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An Optimal Algorithm to find Centres and Diameter of a Circular-Arc Graph

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Abstract. The notion of centre of a graph is motivated by a large class of problems specially facility location problems on a network. In this paper, an algorithm is presented to find centres and diameter of a circular-arc graph. If the circular arc representation is given then the proposed algorithm runs in $O(n)$ time.

Keywords: Design and analysis of algorithms; circular-arc graph; centre; radius; diameter

AMS Subject Classifications: 68Q22, 68Q25, 68R10.

1 Introduction

A graph $G = (V, E)$ is called a circular-arc graph if their is a one-to-one correspondence between the vertices of the graph and the arcs of a circular arc family such that there exits an edge between two vertices if and only if their corresponding arcs have non empty intersection. V is the set of all vertices and E is the set of all edges of the graph G.

Circular-arc graph has many applications in real world such as genetics, computer scheduling, circuit design etc.

Turker [10] proposed $O(n^3)$ time algorithm for recognizing a circular-arc graph and constructing in the affirmative case, a circular arc model. Hsu [6] designed an $O(nm)$ time algorithm for this problem. Eschen and Spinrad [3] presented an $O(n^2)$ time algorithm for recognizing a circular-arc graph.

Locating problem is a topic of great importance in the fields such as transportation, communication, service areas and computer sciences. The criteria for the locating problem in the literature are minmax criteria in which the distance to the furthest vertex from the site is minimized and minsum criteria in which the total distance to the vertices from the site is minimized.

In this paper, we first find out the vertices which form a minimum cycle and cover the whole circle. Then consider the other vertices and by checking their position we determine the centres and diameter of the graph.

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For some particular types of graph such as tree [5], outerplanar graph [4], etc. linear time algorithms can be devised to compute the centre. For the interval graph $O(n)$ time sequential algorithm is presented for computing diameter and centre of an interval graph with n vertices by Pal and Bhattacharjee [9]. In [8], Olariu has presented an $O(n+m)$ time sequential algorithm where input is an adjacency list that takes $O(n+m)$ space, where n and m are the number of vertices and edges respectively. The centre problem on circular-arc graph and interval graph is studied in [1]. Chen et al. [2] have introduced an algorithm for solving all-pair shortest paths query problem on interval and circular-arc graphs. A linear time algorithm for solving the centre problem is presented by Lau et. al. [7] for cactus graph.

2 Definitions and Notations

Let $A = \{A_1, A_2, \ldots, A_n\}$ be the circular arc family of a given circular-arc graph $G = (V, E)$. The family of circular arcs are located around a circle $\mathcal{C}$. Every arc can be represented by its two endpoints e.g., A_i can be represented as $[s_i, f_i]$ where s_i is the starting point and f_i is the finishing point of the arc A_i on the circle $\mathcal{C}$. Each endpoint of an arc is assigned to a positive integer called a *coordinate*. A *ray* is a straight line from the centre of $\mathcal{C}$ passing through any coordinate.

Without loss of generality, we assume the following

1. An arc contains both its endpoints and no two arcs share a common endpoint.
2. The graph G is connected and the list of sorted endpoints is given.
3. No single arc in A cover the entire circle $\mathcal{C}$.
4. Arcs and vertices of a circular-arc graph are same thing.
5. The endpoints of the arcs in A are sorted according to the order in which they are visited during the clockwise traversal along circle by starting at an arbitrary arc called A_1.
6. The arcs are sorted in increasing values of s_i's i.e., $s_i > s_j$ for $i > j$.
7. $\bigcup_{i=1}^{n} A_i = \mathcal{C}$ (otherwise, the problem becomes one on interval graph).

The family of arcs A is said to be canonical if
(i) s_i's and f_i's for all $i = 1, 2, \ldots, n$ are distinct integers between 1 to $2n$ and
(ii) point 1 is the starting endpoint of the arc A_1.

If A is not canonical, using sorting one can construct a canonical family of arcs using $O(n \log n)$ time.

A *path* of a graph G is an alternative sequence of distinct vertices and edges, beginning and ending with vertices. The length of a path is the number of edges in the path. A path from vertex i to j is a *shortest path* if there is no other path from i to j with lower length. The shortest distance (i.e., the length of the shortest path) between the vertices i and j is denoted by $d(i, j)$. The *eccentricity* $e(i)$ of a vertex i in a graph is the distance from vertex i to a vertex furthest from i. Vertex j is said to be a *furthest neighbour* of the vertex i if $d(i, j) = e(i)$. The *diameter* of a graph G is the maximum among all eccentricities. The *radius* of a graph is the minimum among all eccentricities. A *centre* of a graph is a vertex whose eccentricity equal to radius.

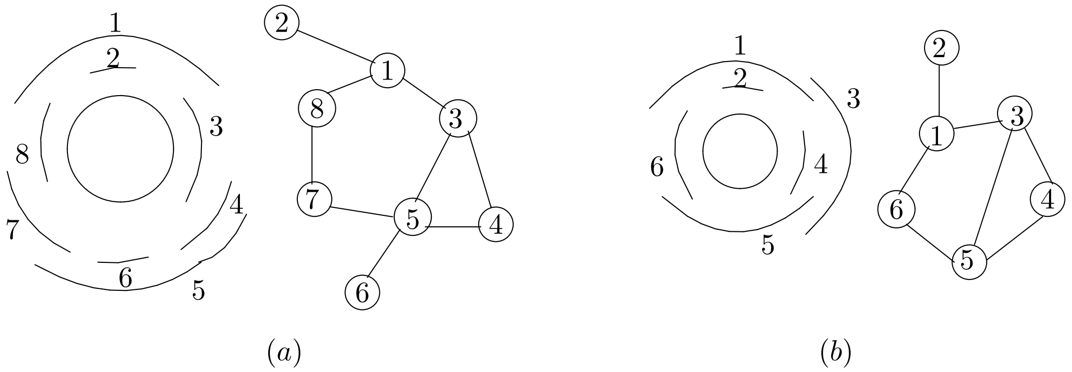

Figure 1: Circular-arc graphs and their circular-arc representation.

The eccentricity, diameter ($\delta(G)$), radius ($\rho(G)$) and centre ($C(G)$) of a graph are defined as follows.

$$\begin{aligned} e(i) &= \max \{d(i,j) \ : \ j \in V\} \\ \delta(G) &= \max \{e(i) \ : \ i \ \in V\} \\ \rho(G) &= \min \{e(i) \ : \ i \in V\} \\ C(G) &= \{u \in V \ : \ e(u) = \rho(G)\}. \end{aligned}$$

The centre of a graph may be a single vertex or more than one vertices.

