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Multi-objective Mean-variance-skewness model for Portfolio Optimization

P. Jana[1]
Department of Mathematics, Scottish Church College
1 & 3 Urquhart Square Calcutta - 700 006 , India

T.K.Roy and S.K. Mazumder
Department of Mathematics , Bengal Engineering & Science University
Howrah - 711103., India

Abstract : Multi-objective non-linear programs occur in various fields of operation research. One of the application of such program portfolio selection problem. In recent years portfolio optimization models that consider more criteria than the standard expected return and variance objectives of the widely used Markowitz model. In this paper first a mean-variance-skewness model is proposed for portfolio selection and next added another entropy objective function to generate well-diversified asset portfolio within optimal asset allocation. Fuzzy programming technique is used to solve the problems. A numerical example is used to illustrate that the method can be efficiently used in practice.

Keywords : Multi-objective, Entropy, Portfolio, Mean-variance-skewness.

1. Introduction :

The Markowitz (1952) mean-variance (MV) optimization is the most common formulation of portfolio selection problems. However, there is controversy over the issue of whether higher moments should be considered in portfolio selection. Many researchers Many researchers [Arditti (1967), Konno *et all* (1993), Pornchai *et all* (1997)] argued that the higher moments cannot be neglected unless there are reasons to believe that the asset returns are symmetrically distributed around the mean and the expected utility function is quadratic. Levy and Sanat (1972) pointed out that the assumption of a quadratic utility function is appropriate only for relatively lower returns, which precludes its use for many types of investment. Samuelson (1970) also showed that the

[1]e-mail: p_jana1974@dataone.in
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higher moment is relevant to investors' decision-making in portfolio selection and almost all investors would prefer a portfolio with a larger third moment if the first and second moments are the same. Above discussions motivated us to add the third moment of return of a portfolio into general MV model.

Again MV approach often leads to portfolio highly concentrated on a few asset. Also, this method leads to negative values for some portfolio weights (short sales) while in practice most investors are not allowed to sell short. Since maximizing Shannon's entropy subject to the moment constraint implies estimating probability that is the closest to uniform, well-diversified optimal portfolio can be achieved. Entropy is a well accepted measure of diversity. Usefulness of entropy optimization models in portfolio selection are illustrated in three well-known books [Fang *et all* (1997), Kapur (1993), Kapur and Kesavan (1992)]

In 1970 Bellman and Zadeh (1970) proposed fuzzy decision theory. Zimmermann (1978) first applied the fuzzy set theory concept with some suitable membership functions to solve the multi-objective transportation problem. He showed that solutions obtained by fuzzy programming are always efficient. Wang et all (2003) presented single objective portfolio optimization model using fuzzy decision theory. Our MVS model for portfolio selection is formed as four objective nonlinear programming problem. We give an approximation of replacing the term 'variance' by absolute deviation' and the term 'skewness' by the expectation of a piecewise linear function. Fuzzy multi-objective programming technique is used to solve this type of problem.

This paper organized as follows. The MVS and Entropy model is presented in section 2 and approximation of replacing the term 'skewness' shown in section 3. Fuzzy programming technique is given in section 4 and numerical example is given in section 5. Some conclusions are finally given in section 6.

2. Model

Portfolio optimization problem or asset allocation problems look at the 'best' way for an investor or fund manager to allocate funds between a number of different asset. A security market with n risk and a risk-less asset offering a fixed rate of return is considered. Criterion for an investor is to maximize the expected utility by allocating the wealth among the securities at the end of the period.

Notations are as follows :

x_i Proportion invested in risky asset i, $i = 1, 2, \cdots, n$,
x_{n+1} Proportion invested in the risk-less asset
R_i Random rate of return on the risky asset i, $i = 1, 2, \cdots, n$,
r_{n+1} Rate of return on risk-less asset
r_i $E(R_i)$, Expected rate of return on the risky asset i, $i = 1, 2, \cdots, n$,
σ_{ij} $\text{Cov}(R_i, R_j)$, covariance between R_i and R_j, $i, j = 1, 2, \cdots, n$,
γ_{ijk} $E[(R_i - r_i)(R_j - r_j)(R_k - r_k)]$, central third moment of returns, $i, j, k = 1, 2, \cdots, n$,

