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Perishable Inventory System with Random Supply Quantity and Negative Demands

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Abstract

In this paper we consider a continuous review perishable inventory system with a modified (s, S) policy which allows a finite number of orders to be placed and full backlogging of demands. However the demand that occurred during the stock out period has the option to join the system with prefixed probability. Moreover, we assume two streams of demands called regular and negative. The negative demand during the stock out periods removes one of the waiting demand if any. The limiting distribution of the inventory level is shown to have matrix geometric form. The measures of system performance in the steady state are derived.

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Keywords : Stochastic Inventory, Random life time, Modified (s, S) Policy, Positive lead time, Negative demand.

1 Introduction

Inventory systems with stochastic input and output processes have been attracting the researchers from the mid twentieth century. Hadley and Whitin[1963] used the probabilistic methods to analyse such systems. Sivazlian[1974] used the methods of renewal processes to analyse continuous review (s, S) inventory systems(CRIS). The work of Srinivasan[1979] provided a general analysis of CRIS with arbitrarily distributed inter-demand time points and random lead times. Since then many researchers contributed to the analyses of CRIS (see Kalpakam and Arivarignan [1985, 1989, 1993], Kalpakam and Sapna[1994], Arivarignan[1994] and Arivarignan and Elango[2003]).

Weiss[1980] extended the notion of perishable inventories to the realm of continuous review system by obtaining $(0, S)$ policy as the optimal ordering policy for a model with Poisson demand and instantaneous supply of ordered items. He assumed that the items fail after a fixed lifetime. Kalpakam and Arivarignan[1988] extended this work to include exponential life time for items.

Studies on CRIS of perishable items include Schmidt and Nahmias[1985] and Kalpakam and Sapna[1994]. Arivarignan[1994] showed that in the case of perishable items and instantaneous supply of ordered items, $(-r, S)$ policy (which places an order at the r-th demand after the depletion of stock) is optimal.

Kalpakam and Arivarignan[1989] introduced a modified (s, S) ordering policy which allows more than one pending orders at a time but fixes an upper bound for it. This policy allowed full backlogging of demands and they assumed that the number of nondefective items in a supply is a random variable. The modified (s, S) policy was adopted by Liu and Lian[1999] for a perishable inventory model.

In this work, we extend the work of Kalpakam and Arivarignan[1989] by including perishable items and negative demands.

The plan of the manuscript is as follows. The section 2 describes the model assumptions. The section 3 presents the steady state distribution of the inventory level and expresses it in a matrix-geometric form. Finally the measures of system performance in the steady state are calculated.

2 The Model

Consider an inventory system which can stock a maximum of S perishable items. The items are removed from the stock as and when items perish (each item has exponential life time) and a demand occurs (the demand time points form a Poisson process). We assume a modified (s, S) ordering policy: An order for Q $(0 < Q < S)$ items is placed whenever Q items are removed from the stock and a maximum of m $(> S/Q)$ orders can be placed. This condition ensures that orders for Q items will be placed until all the available stock is exhausted. The demands that occurred during the stock out periods are fully backlogged and these demands may join the system with probability p and may not join the system with probability $(1-p), (0 \leq p \leq 1)$. We also consider a stream of negative demands which arrive according to an independent Poisson process and it will not demand any item when the inventory level is 0 but remove one of the waiting demands, if any during the stockout period. We assume that not all the supplied items in the lot are in good usable condition. Thus the number of non defective items in the supply of the order is a random quantity.

The modified ordering policy is defined interms of the prefixed reorder levels namely, $S-Q, S-2Q, \cdots, S-mQ$: An order for $Q(= S-s)$ is placed

1. whenever the net inventory level (on hand minus backorders) drops to any one of the reorder levels.

2. at the time of replenishment, if the supply is not sufficient enough to take the net inventory level above the preceding reorder level.