The graph of Figure 1(a) has a single vertex as the centre i.e., vertex 3 while the graph of Figure 1(b) has three vertices in the centre i.e., vertices 1, 3, 6.

To find the diameter of a circular-arc graph a main cycle is constructed from the set of arcs. The method to find such cycle is described in the next section.

3 Method to Find a Main Cycle

A *cycle* of a circular-arc graph is a set of intersecting arcs, those arcs cover the whole circle $\mathcal{C}$. That is, if $A_1, A_2, \ldots, A_r$ be a set of arcs of a cycle then $A_i \cap A_{i+1} \neq \phi$, $A_r \cap A_1 \neq \phi$ and $\bigcup_{i=1}^{r} A_i = \mathcal{C}$. The *main cycle* is a cycle whose cardinality is minimum among all cycles. The main cycle is denoted by M'. Let M be the set of vertices corresponding to the arcs of M'. The set M is also regarded as the main cycle. The *length of a cycle* (C) is the number of arcs on the cycle C and it is denoted by $len(C)$. The length of the main cycle is the minimum among all other cycles. A circular-arc graph may have more than one main cycles and the length of each cycle is equal to $|M|$. If the graph has multiple cycles then any one of them is taken as main cycle.

A method to find a main cycle is described below.

First we draw a ray through the finishing point of any arc of A. Then, consider the arcs which are intersected by this ray. Find out the arc which has right most finishing point. This is the first vertex of the set M. Again, we draw a ray from the centre and through the finishing point of the first vertex of M. Consider the arcs which are intersected by the second ray. Find out the arc which has right most finishing point among the arcs which are intersected by second ray. Define the vertex corresponding to this arc as the second vertex of M. Iterate this process until when any vertex of M is repeated. Finally, the duplicate vertices to be removed from M.

ALGORITHM MC

Input: A set of arcs A of the circular-arc graph G.

Output: A set of vertices M which form a main cycle.

Step 1: Set $M = \phi$. Choose any arc A_i from A.

Step 2: Draw a ray through the finishing point of the arc A_i.

Step 3: Consider the arcs which are intersected by the ray drawn in Step 2, and let these set of arcs be B.

Step 4: Find out the arc which has right most finishing point of the arcs of B. Let this arc be A_i.

Step 5: Set $M = M \cup \{i\}$.

Step 6: Repeat Step 2 to 5 until any vertex of M is repeated.

Step 7: Delete the repeated vertices from M, if any.

END MC.

In this algorithm, the endpoints of each arc are consider to select the numbers of M. The total number of arcs of circular-arc graph is taken as n. Thus the time complexity of Algorithm MC is stated below.

Theorem 1 *A main cycle M of a circular-arc graph can be computed in $O(n)$ time.*

Throughout the paper, we mark the vertices of main cycle M by marking asterisk i.e., $v_1^*, v_2^*, \ldots, v_l^*, u^*, v^*$ etc. are the vertices of M. Let D be the set of the vertices which do not belong to M i.e., $D = V \backslash M$. The unmarked (by asterisk) vertices are taken as the vertices of D.

Lemma 1 *The Algorithm MC correctly computes the main cycle.*

Proof: Any vertex of M is adjacent to its next and previous vertices. The process of finding the vertices of M is terminate when any vertex of M is repeated. Let v_i^* be repeated. The next vertex of v_i^* for the first time and v_i^* are adjacent. Similarly, the previous vertex of v_i^* for second time and v_i^* are adjacent. So, if we delete the vertices from first vertex to first v_i^* from M, then the any two consecutive vertices of the remaining vertices of M are adjacent. Also, any vertex is repeated in clockwise traversal, so the vertices of M covers the whole circle i.e., vertices of M form a cycle.

If the length of the cycle is minimum the cycle becomes main cycle. If the removal of any vertex from M makes another cycle by the remaining vertices, then the cycles constructed by the Algorithm MC do not cover the circle $\mathcal{C}$. Let v_i^*, v_{i+1}^*, v_{i+2}^* are three consecutive vertices. So, v_i^* and v_{i+1}^* are adjacent and v_{i+1}^* and v_{i+2}^* are adjacent. If v_i^*, v_{i+2}^* are adjacent then v_{i+2}^* must cover the finishing point of v_i^*. So, both the vertices v_{i+1}^*, v_{i+2}^* cover the finishing point of v_i^*. If v_{i+1}^* is the next vertex of v_i^*, then finishing point of v_{i+2}^* is less than finishing point of v_{i+1}^*. But it is impossible, because v_{i+2}^* is the next vertex of v_{i+1}^*. So, any two non-consecutive vertices are not adjacent. Therefore, if we delete any vertex from M then vertices of M cannot form a cycle. □

A circular-arc graph may contain more than one main cycle. But, the length of all main cycles are equal whatever may be the starting arc. The Algorithm MC generates only one main cycle.

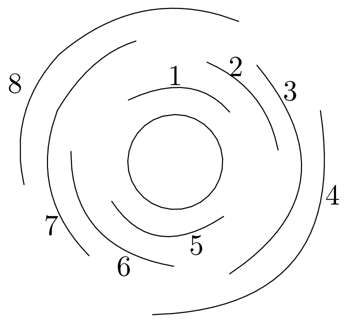

Figure 2: A circular-arc graph representation of a circular-arc graph with two main cycles.

In Figure 2, the main cycle is $\{1, 3, 5, 7\}$ when starting vertex is 1. But, if the starting vertex is 4 then the main cycle is $\{4, 6, 8, 2\}$. In both the cases the length are same.

In geometry, we know that the radius is equal to the half of the diameter. But, in circular-arc graph it may or may not be that the radius is half of the diameter. Let $d(u, v)$ be the shortest distance between the vertices u and v. If the graph is non-weighted then $d(u, v)$ is the number of edges of the path starting from the vertex u and ending at the vertex v. If the graph is weighted then $d(u, v)$ is the sum of the weights of the edges on the shortest path.