The return of a portfolio $x = (x_1, x_2, \cdots, x_n)$ is :

$$R(x) = \sum_{i=1}^{n} R_i x_i + r_{n+1} x_{n+1} \tag{1}$$

The expected return , variance and skewness of portfolio $x = (x_1, x_2, \cdots, x_n, x_{n+1})$ are respectively :

$$\begin{aligned}
E(x) &= \sum_{i=1}^{n+1} r_i x_i \\
V(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\
S(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{ijk} x_i x_j x_k
\end{aligned}$$

2..1 Model - I: Portfolio Selection Problem (PSP)

The MVS model proposed here is three-objective programming problem. An optimal portfolio should maximize both expected return and skewness as well as minimizing the variance. So the problem can be stated as :

$$\begin{aligned}
\text{Maximize } S(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{ijk} x_i x_j x_k \\
\text{Maximize } E(x) &= \sum_{i=1}^{n+1} r_i x_i \\
\text{Minimize } V(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\
\text{subject to } & \sum_{i=1}^{n+1} x_i = 1 \quad , \quad x_i \geq 0
\end{aligned}$$

The Markowitz mean variance (MV) criterion simply states that an investors should always choose an efficient portfolio. The main problem in optimal MV portfolio is that the portfolios are often extremely concentrated on a few asset, which is a contradiction to the notion of diversification. Therefore there is a scope for introducing another criterion viz one for diversification and the best candidate for this. It is not surprising that entropy is used as the divergence measure of asset portfolio in finance literature. They usually solve quadratic problem for MV portfolio selection and then, apply entropy measure to infer how much portfolio is diversified. In this paper we maximize entropy function

$$En(x) = -\sum_{i=1}^{n+1} x_i \log x_i$$

2..2 Model -II: Portfolio Selection Problem with Diversification (PSPD)

Well-diversified optimal portfolio problem can be stated as

$$\begin{aligned}
\text{Maximize } S(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{ijk}x_i x_j x_k \\
\text{Maximize } E(x) &= \sum_{i=1}^{n+1} r_i x_i \\
\text{Minimize } V(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
\text{Maximize } En(x) &= -\sum_{i=1}^{n+1} x_i \log x_i \\
\text{subject to } & \sum_{i=1}^{n+1} x_i = 1 \quad , \quad x_i \geq 0
\end{aligned}$$

3. Approximate Model

The absolute deviation rather than the variance taken as the measure of risk in portfolio selection was analyzed by, [Konno *etall* (1993), Konno (1990)].
Let R be the random variable. The absolute deviation of R is defined as

$$W\Big(R\Big) = E\Big|R - E(R)\Big|$$

Under the assumption of normal distribution, the absolute deviation is equivalent to the standard deviation as the measure of risk. By [Konno (1990)]

$$W\Big(\sum_{i=1}^{n} R_i x_i\Big) = \sqrt{\frac{2}{\pi}V\Big(\sum_{i=1}^{n} R_i x_i\Big)}$$

With the Konno's technique, we can replace the quadratic inequality constraint by a piecewise linear one. Difficulty in solving the problem is related to the other term *Skewness* which can be approximated by a piecewise linear function.

Let $F(x) = x^3$. The skewness can be written as :

$$S(x) = E\Bigg[F\Bigg(\sum_{i=1}^{n} x_i R_i - E\Big(\sum_{i=1}^{n} x_i R_i\Big)\Bigg)\Bigg]$$

Let $f(x)$ be a piecewise linear function as a local approximation $F(x) = x^3$. If the sequence of points $\{a_i\}$ is given by

$$a_{-(k+1)} < a_{-k} < \cdots, < a_{-2} < a_{-1} < 0 = a_0 < a_1$$

Where $a_{-1} = -a_1$
$f(x)$ can be taken in the following analytical form :

$$f(x) = \begin{cases} a_i^3 + \frac{a_i^3 - a_{i+1}^3}{a_i - a_{i+1}}(x - a_i) & \text{if } a_i \leq x \leq a_{i+1},\ i = -k, -(k-1), \cdots, -1, 0 \\ a_{-k}^3 + \frac{a_{-(k+1)}^3 - a_{-k}^3}{a_{-(k+1)} - a_{-1}}(x - a_{-k}) & \text{if } x \leq a_{-k} \\ a_1^2 x & \text{if } x \geq a_1 \end{cases}$$

