\varphi(u^0 + \epsilon\overline{u} + r_u^1(\epsilon)), u^0 + \epsilon\overline{u} + r_u^1(\epsilon)\right) \in Q_3.$$

Since φ is a differentiable operator, therefore

$$\varphi(u^0 + \epsilon\overline{u} + r_u^1(\epsilon)) = \varphi(u^0) + \epsilon\varphi_u(u^0)\overline{u} + r_y^3(\epsilon)$$

for some $r_y^3(\epsilon)$ such as $\frac{r_y^3(\epsilon)}{\epsilon} \to 0$ with $\epsilon \to 0^+$.

Taking into account (4.18) and (4.19), we get

$$(y^0, u^0) + \epsilon(\overline{y}, \overline{u}) + (r_y^3(\epsilon), r_u^1(\epsilon)) \in Q_3. \tag{4.21}$$

If in (4.20) we have $r_u^1(\epsilon) = r_y^3(\epsilon)$, then it follows from (4.20) and (4.21) that $(\overline{y}, \overline{u})$ is an element of the cone tangent to the set $Q_1 \cap Q_3$ at (y^0, u^0). It completes the proof of the inclusion " $\supset$ ". Further applying Theorem 3.3 [21] we can prove that the adjoint cones $[RTC(Q_1, (y^0, u^0))]^*$ and $[RTC(Q_3, (y^0, u^0))]^*$ are of the same sense.

(e.) Analysis of the Euler-Lagrange Equation.

The Euler-Lagrange Equation for our optimization problem has the form

$$\sum_{i=1}^{4} f_i(\overline{y}, \overline{u}) = 0. \tag{4.22}$$

Taking into account the form of functionals in (4.22), we get

$$f_1^2(\overline{u}) + f_2^1(\overline{y}) = \mu\lambda_1 \int_0^T \int_{\mathbb{R}^n} u^0 \overline{u} dx dt + \mu\lambda_2 \int_{\mathbb{R}^n} (y^0(T) - z_d)\overline{y}(T) dx,$$
$$\forall(\overline{y}, \overline{u}) \in RTC(Q_3, (y^0, u^0)). \tag{4.23}$$

We transform the component with $\overline{y}(T)$ in (4.23) using the adjoint equations (4.10)-(4.12) and the fact that $(\overline{y}, \overline{u}) \in RTC(Q_3, (y^0, u^0))$.

In turn, we get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^n} \left(-\frac{\partial p}{\partial t} + A^*(t)p \right) \overline{y} dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} \left(\frac{\partial \overline{y}}{\partial t} + A(t)\overline{y} \right) p dx dt \\ &+ \int_{\mathbb{R}^n} p(0)\overline{y}(0) dx - \int_{\mathbb{R}^n} p(T)\overline{y}(T) dx \\ &= \int_0^T \int_{R^n} p\overline{u} dx dt - \int_{\mathbb{R}^n} p(T)\overline{y}(T) dx. \end{aligned} \tag{4.24}$$

From (4.24) and (4.11), we obtain

$$\lambda_2 \int_{\mathbb{R}^n} (y^0(T) - z_d)\overline{y}(T) dx = \int_0^T \int_{\mathbb{R}^n} p\overline{u} dx dt.$$

Transforming the component with $\overline{u}$ in (4.23) with the help of the adjoint equation (4.13)-(4.15) and having in mind that $(\overline{y}, \overline{u}) \in RTC(Q_3, (y^0, u^0))$, we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} u^0 \overline{u} dx dt &= \int_0^T \int_{\mathbb{R}^n} u^0 \left(\frac{\partial \overline{y}}{\partial t} + A(t)\overline{y} \right) dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} \left(-\frac{\partial u^0}{\partial t} + A^*(t)u^0 \right) \overline{y} dx dt - \int_{\mathbb{R}^n} u^0(0)\overline{y}(0) dx \\ &+ \int_{\mathbb{R}^n} u^0(T)\overline{y}(T) dx = \int_0^T \int_{\mathbb{R}^n} \xi\overline{y} dx dt + \int_{\mathbb{R}^n} u^0(T)\overline{y}(T) dx \\ &= \int_0^T \int_{\mathbb{R}^n} \xi\overline{y} dx dt - \frac{\lambda_2}{\lambda_1} \int_{\mathbb{R}^n} (y^0(T) - z_d)\overline{y} dx. \end{aligned} \tag{4.25}$$

Replacing the right-hand side of (4.23) by (4.24) and (4.25), we get

$$f_1^2(\overline{u}) + f_2^1(\overline{y}) = \frac{1}{2}\mu \int_0^T \int_{\mathbb{R}^n} (p + \lambda_1 u^0)\overline{u}dxdt + \frac{1}{2}\mu \int_0^T \int_{\mathbb{R}^n} \xi\overline{y}dxdt. \tag{4.26}$$

Further from (4.26) and the definition of the support functional to $\mathcal{U}_{ad}$ and Y_{ad}, respectively at the point u^0 or y^0, we obtain maximum conditions (4.16)-(4.17). This last remark ends the proof of necessity.