In other words we follow a modified $(m+1)$-bin policy in the sense that if we partition the state space E into $(m+1)$ classes, viz., $E_0 = \{S, S-1, \ldots, S-Q+1\}, E_1 = \{S-Q, S-Q-1, \ldots, S-2Q+1\}, \ldots, E_{m-1} = \{S-(m-1)Q, \ldots, S-mQ+1\}$ and $E_m = \{S-mQ, \ldots, \}$, then a reorder for Q items is placed at a demand point if the inventory level moves from E_i

to $E_{i+1}, (i = 0, 1, 2, \ldots, m-1)$, and at a replenishment point if the resupply is not sufficient enough to move the level from one class to the preceeding one.

Notations

0 : Zero matrix

A': Transpose of any matrix A

I : Identity matrix of order Q

e^T: $(1, 1, \ldots, 1)_{1\times Q}$

3 Analysis

Let $I(t)$ denote the net inventory level at time t. Then the process $\{I(t), t \geq 0\}$ has the state space $E = \{S, S-1, S-2, \ldots\}$. Let $Z(t)$ be the number of pending orders at time t which takes values $0, 1, 2, \ldots, m$.

The demand is for single item and the demand occurrence times form a Poisson process with rate λ_1. The negative demands form an independent Poisson process with rate λ_2. The life time of each item has exponential distribution with parameter μ. A reorder will be for a fixed quantity of Q items and the maximum number of orders that can be pending at any time is fixed as m with $S - mQ \leq 0$.

The policy for placing orders is as follows: A reorder will be placed

(i) at a demand epoch t when $I(t) > I(t+) = S - kQ,\ k = 1, 2, \ldots, m$ and

(ii) at a replenishment epoch t when $S - (l+1)Q < I(t) < I(t+) \leq S - lQ,\ l = 1, 2, \ldots, m-1$ and $I(t) < I(t+) \leq S - mQ$

For the input process, the rate of replenishment is assumed to depend not only on the number of outstanding orders but also on the quantity of nondefective items in the supply. We assume for $\beta_{ij} > 0,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, Q$

$$\begin{aligned} P[\text{a replenishment for j-items in}(t, t + \Delta t)|Z(t) = i] &= \beta_{ij}\Delta t + o(\Delta t). \\ P[\text{more than one replenishment in}(t, t + \Delta t)|Z(t) = i] &= o(\Delta t), \\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, Q. \end{aligned}$$

Then we have

$$\begin{aligned} P[\text{a replenishment in}(t, t + \Delta t)|Z(t) = i] &= \beta_i^1\Delta t + o(\Delta t) \\ and \quad P[\text{no replenishment in}(t, t + \Delta t)|Z(t) = i] &= 1 - \beta_i^1\Delta t + o(\Delta t) \end{aligned}$$

where $\beta_i^1 = \sum_{j=1}^{Q} \beta_{ij}$. We also write $\beta_i^j = \sum_{k=j}^{Q} \beta_{ik}$, and $g_i^j = \sum_{k=1}^{j} \beta_{ik}$.

During the stock out periods, the backlogged demand may join the system with probability p or may not join the system with probability $(1 - p), (0 \le p \le 1)$.

It may be observed that the net inventory level at any time uniquely determines the number of orders pending. This fact, with the assumptions on the input process and the Poisson nature of demands, implies that $\{I(t), t \ge 0\}$ is a Markov process. To determine the infinitesimal generator $A = ((a(i, j))),\ \ i, j \in E$, we use the following arguments.

As a demand or a failure of an item takes the inventory level down by one unit, the intensity of transition $a(i, i-1)$ from i to $i-1$ is given by $\lambda_1 + \mu_i$ for $i > 0$ and $p\lambda_1$ for $i \le 0$ where $\mu_i = i\mu$ for $i > 0$ and 0 for $i \le 0$. When the net inventory level i satisfies the condition $S - (l+1)Q < i \le S - lQ,\ \ l$ ($= 1, 2. \ldots, m-1$) orders are pending and the intensity of transitions from i to $i + x$(where x is the quantity supplied, $x = 1, 2, \ldots, Q$) is β_{lx}. When $i \le S - mQ$, the transition to the level $i + x$ ($x = 2, 3, \ldots, Q$) occurs at the rate β_{mx} and the transition to the level $i+1$ occurs at the rate $\beta_{m1} + \lambda_2$. No other type of transition is possible from i to j ($\ne i$). To obtain the intensity of passage $-a(i,i)$ of level i, we make use of the identity