The eccentricity $e(u)$ of the vertex u is defined as

$$e(i) = \max\ \{d(i, j)\ :\ j \in V\}.$$

The diameter $\delta(G)$ and radius $\rho(G)$ of the graph G are defined as

$$\delta(G) = \max\ \{e(i)\ :\ i\ \in V\}$$

and

$$\rho(G) = \min\ \{e(i)\ :\ i \in V\}.$$

Every circular-arc graph has a main cycle. If the main cycle of any circular-arc graph contains every vertex of the graph, then the eccentricities of every vertex are equal. So, the maximum of all eccentricities is equal to the minimum of all eccentricities i.e., diameter is equal to radius. In graph of the Figure 3, main cycle contains every vertex of the graph i.e., $M = \{1, 2, 3, 4, 5, 6\}$. Therefore,

$$\begin{aligned} e(i) &= 3 \text{ for all } i \in V, \\ \rho(G) &= \min_i \{e(i)\} = 3, \\ \delta(G) &= \max_i \{e(i)\} = 3, \\ \rho(G) &= \delta(G) = 3. \end{aligned}$$

So, in this circular-arc graph radius and diameter are equal and is equal to 3.

Let C_n be a cycle of length n, i.e., C_n is a cycle containing n vertices and n edges.

From the above observation, we state the following result.

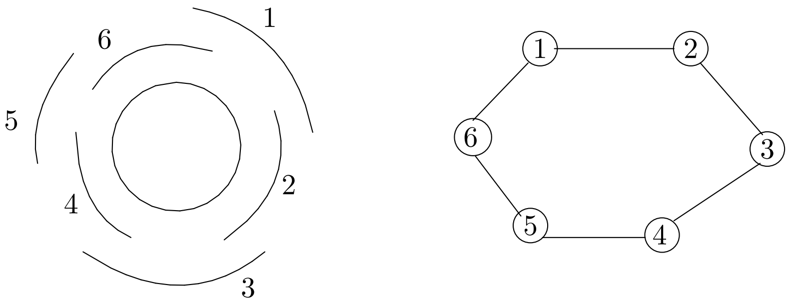

Figure 3: The graph whose radius and diameter are equal.

Lemma 2 *If the circular-arc graph is C_n, i.e., $M = C_n$ then $\delta(G) = \rho(G) = \lfloor \frac{n}{2} \rfloor$.*

Proof: Case – I: n is even.
Let $n = 2m$ and $M = \{v_1^*, v_2^*, \ldots, v_m^*, \ldots, v_{2m}^*\}$. If the main cycle contains every vertex of the circular -arc graph, then there exist only two paths between every pair of vertices of the graph. The distances (not necessarily shortest) between the vertices v_1^* and v_2^* are 1 and $2m-1$. So the shortest distance between v_1^* and v_2^* is equal to 1 i.e., $d(v_1^*, v_2^*) = 1$. There exist $(m-1)$ vertices between v_1^* and v_{m+1}^* in C_n. So, distance between v_1^* and v_{m+1}^* is m. Again, the length of another path from v_1^* to v_{m+1}^* is $(2m-m)$ i.e., m. Therefore, $d(v_1^*, v_{m+1}^*) = m$. Similarly, there exist m vertices between v_1^* and v_{m+2}^* i.e., distance between v_1^* and v_{m+2}^* is $m+1$. But another path from v_1^* and v_{m+2}^* contains $(m-2)$ vertices i.e., another distance between them is $(m-2)$. So, $d(v_1^*, v_{m+2}^*) = (m-1)$. Therefore, eccentricity of v_1^* is m. Similarly, eccentricities of all other vertices of the graph are equal to m i.e., $\frac{n}{2}$. Therefore, diameter and radius of the graph are equal to $\frac{n}{2}$.

Case– II: When n is odd.
Let $n = 2m+1$ and $M = \{v_1^*, v_2^*, \ldots, v_m^*, \ldots, v_{2m+1}^*\}$. In this case also, $d(v_1^*, v_2^*) = 1$, $d(v_1^*, v_3^*) = 2$. One path from v_1^* to v_{m+1}^* contains $(m-1)$ vertices, so the shortest distance between v_1^* and v_{m+1}^* is m. Another path from v_1^* to v_{m+1}^* contain m vertices, so another distance between them is $m+1$. Therefore, $d(v_1^*, v_{m+1}^*) = m$. Similarly, distances between v_1^* and v_{m+2}^* are m and $m+1$. Therefore shortest distance from v_1^* to v_{m+2}^* is m i.e., $d(v_1^*, v_{m+2}^*) = m$. Again, $d(v_1^*, v_m^*) = m-1$ and $d(v_1^*, v_{m+3}^*) = m-1$. So, the eccentricities of all vertices are m i.e., $\frac{(n-1)}{2}$. Therefore, diameter and radius of the graph are equal to $\frac{(n-1)}{2}$.

Hence in both the cases $\delta(G) = \rho(G) = \lfloor \frac{n}{2} \rfloor$, if the circular-arc graph is C_n. □

Lemma 3 *If $u \in D$ then u is adjacent to at least one vertex of M.*

Proof: By the definition of main cycle, the vertices of M form a cycle and cover the whole circle $\mathcal{C}$. Since the graph is a circular-are graph, then any arc corresponding to the vertex $u \in D$ must lie over the circle. But, the vertices of M cover the whole circle. So, the arc corresponding to the vertex u must has non empty intersection with at least one arc corresponding to a vertex of M. Therefore, u is adjacent to at least one vertex of M. □

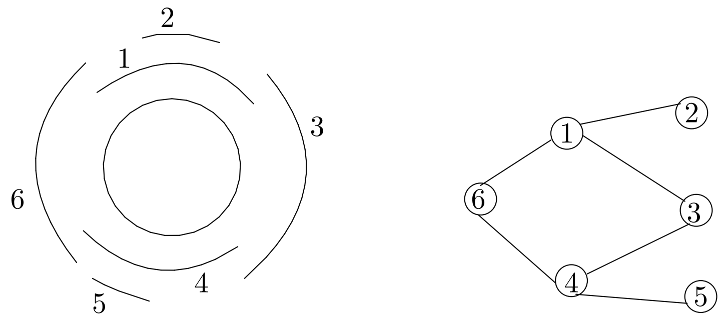

Figure 4: The graph whose diameter is double of the radius.

Motivated by Lemma 2, we define a number r as

$$r = \begin{cases} |M|/2, & \text{if } \ |M| \text{ is even} \\ \dfrac{|M|-1}{2}, & \text{if } \ |M| \text{ is odd.} \end{cases}$$

It can be shown that the upper bound of the distance between any two vertices of M is r.

