AMO - Advanced Modeling and Optimization, Volume 8, Number 1, 2006

A Possibilistic Mean VaR Model for Portfolio Selection*

Guohua Chen[1,3], Shou Chen[1], Yong Fang[2], Shouyang Wang[1,2,**]

[1]School of Business Administration,
Hunan University, Changsha 410082, China
[2]Institute of Systems Science,
Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, China
[3] Department of Mathematics, Hunan Institute of Humanities
Science and Technology, Loudi, Hunan 417000, China

Abstract: This paper deals with a portfolio selection problem with fuzzy return rates. A possibilistic mean VaR model was proposed for portfolio selection. Specially, we present a mathematical programming model with possibilistic constraint. The possibilistic programming problem can be solved by transforming it into a linear programming problem. A numerical example is given to illustrate the behavior of the proposed model.

Keywords: Portfolio selection; VaR; Possibilistic mean; Possibility theory

1. Introduction

In 1952, Markowitz [10] published his pioneering work which laid the foundation of modern portfolio analysis. Markowitz's mean variance model has served as a basis for the development of modern financial theory over the past five decades. Assuming the normality of the returns and quadratic investor's preferences allow the simplification of the problem in a relatively easy to solve quadratic programming problem.

Notwithstanding its popularity, mean variance approach has also been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns. Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean absolute deviation, semi-variance and so on are used.

Since 1960s, fuzzy set theory has been widely used to solve many problems including financial risk management. By using fuzzy approaches, the experts' knowledge and the investors' subjective opinions can be better integrated into a portfolio selection model. Bellman and Zadeh [2] proposed the fuzzy decision theory. Ramaswamy [12] presented a bond portfolio selection model based on the fuzzy decision theory. The approach is such that a given target rate of return is

* Supported by the National Natural Science Foundation of China under Grant No. 70221001.
**The corresponding author, Email: sywang@amss.ac.cn.

achieved for an assumed market scenario. A similar approach for portfolio selection by using the fuzzy decision theory was proposed by Leon et al [8]. By using the fuzzy decision principle, Ostermark[11] proposed a dynamic portfolio management model. Watada [14] presented another type of portfolio selection model based on the fuzzy decision principle. The model is directly related to the mean-variance model, where the goal rate for an expected return and the corresponding risk described by logistic membership functions. Tanaka et al [13] give a special formulation of fuzzy decision problems by the probability events. Carlsson et al [4] studied the portfolio selection model in which the rate of return of security follows the possibility distribution.

This paper is organized as follows. In Section 2, we introduce briefly the mean downside-risk framework and present a mean VaR portfolio selection model with transaction costs. In Section 3, we introduce briefly the possibility theory and propose a possibilistic mean VaR portfolio selection model. In Section 4, an example is given to illustrate the proposed model. A few concluding remarks are finally given in Section 5.

2. Mean VaR portfolio selection model with transaction costs

2.1. Mean downside-risk framework

In practice investors are concerned about the risk that their portfolio value falls below a certain level. That is the reason why different measures of downside-risk are considered in the asset allocation problem. If we denote by v the future portfolio value, i.e., the value of the portfolio by the end of the planning period, then the probability

$$P(v < VaR)$$

that the portfolio value falls below the VaR level, is called the shortfall probability. The conditional mean value of the portfolio given that the portfolio value has fallen below VaR, called the expected shortfall, is defined as

$$E(v \mid v < VaR).$$

Other risk measures used in practice are the mean absolute deviation

$$E(| v - E(v) | \mid v < E(v)),$$

and the semi-variance

$$E((v - E(v))^2 \mid v < E(v)),$$

where we consider only the negative deviations from the mean.

Assume that an investor wants to allocate his/her wealth among n risky securities. If the risk profile of the investor is determined in terms of VaR, a mean-VaR efficient portfolio will be a solution of the following optimization problem [6]:

$$
\begin{aligned}
(P1)\quad \max_{x} \quad & E(v) \\
s.t. \quad & P(v < VaR) \le \beta, \\
& \sum_{j=1}^{n} x_j = 1, \\
& l_j \le x_j \le u_j, \qquad j = 1, \cdots, n,
\end{aligned}
$$

where $x_j\,(j=1,\cdots,n)$ represents the proportion of the total amount of money devoted to security j, l_j and u_j $(j=1,\cdots,n)$ represent the minimum and maximum proportion of the total amount of money devoted to security j, respectively. Let $r_j\,(j=1,\cdots,n)$ be the random variable representing the rate of return of security j, then $v=\sum_{j=1}^{n} r_j x_j$. In this model, the investor is trying to maximize the future value of his/her portfolio, which requires the probability that the future value of his portfolio falls below VaR not to be greater than β.