$$a(i,i) = -\sum_{j \neq i} a(i,j)$$

Hence we have $a(i,j)$

$$= \begin{cases} -\lambda_{1i}, & i=j, & i = S-Q+1, S-Q+2, \ldots, S, \\ -(\lambda_{1i} + \beta_q^{(1)}), & j=i, & i = S-(q+1)Q+1, \ldots, S-qQ, \\ & q = 1,2,\ldots,m-1 & \\ -(\lambda_1 + \beta_m^{(1)}), & j = i, & i = S-mQ, S-mQ-1, \ldots, \\ \lambda_{1i}, & j = i-1, & i = S, S-1, \ldots \\ \beta_{q(j-i)}, & j = i+1, i+2, \ldots, i+Q, & i = S-(q+1)Q+1, \ldots, S-qQ \\ & q = 1,2,\ldots,m-1 & \\ \beta_{m(j-i)}, & j = i+2, i+3, \ldots, Q, & i = S-mQ, S-mQ-1, \ldots \\ \beta_{m(j-i)} + \lambda_2, & j = i+1, & i = S-mQ, S-mQ-1, \ldots \\ 0, & \text{otherwise.} & \end{cases}$$

where $\lambda_{1i} = \begin{cases} \lambda_1 + i\mu, & \text{if } \ i > 0 \\ p\lambda_1, & \text{if } \ i \leq 0. \end{cases}$

Define $\boldsymbol{q} = (S-qQ, S-qQ-1, \ldots, S-(q+1)Q+1), \ \ q = 0,1,2,\ldots.$ The infinitesimal matrix A (where the row (column) numbers are $S, S-1, \ldots$) can be conveniently expressed as a block-partitioned matrix with blocks of size $Q \times Q$, as follows,

$$A = \begin{pmatrix} A_0 & B_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ C_1 & A_1 & B_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & A_2 & B_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & C_{m-1} & A_{m-1} & B_{m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & C_m & A_m & B_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & C_m & A_m & B_m \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where

$$A_0 = \begin{pmatrix} -\lambda_{1S} & \lambda_{1S} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda_{1(S-1)} & \lambda_{1(S-1)} & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & -\lambda_{1(S-Q+2)} & \lambda_{1(S-Q+2)} \\ 0 & 0 & 0 & \ldots & 0 & -\lambda_{1(S-Q+1)} \end{pmatrix}$$

$$A_i = \begin{pmatrix} -(\lambda_{1(S-iQ)} + \beta_i^{(1)}) & \lambda_{1(S-iQ)} & \dots & 0 \\ \beta_{i1} & -(\lambda_{1(S-iQ-1)} + \beta_i^{(1)}) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \beta_{i(Q-2)} & \beta_{i(Q-3)} & \dots & \lambda_{1(S-(i+1)Q+2)} \\ \beta_{i(Q-1)} & \beta_{i(Q-2)} & \dots & -(\lambda_{1(S-(i+1)Q+1)} + \beta_i^{(1)}) \end{pmatrix},$$

$$1 \le i \le m-1$$

$$B_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_{1(S-(i+1)Q+1)} & 0 & \dots & 0 & 0 \end{pmatrix}, \ 0 \le i \le m-1$$

$$C_i = \begin{pmatrix} \beta_{iQ} & \beta_{i(Q-1)} & \dots & \beta_{i1} \\ 0 & \beta_{iQ} & \dots & \beta_{i2} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \beta_{iQ} \end{pmatrix}, 1 \le i \le m-1,$$

$$A_m = \begin{pmatrix} -(p\lambda_1 + \lambda_2 + \beta_m^{(1)}) & p\lambda_1 & \dots & 0 & 0 \\ \beta_{m1} + \lambda_2 & -(p\lambda_1 + \lambda_2 + \beta_m^{(1)}) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \beta_{m(Q-1)} & \beta_{m(Q-2)} & \dots & \beta_{m1} + \lambda_2 & -(p\lambda_1 + \lambda_2 + \beta_m^{(1)}) \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ p\lambda_1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$C_m = \begin{pmatrix} \beta_{mQ} & \beta_{m(Q-1)} & \dots & \beta_{m1} + \lambda_2 \\ 0 & \beta_{mQ} & \dots & \beta_{m2} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \beta_{mQ} \end{pmatrix}.$$

We note that all the submatrices are square matrices of size Q.