Hence we have the following approximate model :

3..1 Model - III

$$\text{Maximize } E\left[f\left(\sum_{i=1}^{n} x_i R_i - \sum_{i=1}^{n} x_i r_i\right)\right]$$

$$\text{Minimize } E\Big|\sum_{i=1}^{n} x_i R_i - \sum_{i=1}^{n} x_i r_i\Big|$$

$$\begin{aligned}
\text{Maximize } E(x) &= \sum_{i=1}^{n+1} r_i x_i \\
\text{Maximize } En(x) &= -\sum_{i=1}^{n+1} x_i \log x_i \\
\text{subject to} \quad & \sum_{i=1}^{n+1} x_i = 1 \quad , \quad x_i \geq 0
\end{aligned}$$

Here we use the function $f(x)$ to approximate the skewness. It is obvious that function $f(x)$ is linear and the difference function $F(x)$ and $f(x)$ ie. $g(x) = F(x) - f(x)$ is continuous and equal to 0 at points a_i , $i = -(k+1), -k, \cdots, 0, 1$. Therefor if x varies along the segment $[a_{-(k+1)},\ a_1]$, then $f(x)$ can be regarded as a good local linear approximation of $F(x)$ if only points a_i , $i = -(k+1), -k, \cdots, 0, 1$ are properly selected. Practically, because the real distribution of the portfolio return is not known, we need historical data to give the estimation. The historical returns have an upper bound and lower bound. The bounds are a reference for us to determine the variable range and select a sequence of points $\{a_i\}$. A numerical example will be given in section 6 to illustrate this technique.

$f(x)$ is a concave function and can be written in the form

$$f(x) = f_0(x) + f_{-1}(x) + \cdots + f_{-k}(x)$$

where

$$f_0(x) = a_1^2 x$$

and

$$f(x) = \begin{cases} \left(\frac{a_i^3 - a_{i-1}^3}{a_i - a_{i-1}} - \frac{a_i^3 - a_{i+1}^3}{a_i - a_{i+1}}\right)(x - a_i) & \text{if } x \leq a_i, \\ 0 \quad , & \text{if } x > a_i, \\ \qquad\qquad i = -1, -2, \cdots, -k. & \end{cases}$$

Let R_{it} = random rate of return on the risky asset i at any time t ($i = 1, 2, \cdots, n \quad ; \quad t = 1, 2, \cdots, T$)

$$r_i = \frac{1}{T}\sum_{t=1}^{T} R_{it}$$

$$\text{Let} \quad b_t = \frac{1}{2}\left|\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i\right| + \frac{1}{2}\left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i\right)$$
$$c_t = \frac{1}{2}\left|\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i\right| - \frac{1}{2}\left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i\right)$$
$$b_t\, c_t = 0 \quad , \quad b_t \geq 0 \ , \ c_t \geq 0$$

Then Model III can be represented as

3..2 Model -IV

$$\text{Maximize } S_1 = \frac{1}{T-1} \sum_{t=1}^{T} \sum_{j=-k}^{-1} \left(\frac{a_j^3 - a_{j-1}^3}{a_j - a_{j-1}} - \frac{a_j^3 - a_{j+1}^3}{a_j - a_{j+1}} \right) y_{jt}$$
$$\text{Maximize } E(x) = \sum_{i=1}^{n+1} r_i x_i$$
$$\text{Minimize } A_1 = \frac{1}{T-1} \sum_{t=1}^{T} (b_t + c_t)$$

$$\text{Maximize } En(x) = -\sum_{i=1}^{n+1} x_i \log x_i$$
$$\text{subject to} \quad b_t - c_t = \sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i$$
$$y_{jt} \leq 0 \ , \qquad y_{jt} - \left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i - a_j \right) \leq 0$$
$$\sum_{i=1}^{n+1} x_i = 1 \ , \qquad x_i \geq 0 \quad , \quad b_t\, c_t = 0 \quad , \quad b_t \geq 0 \ , \ c_t \geq 0$$
$$j = -k, -(k-1), \cdots, -1 \quad , \quad i = 1, 2, \cdots, n \quad , \ t = 1, 2, \cdots, T$$