2.2. Model formulation

Transaction cost is one of the main sources of concern to portfolio managers. Arnott and Wagner [1] found that ignoring transaction costs would result in an inefficient portfolio. Yoshimoto's empirical analysis [15] also drew the same conclusion. In this paper, we consider the proportional transaction costs. Assume the rate of transaction cost of security j $(j=1,\cdots,n)$ is c_j, thus the transaction cost of security j is $c_j x_j$. The transaction cost of the portfolio $x=(x_1,x_2,\cdots,x_n)$ is $\sum_{j=1}^{n} c_j x_j$. Considering the proportional transaction costs and the shortfall probability constraint, we propose the following mean VaR portfolio selection model with transaction costs.

$$
\begin{aligned}
(P2)\quad \max_{x}\quad & E(v)-\sum_{j=1}^{n} c_j x_j \\
s.t.\quad & P(v<VaR)\le \beta, \\
& \sum_{j=1}^{n} x_j = 1, \\
& l_j \le x_j \le u_j, \qquad j=1,\cdots,n.
\end{aligned}
$$

3. Possibilistic Mean VaR Portfolio Selection Model

3.1. Possibility theory

Possibility theory was proposed by Zadeh [16] and advanced by Dubois and Prade [5] where fuzzy variables are associated with possibility distributions in a similar way that random variables are associated with probability distributions in the probability theory. The possibility distribution function of a fuzzy variable is usually defined by the membership function of the corresponding fuzzy set. We call a fuzzy number any fuzzy subset $\tilde{a}$ of R with membership function $\mu_{\tilde{a}}: R\to[0,1]$. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership function $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$, respectively. Based on the concepts and techniques of possibility theory founded by Zadeh [7], the possibility of $\tilde{a} \le \tilde{b}$ is defined as follows [5]:

$$Pos\{\tilde{a} \le \tilde{b}\} = \sup\{\min\{(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in R, x \le y\},$$

where the abbreviation Pos represents possibility. This means that the possibility of $\tilde{a} \le \tilde{b}$ is the possibility that there exists at least one pair of values x, y belong to R such that $x \le y$, and the values of $\tilde{a}$ and $\tilde{b}$ are x and y, respectively. Analogously, the possibility of $\tilde{a} < \tilde{b}$ is defined by:

$$Pos\{\tilde{a} < \tilde{b}\} = \sup\{\min\{(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in R, x < y\} \ .$$

Furthermore, the possibility of $\tilde{a} = \tilde{b}$ is defined by :

$$Pos\{\tilde{a} = \tilde{b}\} = \sup\{\min\{(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)) \mid x \in R\}.$$

Specially, when $\tilde{b}$ is crisp, i.e., b, we have:

$$Pos\{\tilde{a} \le b\} = \sup\{\min\{\mu_{\tilde{a}}(x) \mid x \in R, x \le b\ \},$$

$$Pos\{\tilde{a} < b\} = \sup\{\min\{\mu_{\tilde{a}}(x) \mid x \in R, x < b\},$$

$$Pos\{\tilde{a} = b\} = \mu_{\tilde{a}}(b).$$

3.2. Model formulation

In standard portfolio models uncertainty is equated with randomness, which actually combines both objectively observable and testable random events with subjective judgments of the decision maker into probability assessments. A purist on theory would accept the use of probability theory to deal with observable random events, but would frown upon the transformation of subjective judgments to probabilities. So in this paper we will assume that the rates of return on securities are modeled by possibility distributions rather than probability distributions. Applying possibilistic distribution may have two advantages [10]: (1) the knowledge of the expert can be easily introduced to the estimation of the return rates and (2) the reduced problem is more tractable than that of the stochastic programming approach. Denote the rate of return on security by the trapezoidal fuzzy number $\tilde{r} = (r_1, r_2, r_3, r_4)$ where $r_1 < r_2 \le r_3 < r_4$. The membership function of the fuzzy number $\tilde{r}$ can be denoted by:

$$\mu(x)=\begin{cases}\dfrac{x-r_1}{r_2-r_1}, & r_1\le x\le r_2;\\ 1, & r_2\le x\le r_3;\\ \dfrac{x-r_4}{r_3-r_4}, & r_3\le x\le r_4;\\ 0, & \text{otherwise.}\end{cases}$$

we mention that the trapezoidal fuzzy number is a triangular fuzzy number if $r_2=r_3$. Now let us consider two trapezoidal fuzzy numbers $\tilde{r}=(r_1,r_2,r_3,r_4)$ and $\tilde{b}=(b_1,b_2,b_3,b_4)$, then we have the results: [9]