The conditions (4.7)-(4.17) are also sufficient for the Pareto optimality for the problem (4.1)-(4.6). It follows immediately from the fact that the stated optimization problem is convex, I_1, I_2 are convex, continuous and so the Slater condition is fulfilled. The uniqueness of the optimal pair y^0, u^0 follows from the strict convexity of the scalar performance index.■

Comments.

The main result of the paper contains necessary and sufficient conditions of optimality (of Pontryagin's type) for infinite order parabolic system that give characterization of Pareto optimal control. But it is easily seen that obtaining analytical formulas for optimal control is very difficult. This results from the fact that state equations (4.7)-(4.9), adjoint equations (4.10)-(4.15) and maximum conditions (4.16)-(4.17) are mutually connected that cause that the usage of derived conditions is difficult. Therefore we must resign from the exact determining of the optimal control and therefore we are forced to use approximations methods. Those problems need further investigations and form tasks for future research.

Also it is evident that by modifying:

- the boundary conditions,
- the nature of the control (distributed, boundary),
- the nature of the observation,
- the initial differential system,

an infinity of variations on the above problem are possible to study with the help of Dubovitskii-Milyutin formalism.

Acknowledgement. The author is grateful for the editors of AMO-Advanced Modelling and Optimization Journal for their cooperation.

REFERENCES

[1] **Bahaa, G. M.,**" Time-Optimal control problem for parabolic equations with control constraints and infinite number of variables," *IMA Journal of Mathematical Control and Information*, **22**, 2, (2005), 364-375.

[2] **Bahaa, G. M.,**" Time-Optimal control problem for infinite order parabolic equation with control constraints," *Differential Equations and Control Processes, Electronic Journal, http://www.neva.ru/journal* **4**,(2005), 64-81.

[3] **Bahaa, G. M.,**" Boundary control for cooperative parabolic systems governed by Schrödinger operator", *Differential Equations and Control Processes, Electronic Journal, http://www.neva.ru/journal* **1**,(2006), 78-88.

[4] **Bahaa, G. M.,**" Boundary control for cooperative elliptic systems governed by Schrödinger operator", *Differential Equations and Control Processes, Electronic Journal, http://www.neva.ru/journal* **4**,(2006), 17-28.

[5] **Bahaa, G. M.,**" Optimal control for cooperative parabolic systems governed by Schrödinger operator with control constraints " *IMA J. Math. Control Inform. Advance Access published March* **23**, 3, (2006), 1-12.

[6] **Dubinskii, Ju. A.,** "Non-triviality of Sobolev spaces of infinite order for a full Euclidean space and a Tours,"*Math . USSR. Sb.* **29**, (1976), 393-401.

[7] **Dubinskii, Ju. A.,** " Sobolev spaces of infinite order and behavior of solution of some boundary value problems with unbounded increase of the order of the equation," *Math. USSR. Sb.* **27**, (1975), 143-162.

[8] **Dubovitskii, A., Milyutin, M.,**" Extremal problems with constraints," *Doklady Akademii Nauk SSSR,* **149**, 4, (1963), 759-762.

[9] **Dubovitskii, A., Milyutin, M.,**" Extremal problems with constraints," *Zhurnal Vycislitelnoi Mathematiki i Matematiceskoi Fiziki,* **5**, 5, (1965), 395-453.

[10] **EL-Saify H. A., and Bahaa, G. M.,** "Optimal control for $n \times n$ systems of hyperbolic types," *Revista de Matemáticas Aplicadas*, **22**, 1&5, (2001), 41-58.

[11] **El-Saify, H. A. and Bahaa, G. M.,**" Optimal control for $n \times n$ hyperbolic systems involving operators of infinite order,"*Mathematica Slovaca*, **62**, 4, (2002), 409-424.

[12] **El-Saify, H. A., and Bahaa, G. M.,**"Optimal control for $n \times n$ coupled systems of Petrowsky type with an infinite number of Variables," *Mathematica Slovaca,* **53**, 3, (2003), 291-311.

[13] **El-Saify, H. A., Serag, H. M, and Bahaa, G. M.,** "On optimal control for $n \times n$ elliptic system involving operators with an infinite number of variables," *Advances in Modelling & Analysis,* **37**, 4, (2000), 47-61.