Lemma 4 *If $v_i^*, v_j^* \in M$ then $d(v_i^*, v_j^*) \leq r$ i.e., distance between any two vertices of M is at most r.*

Proof: If $v_i^*, v_j^* \in M$ are adjacent then $d(v_i^*, v_j^*) = 1$. If there exists one vertex between them in M then $d(v_i^*, v_j^*) = 2$. If there exist $(r-1)$ vertices between v_i^* and v_j^*, then distance between v_i^* and v_j^* is r. v_i^*, v_j^* are the vertices of M so there must exist another path of the distance $r+1$ or r. Therefore $d(v_i^*, v_j^*) = r$. Similarly, if there exist r vertices between v_i^* and v_j^* in M, then distance between v_i^* and v_j^* is $r+1$. Another distance is $r-1$ or r. So, $d(v_i^*, v_j^*) \leq r$. Therefore, distance between any two vertices of M is at most r. $\square$

Throughout the paper, we denote a path between the vertices u and v of length more than one by the symbol $u \xrightarrow{*} v$.

Lemma 5 *Distance between any two vertices of D is not more than $r+2$.*

Proof: Let $u, v \in D$. If u and v are adjacent then $d(u, v) = 1$. If u and v are not adjacent but they are both adjacent to a vertex of M, then $d(u, v) = 2$. Let v_i^* and v_j^* are two vertices of M and $d(v_i^*, v_j^*) = r$. If u is adjacent to only v_i^* and v is adjacent to only v_j^* then there should be a path $u \rightarrow v_i^* \xrightarrow{*} v_j^* \rightarrow v$ connecting u and v. In this case, $d(u, v) = r+2$. If there exists any vertex $w \in D$ which is adjacent to only v but not with v_j^*, then shortest distance between w and u becomes $r+3$. But from the Lemma 4, we know that every vertex of D is adjacent at least one vertex of M. So, w must adjacent to v_j^*, then $d(w, u) = r+2$. Therefore, distance between any two vertices of D is not more than $r+2$. $\square$

In Figure 4, the distance between the vertices 1 and 4 is 2. Vertex 2 is adjacent to vertex 1 and vertex 5 is adjacent to only vertex 4. So, the distance between 2 and 5 is 2+2=4.

Lemma 6 *The diameter of a circular-arc graph is at most $r+2$ i.e., $\delta(G) \leq r+2$.*

Proof: Let $d(v_i^*, v_j^*) = r$, where v_i^*, v_j^* are vertices of M. If $u, v \in D$ are two vertices such that u is adjacent to v_i^* and v is adjacent to only v_j^*, then $d(u, v) = r + 2$. By the Lemma 5, the distance between any two vertices is not more than $r + 2$. So, $e(v) = r + 2$ and $e(u) = r + 2$. Also $d(u, v_j^*) = r + 1$ and $d(v_i^*, v) = r + 1$. If w is adjacent to any vertex of M other than v_j^* then $d(v_i^*, w) \leq r + 1$. So, $e(v_i^*) \leq r + 1$ i.e., eccentricity of any vertex of M is not more than $r + 1$. Therefore, diameter of the graph is not more than $r + 2$. □

Lemma 7 *If u be any vertex of D then $r \leq e(u) \leq r + 2$.*

Proof: Let $v_{i-1}^*, v_i^*, v_{i+1}^*$ be three adjacent vertices of M and $d(v_i^*, v_j^*) = r$. If u is adjacent to all the vertices $v_{i-1}^*, v_i^*, v_{i+1}^*$, then the paths from u to v_j^* are $u \to v_{i+1}^* \xrightarrow{*} v_j^*$ and $u \to v_{i-1}^* \xrightarrow{*} v_j^*$. So, the shortest distance between u and v_j^* is r i.e., $d(u, v_j^*) = r$. If there is no vertex $v \in D$ adjacent to v_j^* then $e(u) = r$.

Let $d(v_i^*, v_j^*) = r$. u is adjacent to only v_i^*, but there exists no vertex $v \in D$ adjacent to v_j^* then the path from u to v_j^* is $u \to v_i^* \xrightarrow{*} v_j^*$ and $d(u, v_j^*) = r + 1$. So, $e(u) = r + 1$.

u is adjacent to only vertex v_i^* and v is adjacent to only vertex v_j^*. There exists a path $u \to v_i^* \xrightarrow{*} v_j^* \to v$ and $d(u, v) = r + 2$. By the Lemma 4, distance between any two vertices of D is not more than $r + 2$. So, $e(u) = r + 2$. Therefore, $r \leq e(u) \leq r + 2$. □

From Lemma 2, Lemma 6 and Lemma 7 we can conclude the following result.

Lemma 8 *For any circular-arc graph $r \leq \delta(G) \leq r + 2$, where $r = \lfloor \frac{|M|}{2} \rfloor$.*

4 Computation of Eccentricity

If the vertices u^*, v^* on the main cycle having distance r then we called they form a *pair*, i.e., if (u^*, v^*) is a pair then $d(u^*, v^*) = r$. From Lemma 4 we know that maximum of the distances between any two vertices of M is r. From Lemma 8 we observed that there may exist a vertex of D whose eccentricity is r. Thus a vertex of D may be a centre of the graph.

The number of vertices at a distance r from a vertex of M is either one or two, depending on the size of M. This observation is proved in the following lemma.

Lemma 9 *Let $d(v_i^*, v_j^*) = r$ and v_i^* is fixed. If $|M|$ is even then v_j^* is unique and if $|M|$ is odd then there are two vertices v_j^* and v_k^* such that $d(v_i^*, v_j^*) = d(v_i^*, v_k^*) = r$.*

Proof: Let $|M|$ be even and $d(v_i^*, v_j^*) = r$. Then the length of the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^*$ is r and length of the another path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$ is also equal to r. So, the number of vertices in both the paths is $(r - 1)$. If $v_k^* \in M$ is a vertex in the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^*$ then the length of the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_k^*$ is less than r and length of the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^* \xrightarrow{*} v_k^*$ is greater than r. So, the shortest distance between the vertices v_i^* and v_k^* is less than r. Similarly, if v_k^* is a vertex in the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$ then $d(v_i^*, v_k^*)$ is less than r. Therefore, there is unique vertex v_j^* whose distance from v_i^* is r.