1) $$Pos\{\tilde{r}\le\tilde{b}\}=\begin{cases}1, & r_2\le b_3;\\ \dfrac{b_4-r_1}{b_4-b_3+r_2-r_1}, & r_2\ge b_3, r_1\le b_4;\\ 0, & r_1\le b_4.\end{cases}$$

Specially, when $\tilde{b}$ is crisp 0, then we have

$$Pos\{\tilde{r}\le 0\}=\begin{cases}1, & r_2\le 0;\\ \dfrac{r_1}{r_1-r_2}, & r_1\le 0\le r_2;\\ 0, & r_1\ge 0.\end{cases}$$

2) The sum of two trapezoidal fuzzy numbers is also a trapezoidal fuzzy number; the product of a trapezoidal fuzzy number and a scalar number is also a trapezoidal fuzzy number. Considering two trapezoidal fuzzy numbers $\tilde{r}=(r_1,r_2,r_3,r_4)$ and $\tilde{b}=(b_1,b_2,b_3,b_4)$, we have

$$\tilde{r}+\tilde{b}=(r_1+b_1,r_2+b_2,r_3+b_3,r_4+b_4);$$

$$k\cdot\tilde{r}=\begin{cases}(kr_1,kr_2,kr_3,kr_4), & k\ge 0;\\ (kr_4,kr_3,kr_2,kr_1), & k<0.\end{cases}$$

$$\tilde{r}-\tilde{b}=(r_1-b_1,r_2-b_2,r_3-b_3,r_4-b_4).$$

We have the following lemma:

Lemma 1. Assume that the trapezoidal fuzzy number $\tilde{r}=(r_1,r_2,r_3,r_4)$, then for any given confidence level $\beta(0\le\beta\le 1)$, $\mathrm{Pos}\{\tilde{r}\le 0\}\le\beta$ if and only if $(1-\beta)r_1+\beta r_2\ge 0$.

Proof: If $\mathrm{Pos}\{\tilde{r}\le 0\}\le\beta$ then we have either $r_1\ge 0$ or $\dfrac{r_1}{r_1-r_2}\le\beta$. If $r_1\ge 0$ then

$r_2 > r_1 \geq 0$, so we have $(1-\beta)\mathrm{r}_1 + \beta r_2 \geq 0$; if $\frac{r_1}{r_1 - r_2} \leq \beta$ then $\mathrm{r}_1 \geq \beta(r_1 - r_2)$ by the fact that $\mathrm{r}_1 < r_2$. Hence we have $(1-\beta)\mathrm{r}_1 + \beta r_2 \geq 0$ for all cases.

If $(1-\beta)\mathrm{r}_1 + r_2 \geq 0$, the argument breaks down into two cases when $\mathrm{r}_1 \geq 0$, we have $\mathrm{Pos}\{\tilde{\mathrm{r}} \leq 0\} = 0$ which implies that $\mathrm{Pos}\{\tilde{\mathrm{r}} \leq 0\} \leq \beta$, when $\mathrm{r}_1 < 0$, we have $\mathrm{r}_1 - r_2 < 0$. We can rearrange $(1-\beta)\mathrm{r}_1 + \beta r_2 \geq 0$ as $\frac{r_1}{r_1 - r_2} \leq \beta$, i.e., $\mathrm{Pos}\{\tilde{\mathrm{r}} \leq 0\} \leq \beta$.

The α-level set of a fuzzy number $\tilde{r} = (r_1, r_2, r_3, r_4)$ is a crisp subset of R and is denoted by $[\tilde{r}]^\alpha = \{x \mid \mu(x) \geq \alpha, x \in R\}$, then

$$[\tilde{r}]^\alpha = \{x \mid \mu(x) \geq \alpha, x \in R\} = [r_1 + \alpha(r_2 - r_1), \mathrm{r}_4 - \alpha(r_4 - r_3)].$$

Carlsson et al [3] introduced the notation of crisp possibilitic mean value of continuous possibility distributions, which are consistent with the extension principle. Let $[\tilde{r}]^\alpha = [a_1(\alpha), a_2(\alpha)]$, then the crisp possibilistic mean value of $\tilde{r} = (r_1, r_2, r_3, r_4)$ as

$$\tilde{E}(\tilde{r}) = \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha .$$