3.1 The Steady-State Analysis

Before we consider the steady state distribution of the inventory level process, we first obtain the necessary condition for the stability of the process. Define $\tilde{A} = B_m + A_m + C_m$ which is given by

$$\begin{pmatrix} -(p\lambda_1+\lambda_2+\beta_m^{(1)})+\beta_{mQ} & p\lambda_1+\beta_{m(Q-1)} & \dots & \beta_{m1}+\lambda_2 \\ \beta_{m1}+\lambda_2 & -(p\lambda_1+\lambda_2+\beta_m^{(1)})+\beta_{mQ} & \dots & \beta_{m2} \\ \beta_{m2} & \beta_{m1}+\lambda_2 & \dots & \beta_{m3} \\ \vdots & \vdots & \dots & \vdots \\ p\lambda_1+\beta_{m(Q-1)} & \beta_{m(Q-2)} & \dots & -(p\lambda_1+\lambda_2+\beta_m^{(1)})+\beta_{mQ} \end{pmatrix}$$

As this matrix is a generator of some continuous time Markov chain, its steady state probability vector $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_Q)$ satisfying $\Phi\tilde{A} = 0$, and $\Phi e = 1$ is given by

$$\Phi = \frac{1}{Q}e.$$

This can be verified by noting that $\tilde{A}$ is a circulant matrix,

Lemma 1 *The stability condition of the inventory process* $\{I(t), t \geq 0\}$ *is given by*

$$p\lambda_1 - \lambda_2 < \sum_{j=1}^{Q} j\beta_{mj}. \tag{1}$$

Proof

This result follows from the well-known result of Neuts [1981] on the positive recurrence of A, namely, $\Phi B_m e < \Phi C_m e$ which reduces to (1).

Consider

$$\lim_{t\to\infty} Pr[I(t) = j | I(0) = i], \quad i, j \in E \tag{2}$$

It can be seen from the structure of A that the homogeneous Markov process of $\{I(t), t \geq 0\}$ is irreducible. This fact with the stability condition (1) implies that (2) exists and is independent of i, which we denote by Π_j. Let $\Pi = (\Pi_0, \Pi_1, \dots)$ with $\Pi_q = (\ \pi_{S-qQ},\ \pi_{S-qQ-1},\ \dots,\ \pi_{S-(q+1)Q+1}\ )$, denote the steady state probability vector of A, i.e., Π satisfies

$$\Pi A = 0, \Pi e = 1$$

Theorem 1 *When the stability condition (1) holds good,the steady-state probability vector* Π^i *is given by*

$$\begin{aligned} \Pi_i &= \Pi_0 D_i, \quad i = 0, 1, \ldots, m & (3) \\ \Pi_i &= \Pi_0 D_m R^{i-m}, \quad i = m+1, m+2, \ldots & (4) \end{aligned}$$

where

$$\begin{aligned} D_i &= -(D_{i-2}B_{i-2} + D_{i-1}A_{i-1})C_i^{-1}, \quad i = 1, 2, \ldots, m & (5) \\ & \textit{with} \quad D_{-1} = 0 \quad \textit{and} \quad D_0 = I, \end{aligned}$$

and the matrix R *satisfies the matrix quadratic equation*

$$R^2 C_m + R A_m + B_m = \boldsymbol{0} \qquad (6)$$

and the vector Π^0 *is obtained by solving*

$$\begin{aligned} \Pi_0 \left[D_{m-1}B_{m-1} + D_m[A_m + RC_m] \right] &= \boldsymbol{0} & (7) \\ \textit{and} \quad \Pi_0 \left[\sum_{i=1}^{m} D_i + D_m R (I-R)^{-1} \right] \boldsymbol{e} &= 1 & (8) \end{aligned}$$