4. Mathematical Analysis: Multi-objective Non-Linear Programming (MONLP) Problem

Here we discuss the general form of the MONLP problem and technique to solve this type of problem

4..1 Multi-objective Non-Linear Programming (MONLP) Problem

A general MONLP problem may be taken in the following Vector Minimization Problem (VMP) :
Minimize k non-linear objective functions

$$\text{Minimize} \quad Z(x) = \Big[Z_1(x),\ Z_2(x)\, , \cdots, Z_k(x) \Big] \tag{2}$$

Subject to the inequality constraints

$$\text{Subject to} \quad x \in X = \{x: \ g_j(x) \le b_j, \ (j = 1, 2, \cdots, m), \ l_i \le x_i \le u_i \ (i = 1, 2, \cdots, n)\}$$

A direct application of the optimality for single objective non-linear programming to the MONLP leads us to the following complete optimality concept.

4..2 Fuzzy programming technique to solve MONLP problem

To solve the MONLP (2) problem, following steps are used:

Step 1 : Solve the MONLP (2) as a single objective non-linear programming problem using only one objective at a time and ignoring the others. These solutions are known as ideal solutions.
Step 2 : From the results of step 1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, pay-off matrix can be formulated as follows:

	$Z_1(x)$	$Z_2(x)$	......	$Z_k(x)$
x^1	$Z_1^*(x^1)$	$Z_2(x^1)$	......	$Z_k(x^1)$
x^2	$Z_1(x^2)$	$Z_2^*(x^2)$	......	$Z_k(x^2)$
..	...	...	......	...
x^k	$Z_1(x^k)$	$Z_2(x^k)$	......	$Z_k^*(x^k)$

where $x^1, \ x^2, \ .....,x^k$ are the ideal solution of the k objective function.

$$\begin{aligned} U_r &= \max\Big\{Z_r(x^1), \, Z_r(x^2), \cdots, Z_r(x^k)\Big\} \\ L_r &= \min\Big\{Z_r(x^1), \, Z_r(x^2), \cdots, Z_r(x^k)\Big\} \end{aligned}$$

Step 3 : Using aspiration levels of objective functions of the VMP (2) written as follows :

Find x so as to satisfy

$$Z_r(x) \ \tilde{\le} \ L_r \ , \quad (r = 1, 2, \cdots, k) \qquad , \quad x \in X \tag{3}$$

Here objective functions (2) are considered as fuzzy constrains and which are quantified by the membership function

$$\begin{aligned} \mu_r(Z_r(x)) &= 0 \qquad \text{if } Z_r(x) \ge U_r(x) \\ &= d_r(x) \quad \text{if } L_r(x) \le Z_r(x) \le U_r(x) \\ &= 1 \qquad \text{if } Z_r(x) \le L_r(x) \end{aligned} \tag{4}$$

Here $d_r(x)$ is a strictly monotonic decreasing function with respect to $Z_r(x)$. Having elicited the membership functions (as in (4)) $\mu_r(Z_r(x)$ for $r = 1, 2, \cdots, k$, a general aggregation function which is in the following form.

$$\mu_{\widetilde{D}}(x) = \mu_{\widetilde{D}}\Big(\mu_1(Z_1(x)), \, \mu_2(Z_2(x)), \cdots, \mu_k(Z_k(x))\Big)$$

So a fuzzy multi-objective decision making problem can be defined as

$$\begin{aligned} \text{Maximize} \quad & \mu_{\widetilde{D}}(x) \\ \text{Subject to} \quad & x \in X \end{aligned} \tag{5}$$

Fuzzy decision [Bellman and Zadeh's (1970)] based on minimum operator [like Zimmermann [1978]), the problem (3) is reduced to

$$\begin{aligned} \text{Maximize} \quad & \lambda \\ \text{Subject to} \quad & \mu_r(Z_r(x)) \ \geq \lambda, \quad \text{for} \quad r = 1, 2, \cdots, k \\ x \in X \quad & \text{and} \quad 0 \leq \lambda \leq 1 \end{aligned} \tag{6}$$

Step 4: Solve (6) to get Pareto optimal solution

Some basic definitions on Pareto optimal solutions are introduced below.