Let $|M|$ be odd and $d(v_i^*, v_j^*) = r$. If the length of the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^*$ is r then the length of the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$ is must equal to $r + 1$. So, there are $(r - 1)$ vertices in the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^*$ and r vertices in the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$. If $v_k^* \in M$ is a vertex

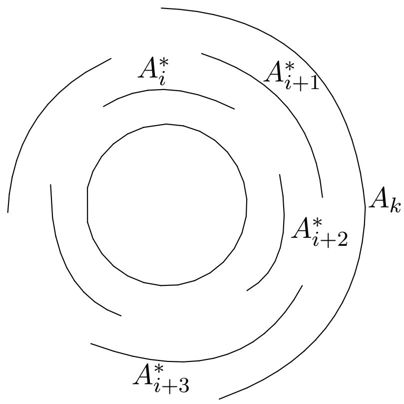

Figure 5:

in the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^*$, then $d(v_i^*, v_k^*)$ is less than r. But, if v_k^* is a vertex in the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$ and adjacent to v_j^*, then the length of the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_k^*$ is r and the length of the path $v_i^* \to v_{i+1}^* \xrightarrow{*} v_j^* \to v_k^*$ is $(r+1)$. So, $d(v_i^*, v_k^*) = r$. If v_k^* is a vertex in the path $v_i^* \to v_{i-1}^* \xrightarrow{*} v_j^*$ and v_k^*, v_j^* are non-adjacent, then $d(v_i^*, v_k^*)$ is less than r. Therefore, there are two vertices v_j^*, v_k^* such that $d(v_i^*, v_j^*) = d(v_i^*, v_k^*) = r$. □

The number of the vertices of M may be even or odd. Depending on the size of M, we propose some results to determine the eccentricities of all vertices of V. In the following we consider $|M|$ is even.

Lemma 10 *Let (v_i^*, v_j^*) and (v_{i+1}^*, v_k^*) be two pairs. If v_i^* and v_{i+1}^* are adjacent then v_j^* and v_k^* are adjacent.*

Proof: If (v_i^*, v_j^*) is a pair, then distance between v_i^* and v_j^* is r. The vertices v_i^* and v_{i+1}^* are adjacent. So, the shortest distance between v_{i+1}^* and v_j^* is $r-1$. Also v_k^* and v_{i+1}^* are belong to a pair, so the distance between v_{i+1}^* and v_k^* is r. Therefore the distances from the vertex v_{i+1}^* to the vertices v_j^* and v_k^* are $r-1$ and r respectively. Therefore, v_j^* and v_k^* are two adjacent vertices. □

Lemma 11 *Any vertex of D cannot adjacent to more than three vertices of main cycle.*

Proof: Let A_k be the arc corresponding to the vertex $v_k \in D$. If possible, let the arc A_k intersect with four arcs $A_i^*, A_{i+1}^*, A_{i+2}^*, A_{i+3}^*$ (see Figure 5). Then the arc A_k covers, at least, the finishing point of the arc A_i^* and starting point of A_{i+3}^*. So, starting point of A_{i+3}^* is less than the finishing point of A_k i.e., $s_{i+3}^* < f_k$. By definition, it is easy to see that A_{i+1}^*, A_{i+3}^* are non intersecting arcs. So, finishing point of A_{i+1}^* is less than starting point of A_{i+3}^* i.e., $f_{i+1}^* < s_{i+3}^*$. Therefore $f_{i+1}^* < f_k$. Both the arcs A_{i+1}^*, A_k cover the finishing point of A_i^* and finishing point of A_{i+1}^* is less than finishing point A_k. Thus the arc A_k is selected as the next arc of A_i^* in main cycle M'. But, v_k is not a member of M' . Therefore, A_k cannot intersect with four arcs $A_i^*, A_{i+1}^*, A_{i+2}^*, A_{i+3}^*$. □

Lemma 12 *If the arc A_k corresponding to the vertex $v_k \in D$ intersect with three consecutive arcs $A_i^*, A_{i+1}^*, A_{i+2}^*$, then the eccentricity of v_k is equal to the eccentricity of v_{i+1}^*.*

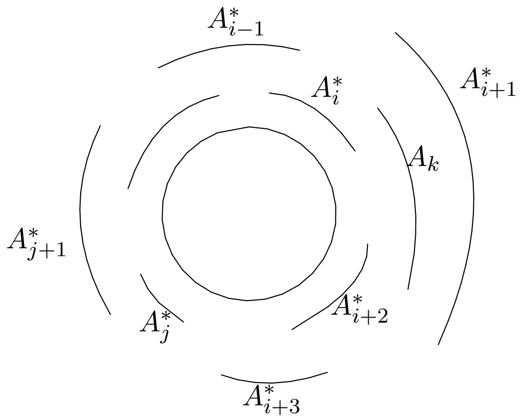

Figure 6:

Proof: Let (v^*_{i+1}, v^*_{j+1}) be a pair. So, the distance between the vertices (v^*_{i+1}, v^*_{j+1}) is r. If $|M|$ is even, then lengths of both the paths $A^*_{i+1} \rightarrow A^*_{i+2} \xrightarrow{*} A^*_{j+1}$ and $A^*_{i+1} \rightarrow A^*_i \xrightarrow{*} A^*_{j+1}$ are r (see Figure 6). Therefore, the length of the path $A^*_{i+2} \rightarrow A^*_{i+3} \xrightarrow{*} A^*_{j+1}$ is $r-1$ and the length of the path $A^*_i \rightarrow A^*_{i-1} \xrightarrow{*} A^*_{j+1}$ is $r-1$. The arc A_k intersect with both the arcs A^*_i and A^*_{i+2}. So, lengths of the paths $A_k \rightarrow A^*_{i+2} \xrightarrow{*} A^*_{j+1}$ and $A_k \rightarrow A^*_i \xrightarrow{*} A^*_{j+1}$ are r i.e., $d(v_k, v^*_{j+1}) = r$. Then, the vertex v^*_{i+1} can be replaced by v_k. Therefore, the eccentricity of v_k is equal to the eccentricity of v^*_{i+1}.