Proof: The equation $\Pi A = 0$ yields

$$\begin{aligned} \Pi_0 A_0 + \Pi_1 C_1 &= 0 & (9) \\ \Pi_i B_i + \Pi_{i+1} A_{i+1} + \Pi_{i+2} C_{i+2} &= 0, \quad i = 0, 1, \ldots, m-2 & (10) \\ \Pi_{m-1} B_{m-1} + \Pi_m A_m + \Pi_{m+1} C_m &= 0, \quad i = m-1 & (11) \\ \Pi_i B_m + \Pi_{i+1} A_m + \Pi_{i+2} C_m &= 0, \quad i = m, m+1, m+2, \ldots & (12) \end{aligned}$$

It can be shown by mathematical induction that

$$\Pi_i = \Pi_0 D_i, \quad i = 1, 2, \ldots, m$$

where D_i are given by (5).

Next we look for solution of the form

$$\begin{aligned}\Pi_i &= \Pi_m R^{i-m}, \\ &= \Pi_0 D_m R^{i-m}, \quad i = m+1, m+2, \ldots,\end{aligned}$$

where R is a non-negative square matrix of order Q whose spectral radius is less than one, which is ensured by the stability condition (1). Equation (12) yields

$$B_m + RA_m + R^2 C_m = 0.$$

Thus R is a solution of the above matrix quadratic equation.

Post multiplying the above equation by e, we obtain, after rearranging the terms

$$(I - R)(\lambda - R\Gamma) = 0 \tag{13}$$

where

$$\begin{aligned}\lambda &= (0, 0, \ldots, p\lambda_1)', \\ \Gamma &= (\beta_m^{(1)} + \lambda_2, \beta_m^{(2)}, \ldots, \beta_m^{(Q)})'.\end{aligned}$$

Since $\mathrm{sp}(R) < 1, (I - R)$ is non-singular. Hence we have from (13),

$$R\Gamma = \lambda. \tag{14}$$

From (11), we get

$$\Pi_0 \left[D_{m-1} B_{m-1} + D_m [A_m + RC_m]\right] = \mathbf{0}$$

Moreover

$$\begin{aligned}\Pi e &= 1 \\ \Rightarrow \Pi_0 \left[\sum_{i=0}^{m} D_i + D_m R (I - R)^{-1}\right] e &= 1\end{aligned}$$

Thus Π_0 is obtained by solving

$$\Pi_0 \left[D_{m-1}B_{m-1} + D_m[A_m + RC_m]\right] \quad = \quad \mathbf{0}$$

and the normalizing condition

$$\Pi_0 \left[\sum_{i=0}^{m} D_i + D_m R(I-R)^{-1}\right] e \quad = \quad 1.$$

This completes the proof of the theorem.

4 Computation of Matrix R

In this section we present an efficient algorithm for computing the rate matrix R which is the main ingredient for discussing qualitative behavior of the model under study. Due to the special structure of coefficient matrices appearing in (6), the matrix R of dimension Q can be efficiently computed as follows. First note that $B_m\mathbf{e}$ is of the form

$$B_m\mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ p\lambda_1 \end{pmatrix}$$

Due to special structure of B_m matrix the matrix R has only one (row) non zero entries as shown below

$$R = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ r_1 & r_2 & \dots & r_Q \end{pmatrix}.$$

In terms of these entries r_j, equation (6) reduces to

$$r_Q R C_m + R A_m + B_m \quad = \quad \mathbf{0}$$

which becomes, for $i = 1$

$$p\lambda_1 - r_1(p\lambda_1 + \lambda_2 + \beta_m^1) + r_2\lambda_2 + \sum_{i=1}^{Q} r_{i+1}\beta_{mi} + r_Q r_1 \beta_{mQ} \quad = \quad \mathbf{0}$$

for $i = 2, 3, \dots, Q-1$

$$r_{i-1}p\lambda_1 - r_i(p\lambda_1 + \lambda_2 + \beta_m^1) + r_{i+1}\lambda_2 + \sum_{j=i+1}^{Q} r_j\beta_{m(j-i)} + r_Q \sum_{j=Q-i+1}^{Q} r_{j-Q+i}\beta_{mj} \quad = \quad \mathbf{0}$$

and for $i = Q$

$$r_{Q-1}p\lambda_1 - r_Q(p\lambda_1 + \lambda_2 + \beta_m^1) + r_Q r_1\lambda_2 + \sum_{j=1}^{Q} r_Q r_j\beta_{mj} \quad = \quad \mathbf{0}$$

Solving the above homogeneous nonlinear equations, we can obtain R.