Definition 1 (Complete Optimal Solution)

x^* is said to be a complete optimal solution to the MONLP (2) if and only if there exists $x^* \in X$ such that $Z_r(x^*) \leq Z_r(x)$, for $r = 1, 2, \cdots, k$ and for all $x \in X$.

However, when the objective functions of the MONLP conflict with each other, a complete optimal solution does not always exist and hence the Pareto Optimality Concept arises and it is defined as follows.

Definition 2 (Pareto Optimal Solution)

x^* is said to be a Pareto optimal solution to the MONLP (2) if and only if there does not exist another $x \in X$ such that $Z_r(x^*) \geq Z_r(x)$, for $r = 1, 2, \cdots, k$ and $Z_j(x) \neq Z_j(x^*)$ for at least one j, $j \in \{1, 2, \cdots, k\}$

4..3 Weighted Fuzzy Non-linear Programming

Decision maker preferences positive weights w_i regarding the relative importance of each objective function $Z_r(x)$, for $r = 1, 2, \cdots, k$. These weights can be normalized by taking

$\sum_{i=1}^{k} w_i = 1$. So introducing normalized weights in fuzzy NLP, (6) becomes

$$\begin{aligned} \text{Maximize} \quad & \lambda \\ \text{Subject to} \quad & w_r \mu_r(Z_r(x)) \ \geq \lambda, \quad \text{for} \quad r = 1, 2, \cdots, k \\ x \in X \quad & \text{and} \quad 0 \leq \lambda \leq 1 \\ \text{where} \quad & \sum_{r=1}^{k} w_i = 1 \end{aligned} \tag{7}$$

4..4 Fuzzy programming technique to solve Multi-objective Portfolio Optimization Model (MOPOM

Model-IV can be formulated as Vector Minimization problem (VMP)

$$\text{Minimize } [-S_1] \quad = \quad -\frac{1}{T-1} \sum_{t=1}^{T} \sum_{j=-k}^{-1} \left(\frac{a_j^3 - a_{j-1}^3}{a_j - a_{j-1}} - \frac{a_j^3 - a_{j+1}^3}{a_j - a_{j+1}} \right) y_{jt}$$

$$
\begin{aligned}
\text{Minimize } [-E(x)] &= -\sum_{i=1}^{n+1} r_i x_i \\
\text{Minimize } A_1 &= \frac{1}{T-1}\sum_{t=1}^{T}(b_t + c_t)
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\text{Minimize } [-En(x)] &= \sum_{i=1}^{n+1} x_i \log x_i \\
\text{subject to} \quad b_t + c_t &= \sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i \\
y_{jt} \le 0 \ , \qquad & y_{jt} - \left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i - a_j\right) \le 0 \\
\sum_{i=1}^{n+1} x_i = 1 \ , \qquad & x_i \ge 0 \quad , \quad b_t\, c_t = 0 \quad , \quad b_t \ge 0 \ , \ c_t \ge 0 \\
j = -k, -(k-1), \cdots, -1 \quad , \quad & i = 1, 2, \cdots, n \quad , \ t = 1, 2, \cdots, T
\end{aligned}
$$

To solve VMP form (8), step-1 of (4.2) is used. After that, pay-off matrix is formulated as follows :

	$S_1(x)$	$E(x)$	$A_1(x)$	$E_n(x)$
x^1	$S_1(x^1)$	$E(x^1)$	$A_1(x^1)$	$E_n(x^1)$
x^2	$S_1(x^2)$	$E(x^2)$	$A_1(x^2)$	$E_n(x^2)$
x^3	$S_1(x^3)$	$E(x^3)$	$A_1(x^3)$	$E_n(x^3)$
x^4	$S_1(x^4)$	$E(x^4)$	$A_1(x^4)$	$E_n(x^4)$