□

Lemma 13 *Let (v^*_i, v^*_j) and (v^*_{i+1}, v^*_{j+1}) be two pairs. If the arc A_k corresponding to the vertex $v_k \in D$ intersect with two consecutive arcs A^*_i, A^*_{i+1} and there exists no arc A_l corresponding vertex $v_l \in D$ intersected with A^*_j, A^*_{j+1}, then the eccentricity of v_k is r.*

Proof: Let (v^*_i, v^*_j) and (v^*_{i+1}, v^*_{j+1}) be two pairs. So, the lengths of the paths $A^*_i \rightarrow A^*_{i+1} \xrightarrow{*} A^*_j$ and $A^*_{i+1} \rightarrow A^*_{i+2} \xrightarrow{*} A^*_{j+1}$ are equal to r. The length of the path $A^*_i \rightarrow A^*_{i+1} \xrightarrow{*} A^*_{j+1}$ is $r+1$ and the length of the path $A^*_i \rightarrow A^*_{i-1} \xrightarrow{*} A^*_{j+1}$ is $r-1$ (see Figure 7). Similarly, the lengths of the paths $A^*_{i+1} \rightarrow A^*_{i+2} \xrightarrow{*} A^*_j$ and $A^*_{i+1} \rightarrow A^*_i \xrightarrow{*} A^*_j$ are $r-1$ and $r+1$ respectively. So, the shortest distance between the vertices v^*_i, v^*_{j+1} is $r-1$ and shortest distance between the vertices v^*_{i+1}, v^*_j is $r-1$. The arc A_k is intersected with both the arcs A^*_i and A^*_{i+1}. So, the lengths of the paths $A_k \rightarrow A^*_{i+1} \xrightarrow{*} A^*_j$ and $A_k \rightarrow A^*_i \xrightarrow{*} A^*_{j+1}$ are equal to r. So, the shortest distance between the vertices v_k and v^*_j is r and shortest distance between the vertices v_k and v^*_{j+1} is r. If there exists no vertex $v_l \in D$ adjacent to v^*_j and v^*_{j+1}, so there exists no path from v_k with length greater than r. Therefore, the eccentricity of v_k is r. □

Lemma 14 *Let (v^*_i, v^*_j) and (v^*_{i+1}, v^*_{j+1}) be two pairs. The arc A_k corresponding to the vertex $v_k \in D$ is intersected with two intersecting arcs A^*_i, A^*_{i+1} and another arc A_l corresponding to the vertex $v_l \in D$ is intersected with A^*_j, A^*_{j+1}. Then the eccentricities of v_k and v_l are equal to $r+1$.*

Proof: From Lemma 13 the length of the path $A^*_i \rightarrow A^*_{i-1} \xrightarrow{*} A^*_{j+1}$ is $r-1$ and the length of the path $A^*_{i+1} \rightarrow A^*_{i+2} \xrightarrow{*} A^*_j$ is $r-1$. A_k is intersected with A^*_i, A^*_{i+1} and A_l is intersected

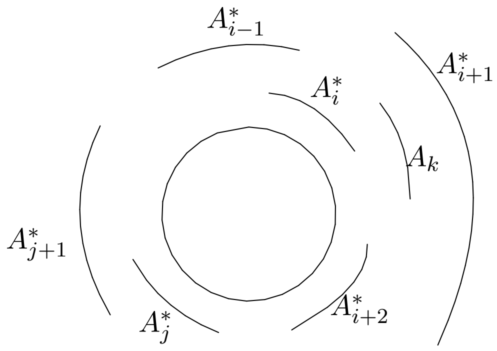

Figure 7:

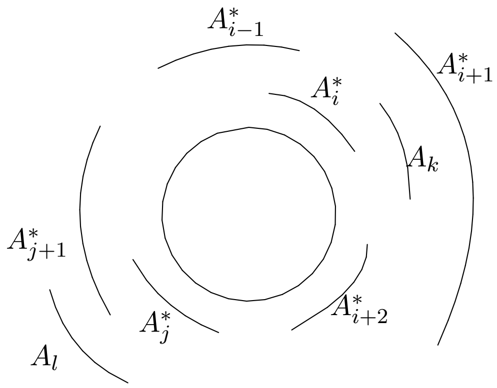

Figure 8:

with A^*_j, A^*_{j+1}. So, the length of the path $A_k \to A^*_i \to A^*_{i-1} \xrightarrow{*} A^*_{j+1} \to A_l$ is $r+1$ (see Figure 8). Also the length of the path $A_k \to A^*_{i+1} \to A^*_{i+2} \xrightarrow{*} A^*_j \to A_l$ is $r+1$ i.e., $d(v_k, v_l) = r+1$. Therefore, the eccentricities of v_k and v_l are equal to $r+1$. □

Lemma 15 *Let the arc A_k corresponding to the vertex $v_k \in D$ be intersected with only one arc A^*_i. Then the eccentricities of v_k and v^*_j are equal to $r+1$, where (v^*_i, v^*_j) is a pair.*

Proof: To prove this lemma we refer Figure 9. By the definition of the pair $d(v^*_i, v^*_j) = r$. If $|M|$ is even then the lengths of both the paths $A^*_i \to A^*_{i+1} \xrightarrow{*} A^*_j$ and $A^*_i \to A^*_{i-1} \xrightarrow{*} A^*_j$ are equal to r. Since A_k is intersected with only the arc A^*_i, the length of the path $A_k \to A^*_i \to A^*_{i-1} \xrightarrow{*} A^*_j$ is $r+1$. Similarly, the length of the path $A_k \to A^*_i \to A^*_{i+1} \xrightarrow{*} A^*_j$ is $r+1$. Then the shortest distance between the vertices v_k and v^*_j is $r+1$. Therefore eccentricities of the vertices v_k and v^*_j are equal to $r+1$. □

The above results are discussed by assuming that $|M|$ is even. Similar, but minor modified results are discussed in the following. Here we assume that $|M|$ is odd.