[14] **El-Zahaby, S. A.** "Optimal control of systems governed by infinite order operators," *Proceeding (Abstracts) of the International Conference of Mathematics (Trends and Developments) of the Egyptian Mathematical Society, Cairo, Egypt, 28-31 Dec. (2002), and Submitted for Publication in Journal of Egyptian Mathematical Society.*

[15] **Gali, I. M. and El-Saify, H. A.,** " Control of system governed by infinite order equation of hyperbolic type,"*Proceeding of the International Conference on "Functional-Differential Systems and Related Topics,"* **III**, *Poland,* (1983),

99-103.

[16] **Gali, I. M., El-Saify, H. A. and El-Zahaby, S. A.,** " Distributed Control of a system governed by Dirichlet and Neumann problems for elliptic equations of infinite order," *Proceeding of the International Conference on "Functional-Differential Systems and Related Topics", **III**, Poland,* (1983), 83-87.

[17] **Girsanov, I. V.**"**Lectures on Mathematical Theory of Extremum Problems**,"*New York: Springer-Verlag*, (1972).

[18] **Ioffe A.D., Tikhomirov W. M.,**"**Theory of Extremal Problems**,"*North Holland, Amsterdam,* (1979).

[19] **Kotarski, W.,**" Optimal control of a system governed by a parabolic equation with an infinite number of variables and time delay," *Journal of Optimization Theory and Applications*, **63**, (1984), 91-101.

[20] **Kotarski, W.,**" On some specification of Milyutin-Dubovitskii's theorem for Pareto optimal problems," *Nonlinear Analysis, Theory, Methods and Applications*, **14**, 3, (1990), 287-291.

[21] **Kotarski, W.,**" **Some Problems of Optimal and Pareto Optimal Control for Distributed Parameter Systems**," *Reports of Silesian University, No.1628, Katowice, Poland*, (1987), 1-93.

[22] **Kotarski, W., and Bahaa, G. M.**"Optimal control problem for infinite order hyperbolic system with mixed control-state constraints" *European Journal of Control,* **11**, (2005), 150-156.

[23] **Kotarski, W.,and Bahaa, G.M.,**"Optimality conditions for infinite order hyperbolic problem with non-standard functional and time delay," Journal of Information and Optimization Sciences, **28**, No. 2, (2007), 1-20.

[24] **Kotarski, W., El-Saify, H. A. and Bahaa, G. M.,** "Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay," *IMA Journal of Mathematical Control and Information*, **19**, 4, (2002), 461-476.

[25] **Kotarski, W., El-Saify, H. A. and Bahaa, G. M.,**" Optimal control problem for a hyperbolic system with mixed control-state constraints involving operator of infinite order," *International Journal of Pure and Applied Mathematics*, **1**, 3, (2002), 241-254.

[26] **Kotarski, W., El-Saify, H. A. and Bahaa, G. M.,**" Optimality conditions for $n \times n$ parabolic systems with control constraints and general performance index." *Proceeding (Abstracts) of the International Conference of Mathematics (Trends and Developments) of the Egyptian Mathematical Society, Cairo, Egypt, 28-31 Dec. (2002), and Submitted for Publication in Journal of Egyptian Mathematical Society.*

[27] **Kowalewski, A. and Kotarski, W.,**“ Application of the Milyutin-Dubovicki's method to solving an optimal control problem of hyperbolic systems,” *Problems of Control and Information Theory*, **9**, (1980), 183-193.

[28] **Lasiecka, I.,**“ Conical approximations of sets with application to optimization,” *Technical University, Warsaw, Poland, Ph.D. Thesis 1974 (in Polish).*

[29] **Laurent, P.J.,** “ **Approximation et Optimization**,” *Paris: Hermann*, (1972).

[30] **Lions, J. L.,** “**Optimal Control of Systems Governed by Partial Differential Equations**,” *Springer-Verlag, Band,* 170, (1971).

[31] **Luenberger, D. G.,**“ **Optimization by Vector Space Methods**,” *Wiley, New York*, (1969).

[32] **Tröltzsch, F.** “ **Optimality Conditions for Parabolic Control Problems and Applications**,” *Teubner-Texte zür Mathematik, Band* **62***, Leipzig*, (1984).

[33] **Walczak, S.**“ One some control problems,” *Acta Universitatis Lodziensis. Folia Mathematica,* **1**, (1984), 187-196.

[34] **Walczak, S.**“ Some properties of cones in normed spaces and their application investigating extremal problems,” *Journal Of Optimization Theory and Applications,* **42**, 2, (1984), 561-582.

[35] **Werner, J.** “**Optimization Theory and Applications**,” *Viewag, Advanced Lectures In Mathematics, Braunschweig, Wiesbaden,* (1984).