Now we find the upper bounds U_{S_1}, U_E, U_{A_1} , U_{E_n} and lower bounds L_{S_1}, L_E, L_{A_1} , L_{E_n}

$$
\begin{aligned}
U_{S_1} &= \max_{i=1,2,3,4}\{S_1(x^i)\} \ , \ L_{S_1} = \min_{i=1,2,3,4}\{S_1(x^i)\} \\
U_E &= \max_{i=1,2,3,4}\{E(x^i)\} \ , \ L_E = \min_{i=1,2,3,4}\{E(x^i)\} \\
U_{A_1} &= \max_{i=1,2,3,4}\{A_1(x^i)\} \ , \ L_{A_1} = \min_{i=1,2,3,4}\{A_1(x^i)\} \\
U_{E_n} &= \max_{i=1,2,3,4}\{E_n(x^i)\} \ , \ L_{E_n} = \min_{i=1,2,3,4}\{E_n(x^i)\}
\end{aligned}
$$

For simplicity we use linear membership $\mu(-S_1(x))$, $\mu(-E(x))$, $\mu(A_1(x))$ and $\mu(-E_n(x))$ for the objective functions $S_1(x)$, $E(x)$, $A_1(x)$ and $E_n(x)$ respectively defined as follows :

$$
\mu(-S_1(x)) = \begin{cases} 0 \ , & \text{if } \ S_1(x) \le L_{S_1} \\ \left(\frac{S_1(x) - L_{S_1}}{U_{S_1} - L_{S_1}}\right), & \text{if } \ L_{S_1} < S_1(x) < U_{S_1} \\ 1 \ , & \text{if } \ S_1(x) \ge U_{S_1} \end{cases}
$$

$$\mu(-E(x)) = \begin{cases} 0 , & \text{if } E(x) \leq L_E \\ \left(\frac{E(x)-L_E}{U_E-L_E}\right), & \text{if } L_E < E(x) < U_E \\ 1 , & \text{if } E(x) \geq U_E \end{cases}$$

$$\mu(A_1(x)) = \begin{cases} 0 , & \text{if } A_1(x) \geq U_{A_1} \\ \left(\frac{U_{A_1}-A_1(x)}{U_{A_1}-L_{A_1}}\right), & \text{if } L_{A_1} < A_1(x) < U_{A_1} \\ 1 , & \text{if } A_1(x) \leq L_{A_1} \end{cases}$$

$$\mu(-E_n(x)) = \begin{cases} 0 , & \text{if } E_n(x) \leq L_E \\ \left(\frac{E(x)-L_{E_n}}{U_E-L_{E_n}}\right), & \text{if } L_E < E_n(x) < U_{E_n} \\ 1 , & \text{if } E(x) \geq U_{E_n} \end{cases}$$

According to Step 3, having elicited the above membership functions crisp non-linear programming problem of (8) is formulated as follows :

$$\begin{aligned} \text{Maximize} \quad & \lambda \\ \text{Subject to } S_1(x) \quad & \geq \quad L_{S_1} + \lambda\Big(U_{S_1} - L_{S_1}\Big) \\ E(x) \quad & \geq \quad L_E + \lambda\Big(U_E - L_E\Big) \\ A_1(x) \quad & \leq \quad U_{A_1} - \lambda\Big(U_{A_1} - L_{A_1}\Big) \\ E_n(x) \quad & \geq \quad L_{E_n} + \lambda\Big(U_{E_n} - L_{E_n}\Big) \\ b_t - c_t \quad & = \quad \sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i \\ y_{jt} \quad & \leq \quad 0 \ , \ y_{jt} - \left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i - a_j\right) \leq 0 \ , \ 0 \leq \lambda \leq 1 \\ \sum_{i=1}^{n+1} x_i \ = \ 1 \ , \quad & x_i \geq 0 \quad , \quad b_t\, c_t \ = \ 0 \quad , \quad b_t \ \geq \ 0 \ , \ c_t \ \geq \ 0 \\ j \ = \ -k, -(k-1), \cdots, -1 \quad , \quad & i = 1, 2, \cdots, n \quad , \ t = 1, 2, \cdots, T \end{aligned} \tag{9}$$