Lemma 16 *If (v^*_i, v^*_j) and (v^*_i, v^*_k) are two pairs, then vertices v^*_j and v^*_k are adjacent.*

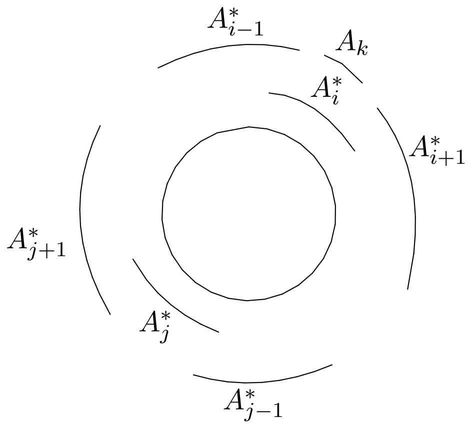

Figure 9:

Proof : By definition, $d(v_i^*, v_j^*) = r$ and $d(v_i^*, v_k^*) = r$. Let v_j^* and v_k^* are non-adjacent vertices. So, there must exist at least one vertex v_l^* between v_j^* and v_k^* in M. Then the distance between v_i^* and v_l^* is $r+1$. But, this result contradicts Lemma 4. So, there is no vertex between v_j^* and v_k^* in M. Therefore, the vertices v_j^* and v_k^* are adjacent. $\square$

Lemma 17 *Let the arc A_k corresponding to the vertex $v_k \in D$ be intersected with three consecutive arcs $A_i^*, A_{i+1}^*, A_{i+2}^*$. Then the eccentricities of v_k is equal to the eccentricity of v_{i+1}^*.*

Proof: Let (v_{i+1}^*, v_j^*) and (v_{i+1}^*, v_{j+1}^*) be two pairs. Then the lengths of the paths $A_{i+1}^* \to A_{i+2}^* \xrightarrow{*} A_j^*$ and $A_{i+1}^* \to A_i^* \xrightarrow{*} A_{j+1}^*$ are equal to r. The length of the path $A_{i+2}^* \to A_{i+3}^* \xrightarrow{*} A_j^*$ is $r-1$ and the path $A_i^* \to A_{i-1}^* \xrightarrow{*} A_{j+1}^*$ is $r-1$. Since A_k is intersect with both A_i^* and A_{i+2}^*, the lengths of the paths $A_k \to A_{i+2}^* \xrightarrow{*} A_j^*$ and $A_k \to A_i^* \to A_{i-1}^* \xrightarrow{*} A_{j+1}^*$ are equal to $r+1$. So, $d(v_k, v_j^*) = r$ and $d(v_k, v_{j+1}^*) = r$. Thus A_j^* and A_{j+1}^* are intersected. Also, $d(v_{i+1}^*, v_{j+1}^*) = r$. Therefore, the eccentricity of v_k is equal to the eccentricity of v_{i+1}^*. $\square$

Lemma 18 *An arc A_k corresponding to the vertex $v_k \in D$ has non-empty intersection with two intersecting arcs A_i^* and A_{i+1}^*. Then the eccentricities of v_k and v_j^* are equal to $r+1$, where (v_i^*, v_j^*) and (v_{i+1}^*, v_j^*) are two pairs.*

Proof: Refer Figure 10 to prove this lemma. If (v_i^*, v_j^*) and (v_{i+1}^*, v_j^*) are two pairs, then the lengths of the paths $A_i^* \to A_{i-1}^* \xrightarrow{*} A_j^*$ and $A_{i+1}^* \to A_{i+2}^* \xrightarrow{*} A_j^*$ are equal to r. A_k is intersect with A_i^* and A_{i+1}^*. So, the length of the path $A_k \to A_i^* \to A_{i-1}^* \xrightarrow{*} A_j^*$ is $r+1$ and length of the path $A_k \to A_{i+1}^* \to A_{i+2}^* \xrightarrow{*} A_j^*$ is $r+1$. So, $d((v_k, v_j^*) = r+1$. Therefore, the eccentricities of v_k and v_j^* are equal to $r+1$. $\square$

Lemma 19 *Let the arc A_k corresponding to the vertex $v_k \in D$ be intersected with only the arc A_i^*. Then the eccentricities of v_k, v_j^* and v_{j+1}^* are equal to $r+1$, where (v_i^*, v_j^*) and (v_i^*, v_{j+1}^*) are two pairs.*

Proof: Let (v_i^*, v_j^*) and (v_i^*, v_{j+1}^*) be two pairs. So, the length of the path $A_i^* \to A_{i-1}^* \xrightarrow{*} A_{j+1}^*$ is r and length of the path $A_i^* \to A_{i+1}^* \to A_{i+2}^* \xrightarrow{*} A_j^*$ is also r (see Figure 11). Similarly, the arc

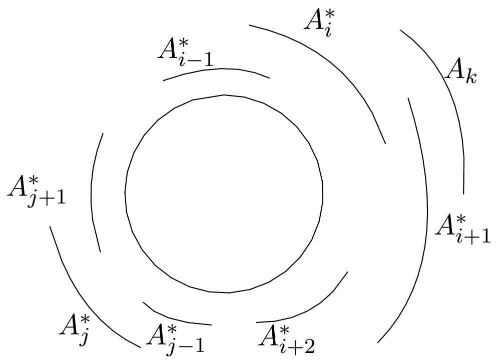

Figure 10:

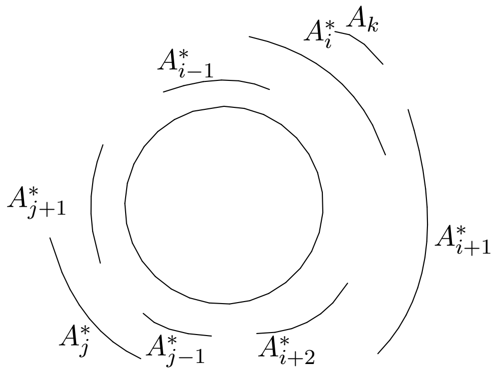

Figure 11:

A_k is intersect with only the arc A_i^*. So, the lengths of the paths $A_k \to A_i^* \to A_{i+1}^* \xrightarrow{*} A_j^*$ and $A_k \to A_i^* \to A_{i-1}^* \xrightarrow{*} A_{j+1}^*$ are $r+1$. Then the shortest distance between v_k and v_j^* is $r+1$ and shortest distance between v_k and v_{j+1}^* is $r+1$. Therefore, the eccentricities of v_k, v_j^* and v_{j+1}^* are equal to $r+1$. $\square$

Algorithm EV

Input: The sets M, D the array $pair(v_i^*)$, $v_i^* \in M$ and the integer r.

Output: The array $e(v_i)$, $v_i \in V$, the eccentricity of all vertices.

Case I. $|M|$ is even.

Step 1: For all $v_i^* \in M$ do

(*i*) if $pair(v_i^*)$ is not adjacent to any vertex of D then $e(v_i^*) = r$.

(*ii*) if there exists a vertex $v_l \in D$ and there exists another vertex $v_k^* \in M$ such that $(v_l, pair(v_i^*)) \in E$ and $(v_l, v_k^*) \in E$ then $e(v_i^*) = r$.