5 Special Cases

5.1 Case 1

The special case $\lambda_1 = \lambda$, $\lambda_2 = 0$, $\mu = 0$ and $p = 1$, corresponds to the inventory model with Poisson demand, non perishable items and the negative demand does not enter the system during stock out periods. The matrix $A^* = B_m + A_m + C_m$ becomes

$$\begin{pmatrix} -\lambda - \beta_m^1 + \beta_{mQ} & \lambda + \beta_{m(Q-1)} & \cdots & \beta_m 1 \\ \beta_{m1} & -\lambda - \beta_m^1 + \beta_{mQ} & \cdots & \beta_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{m(Q-1)} + \lambda & \beta_{m(Q-2)} & \cdots & -\lambda - \beta_m^1 + \beta_{mQ} \end{pmatrix}$$

which is a circulant matrix. Hence the steady state probability vector Φ of $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_Q)$ satisfying $\Phi A^* = 0$, and $\Phi e = 1$ is given by

$$\Phi = \frac{1}{Q}e.$$

Let $\tilde{\Pi} = (\tilde{\Pi}_0, \tilde{\Pi}_1, \dots)$ with $\tilde{\Pi}_q = (\tilde{\pi}_{S-qQ}, \tilde{\pi}_{S-qQ-1}, \dots, \tilde{\pi}_{S-(q+1)Q+1})$, denote the steady state probability vector of A, i.e., $\tilde{\Pi}$ satisfies

$$\tilde{\Pi}A = \mathbf{0}, \tilde{\Pi}\mathbf{e} = 1.$$

The limiting distribution of the inventory level process is given by

$$\begin{aligned} \tilde{\Pi}_i &= \tilde{\Pi}_0\Omega_i, \quad i = 0, 1, \dots, m \\ \tilde{\Pi}_i &= \tilde{\Pi}_0\Omega_m R^{i-m}, \quad i = m+1, m+2, \dots \end{aligned}$$

where

$$
\begin{aligned}
D_i &= -(D_{i-2}B_0 + D_{i-1}A_{i-1})C_i^{-1}, \quad i = 1, 2, \ldots, m \\
&\quad \text{with} \quad D_{-1} = 0 \ \text{ and } \ D_0 = I,
\end{aligned}
$$

and the matrix R satisfies the matrix quadratic equation

$$
R^2 C_m + R A_m + B_0 = \mathbf{0},
$$

and the vector $\tilde{\Pi}_0$ is obtained by solving

$$
\begin{aligned}
\tilde{\Pi}_0 \left[D_{m-1}B_{m-1} + D_m[A_m + RC_m]\right] &= \mathbf{0} \\
\text{and} \quad \tilde{\Pi}_0 \left[\sum_{i=1}^{m} D_i + D_m R(I-R)^{-1}\right] \mathbf{e} &= 1.
\end{aligned}
$$

These results agree with the results of Kalpakam and Arivarignan [7].

5.2 Case 2

The case $\lambda_1 = \lambda, \lambda_2 = 0, p = 1$ and $\beta_{iQ} = \beta$ and $\beta_{ij} = 0$ for $j = 1, 2, \ldots, Q-1, i = 1, 2, \ldots, m$, corresponds to the inventory model with perishable item, Poisson demand, full supply of demands and negative customer does not arrive during the stock out periods. Let $\Xi = (\Xi_0, \Xi_1, \ldots)$ with $\Xi_q = (\xi_{S-qQ}, \xi_{S-qQ-1}, \ldots, \xi_{S-(q+1)Q+1})$, denote the steady state probability vector of A. We have

$$
\begin{aligned}
\Xi_i &= \Xi_0 D_i, \quad i = 0, 1, \ldots, m \\
\Xi_i &= \Xi_0 D_m R^{i-m}, \quad i = m+1, m+2, \ldots
\end{aligned}
$$