and similarly weighted Fuzzy non-linear programming problem is formulated as

$$\begin{aligned} \text{Maximize} \quad & \lambda \\ \text{Subject to} \quad S_1(x) \quad & \geq \quad L_{S_1} + (\lambda/w_{s_1})\Big(U_{S_1} - L_{S_1}\Big) \\ E(x) \quad & \geq \quad L_E + (\lambda/w_E)\Big(U_E - L_E\Big) \\ A_1(x) \quad & \leq \quad U_{A_1} - (\lambda/w_{A_1})\Big(U_{A_1} - L_{A_1}\Big) \\ E_n(x) \quad & \geq \quad L_{E_n} + (\lambda/w_{E_n})\Big(U_{E_n} - L_{E_n}\Big) \end{aligned} \tag{10}$$

$$
\begin{aligned}
b_t - c_t &= \sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i \ , \ w_{s_1} + w_E + w_{A_1} + w_{E_n} = 1 \\
y_{jt} &\le 0 \ , \ y_{jt} - \left(\sum_{i=1}^{n} x_i R_{it} - \sum_{i=1}^{n} x_i r_i - a_j \right) \le 0 \ , \ 0 \le \lambda \le 1
\end{aligned}
$$

$$
\sum_{i=1}^{n+1} x_i = 1 \ , \qquad x_i \ge 0 \quad , \quad b_t \, c_t = 0 \quad , \quad b_t \ge 0 \ , \ c_t \ge 0
$$

$$
j = -k, -(k-1), \cdots, -1 \quad , \quad i = 1, 2, \cdots, n \quad , \ t = 1, 2, \cdots, T
$$

5. Numerical examples

A numerical example is given to illustrated the solution procedure. The returns of 5 stocks at 7 periods are given below

Period	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
1	0.14	−0.12	0.12	0.03	0.01
2	−0.07	0.08	0.14	0.05	−0.10
3	0.29	0.07	−0.10	0.04	0.19
4	0.08	0.12	−0.09	0.08	0.14
5	0.11	0.04	−0.02	0.07	−0.10
6	−0.06	0.03	0.14	0.06	−0.05
7	0.03	0.04	0.13	−0.05	0.07
r_i	0.0743	0.0386	0.0457	0.04	0.0229

Assume that the return of the risk-less asset is $r_7 = 0.035$

To determine the sequence of points $\{a_{-(k+1)},\ a_{-k}\ , ..., a_0\}$,
find the min $\left\{ R_{it} - r_i, \quad i = 1, 2, ...5; \quad t = 1, 2, ..., 7 \right\}$, which is -0.1586
Select $a_{-5} = 0.15$, $a_{-4} = -0.12$,$a_{-3} = 0.09$, $a_{-2} = -0.06$
$a_{-1} = -0.03$, $a_0 = 0$

TABLE - 1

[Pareto optimal solution of Model - IV]

x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*
0.1203	0.1423	0.1250	0.1501	0.1605	0.1372	0.1646

TABLE - 2

[Pareto optimal solution of Model - IV using different weights]

Weights	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*
$w_{S_1} = 1/4 \quad w_E = 1/4$ $w_{A_1} = 1/4 \quad w_{E_n} = 1/4$	0.1203	0.1423	0.1250	0.1501	0.1605	0.1372	0.1646
$w_{S_1} = 0.40 \quad w_E = 0.40$ $w_{A_1} = 0.05 \quad w_{E_n} = 0.15$	0.1412	0.1834	0.1492	0.1179	0.1102	0.1523	0.1458
$w_{S_1} = 0.05 \quad w_E = 0.15$ $w_{A_1} = 0.50 \quad w_{E_n} = 0.30$	0.2578	0.1514	0.0856	0.1789	0.1665	0.0451	0.1147

6. Conclusion :

In this paper we consider MVS model for portfolio selection and added another entropy objective function, taken as Shannons measure of entropy to generate well diversified assets within optimal asset allocation. To overcome the difficulty of calculating skewness and variance we transformed this to a simple approximate model. Fuzzy non-linear programming technique is used to solve the problem. the models are illustrated with numerical example. Portfolio selection is based on uncertainty of returns of securities. Incomplete and asymmetric information is still an important aspect of future research.

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