(*iii*) if there exists a vertex $v_l \in D$ such that $(v_l, pair(v_i^*)) \in E$ but v_l is not adjacent to other vertex of M then $e(v_i^*) = r+1$.

Step 2: For all $v_i \in D$ do

(*i*) if there exist three adjacent vertices $v_j^*, v_{j+1}^*, v_{j+2}^* \in M$ such that v_i is adjacent to $v_j^*, v_{j+1}^*, v_{j+2}^*$ then $e(v_i) = e(v_{j+1}^*)$.

(ii) If there exist two adjacent vertices $v_j^*, v_{j+1}^* \in M$ such that $(v_i, v_j^*) \in E$ and $(v_i, v_{j+1}^*) \in E$, $v_l \in D$ and
(a) $(pair(v_j^*), v_l) \notin E$ and $(pair(v_{j+1}^*), v_l) \notin E$ then $e(v_i) = r$.
(b) $(pair(v_j^*), v_l) \in E$ but $(v_k^*, v_l) \notin E$ for $v_k^* \in M$ then $e(v_i) = r + 1$.
(c) $(pair(v_{j+1}^*), v_l) \in E$ but $(v_k^*, v_l) \notin E$ for $v_k^* \in M$ then $e(v_i) = r + 1$.
(d) $(pair(v_j^*), v_l) \in E$ and $(pair(v_{j+1}^*), v_l) \in E$ then $e(v_i) = r + 1$.
(iii) if v_i is adjacent to a vertex $v_j^* \in M$, and there exists a vertex $v_l \in D$ and
(a) $(pair(v_j^*), v_l) \notin E$ then $e(v_i) = r + 1$.
(b) $(pair(v_j^*), v_l) \in E$ and $(v_k^*, v_l) \in E$ for $v_k^* \in M$, then $e(v_i) = r + 1$.
(c) $(pair(v_j^*), v_l) \in E$ but $(v_k^*, v_l) \notin E$ for $v_k^* \in M$, then $e(v_i) = r + 2$.

Case II. $|M|$ is odd.
Step 3: For all $v_i^* \in M$ do
(i) if $pair(v_i^*)$ is not adjacent to any vertex of D then $e(v_i^*) = r$.
(ii) if there exists a vertex $v_l \in D$ and there exists another vertex $v_k^* \in M$ such that $(v_l, pair(v_i^*)) \in E$ and $(v_l, v_k^*) \in E$ then $e(v_i^*) = r$.
(iii) if there exists a vertex $v_l \in D$ such that $(v_l, pair(v_i^*)) \in E$ but v_l is not adjacent to other vertex of M then $e(v_i^*) = r + 1$.
Step 4: For all $v_i \in D$ do
(i) if there exist three adjacent vertices $v_j^*, v_{j+1}^*, v_{j+2}^* \in M$ such that v_i is adjacent to $v_j^*, v_{j+1}^*, v_{j+2}^*$ then $e(v_i) = e(v_{j+1}^*)$.
(ii) if there exist two adjacent vertices $v_j^*, v_{j+1}^* \in M$ such that $(v_i, v_j^*) \in E$ and $(v_i, v_{j+1}^*) \in E$, $v_l \in D$ and
(a) $(pair(v_j^*), v_l) \notin E$ and $(pair(v_{j+1}^*), v_l) \notin E$ then $e(v_i) = r + 1$.
(b) $(pair(v_j^*), v_l) \in E$ but $(v_k^*, v_l) \notin E$ for $v_k^* \in M$ then $e(v_i) = r + 1$.
(c) $(pair(v_{j+1}^*), v_l) \in E$ but $(v_k^*, v_l) \notin E$ for $v_k^* \in M$ then $e(v_i) = r + 1$.
(d) $(pair(v_j^*), v_l) \in E$ and $(pair(v_{j+1}^*), v_l) \in E$ then $e(v_i) = r + 1$.
(e) $(v_m^*, v_l) \in E$ but v_l is not adjacent to another vertex of M then $e(v_i) = r + 2$, where $v_m^* = pair(v_j^*) = pair(v_{j+1}^*)$.
(iii) if v_i is adjacent to a vertex $v_j^* \in M$, and there exists a vertex $v_l \in D$ and
(a) $(pair(v_j^*), v_l) \notin E$ then $e(v_i) = r + 1$.
(b) $(pair(v_j^*), v_l) \in E$ and $(v_k^*, v_l) \in E$ for $v_k^* \in M$, then $e(v_i) = r + 1$.
(c) $(pair(v_j^*), v_l) \in E$ but v_l is not adjacent to another vertex of M then $e(v_i) = r + 2$.
END EV

The array *pair* of each vertex can be computed in $O(n)$ time. If the endpoints of each arc are available then with the help of array *pair* we can compute the eccentricity of each vertex using $O(n)$ time. We assume that the endpoints of each arc are given as input. Thus we may conclude the following result.

Theorem 2 *The eccentricities of all vertices of a circular-arc graph with n vertices can be determined in $O(n)$ time, if the circular arc representation is given.*

5 The Algorithm and its Complexity

When the eccentricities of all vertices are known then computation of radius, diameter and centre is a trivial task.

In this section, we present an algorithm to find the centre, radius and the diameter of a circular-arc graph.

Algorithm RDC
Input: A set of arcs A_i of a circular-arc graph $G = (V, E)$.
Output: Radius $\rho(G)$, diameter $\delta(G)$, centre $C(G)$.
Step 1: Find the main cycle M using Algorithm MC.
Step 2: Find the eccentricities of all vertices using Algorithm EV.
Step 3: Compute $\rho(G) = min\{e(v_i) : v_i \in V\}$.
Step 4: Compute $\delta(G) = max\{e(v_i) : v_i \in V\}$.
Step 5: Initially, let $C(G) = \phi$ (the null set). For all $v_i \in V$ do
if $e(v_i) = \rho(G)$ then $C(G) = C(G) \cup \{v_i\}$.
End RDC

Theorem 3 *The radius, diameter and centre of a circular-arc graph can be determined using $O(n)$ time, where n represents the number of vertices.*

Proof: Steps 1, and 2 takes $O(n)$ time (Theorem 1, 2). The steps 3, 4, 5 can easily be computed using only $O(n)$ time. Hence the over all time complexity is $O(n)$.

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