where

$$
\begin{aligned}
D_i &= -(D_{i-2}B_{i-2} + D_{i-1}A_{i-1})C^{-1}, \quad i = 1, 2, \ldots, m \\
&\quad \text{with} \quad D_{-1} = 0 \ \text{ and } \ D_0 = I,
\end{aligned}
$$

and the matrix R satisfies the matrix quadratic equation

$$R^2 C_m + RA_m + B_0 \quad = \quad \mathbf{0},$$

and the vector Ξ^0 is obtained by solving

$$\begin{aligned} \Xi_0\left[D_{m-1}B_{m-1} + D_m[A_m + RC_m]\right] &= \mathbf{0} \\ \text{and} \;\; \Xi_0\left[\sum_{i=1}^{m} D_i + D_m R(I-R)^{-1}\right]\mathbf{e} &= 1. \end{aligned}$$

These agree with the results of Liu and Yang [10].

6 System Performance Measures

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

6.1 Mean Inventory Level

Let η_I denote the mean inventory level in the steady state of the system. Then it is given by

$$\eta_I \quad = \quad \sum_{i=1}^{S}(S-i+1)\pi_i.$$

6.2 Mean Reorder Rate

The mean reorder rate η_R in the steady state of the system is given by

$$\begin{aligned} \eta_R \quad = \quad & \lambda_1 \sum_{i=1}^{m} \pi_{S-iQ+1} + \sum_{i=1}^{m-1}\sum_{j=1}^{Q-1} g_i^j \pi_{S-iQ-j} \\ & + \gamma_m^{(1)} \sum_{i=1}^{\infty} \pi_{S-(m+1)Q-i} + \sum_{i=1}^{m}(S-iQ+1)\mu\pi_{S-iQ+1}. \end{aligned}$$

6.3 Expected Backlogging Rate

Let η_B denote the expected backlogging rate in the steady state of the system. Then it is given by

$$\eta_B = \sum_{i=-\infty}^{-1} p\lambda\pi_i.$$

6.4 Expected Deterioration Rate

Let η_D denote the expected deterioration rate in the steady state of the system and it is given by

$$\eta_D = \sum_{i=1}^{S}(S-i+1)\mu\pi_i.$$

6.5 Loss due to Negative Demand

The expected loss rate due to negative demand in the steady state of the system is given by

$$\eta_L = \sum_{i=-\infty}^{-1} \lambda_2\pi_i.$$

6.6 Expected Cost rate

Let

c_i = the inventory holding cost per unit per unit time.
c_r = the reorder cost per unit per unit time.
c_b = the backlogging cost per unit time.
c_d = the deterioration cost per unit per unit time.
c_l = the cost of loss due to negative demand per unit per unit time.

The total expected cost rate per unit per unit time is given by

$$TC(S,Q,m) = c_i\eta_I + c_r\eta_R + c_b\eta_B + c_d\eta_D.$$

Substituting η's, we get

$$
\begin{aligned}
TC(S,Q,m) \quad = \quad & c_i \sum_{i=1}^{S}(S-i+1)\pi_i + c_r\lambda_1 \sum_{i=1}^{m} \pi_{S-iQ+1} + \sum_{i=1}^{m-1}\sum_{j=1}^{Q-1} g_i^j \pi_{S-iQ-j} \\
& + \gamma_m^{(1)} \sum_{i=1}^{\infty} \pi_{S-(m+1)Q-i} + \sum_{i=1}^{m}(S-iQ+1)\mu\pi_{S-iQ+1} \\
& + c_b \sum_{i=-\infty}^{-1} p\lambda\pi_i + c_d \sum_{i=1}^{S}(S-i+1)\mu\pi_i + c_l \sum_{i=-\infty}^{-1} \lambda_2\pi_i.
\end{aligned}
$$

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically.

Conclusion

In this work we considered an inventory model under continuous review, by incorporating the perishable nature of stocked items, by allowing more than one pending order at a given time, by assuming the number of non defective items in the supplied lot is a random variable and by considering the negative demand. We are able to derive the steady state inventory level distribution in matrix geometric form. Finally we have computed measures of system performance and constituted the total expected cost rate under a broad cost structure.

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