It is easy to see that if $\tilde{r} = (r_1, r_2, r_3, r_4)$ is a trapezoidal fuzzy number then

$$\tilde{E}(\tilde{r}) = \int_0^1 \alpha(r_1 + \alpha(r_2 - r_1) + r_4 - \alpha(r_4 - r_3))d\alpha = \frac{r_2 + r_3}{3} + \frac{r_1 + r_4}{6} . \tag{1}$$

Denote the rate of return on security j $(j = 1, \cdots, n)$ by the trapezoidal fuzzy number $\tilde{r}_j = (r_{j1}, r_{j2}, r_{j3}, r_{j4})$ where $r_{j1} < r_{j2} \leq r_{j3} < r_{j4}$. In addition, we denote the VaR level by the trapezoidal fuzzy number $\tilde{b} = (b_1, b_2, b_3, b_4)$. Thus we use the shortfall possibility constraint instead of the shortfall probability constraint and formulate the possibilistic mean VaR portfolio selection model as follows.

$$(P3)\quad \max_{x}\quad \tilde{E}(\sum_{j=1}^{n}\tilde{r}_j x_j)-\sum_{j=1}^{n}c_j x_j$$
$$s.t.\quad Pos(\sum_{j=1}^{n}\tilde{r}_j x_j<\tilde{b})\le\beta,$$
$$\sum_{j=1}^{n}x_j=1,$$
$$l_j\le x_j\le u_j,\qquad j=1,\cdots,n,$$

where $\tilde{E}$ denotes fuzzy mean operator, Pos denotes possibility.

From (1), we have

$$\tilde{\mathrm{E}}(\sum_{j=1}^{n}\tilde{r}_j x_j)=\frac{\sum_{j=1}^{n}r_{j2}x_j+\sum_{j=1}^{n}r_{j3}x_j}{3}+\frac{\sum_{j=1}^{n}r_{j1}x_j+\sum_{j=1}^{n}r_j x_j}{6}.$$

From Lemma 1, we can get that

$\mathrm{Pos}(\sum_{j=1}^{n}\tilde{r}_j x_j<\tilde{b})\le\beta$ is equivalent to $(1-\beta)(\sum_{j=1}^{n}r_{j1}x_j-b_4)+\beta(\sum_{j=1}^{n}r_{j2}x_j-b_3)\ge 0.$

Thus, problem (P3) can be transformed into the following problem:

$$(P4)\quad \max_{x}\quad \frac{\sum_{j=1}^{n}r_{j2}x_j+\sum_{j=1}^{n}r_{j3}x_j}{3}+\frac{\sum_{j=1}^{n}r_{j1}x_j+\sum_{j=1}^{n}r_{j4}x_j}{6}-\sum_{j=1}^{n}c_j x_j$$
$$s.t.\quad (1-\beta)(\sum_{j=1}^{n}r_{j1}x_j-b_4)+\beta(\sum_{j=1}^{n}r_{j2}x_j-b_3)\ge 0,$$
$$\sum_{j=1}^{n}x_j=1,$$
$$l_j\le x_j\le u_j,\qquad j=1,\cdots,n.$$

Problem (P4) is a standard linear programming problem. One can use several algorithms of linear programming to solve it efficiently, for example, the simplex method.

4. Numerical example

In this section, we will give a numerical example to illustrate the proposed possibilistic mean VaR portfolio selection model. Consider a 5-securities problem with the following possibility distributions:

r_1 =(0.04, 0.05, 0.06, 0.07), r_2 =(0.04, 0.06, 0.065, 0.07), r_3 =(0.048, 0.068, 0.075, 0.08),

r_4 =(0.05, 0.065, 0.07, 0.1), r_5 =(0.05, 0.075, 0.085, 0.116).

In the example, the VaR level is given by

$\tilde{\mathrm{b}}$ =(0.04, 0.046, 0.048, 0.05).

The rates of transaction costs of securities are given by

c_1=0, c_2=0.001, c_3=0.001, c_4=0.002, c_5=0.003.

Let β =0.01, from (4) we obtain

$$\begin{aligned} &\max_x \ 0.055x_1 + 0.059x_2 + 0.068x_3 + 0.067x_4 + 0.078x_5 \\ &\text{s.t} \quad 0.0401x_1 + 0.0402x_2 + 0.0482x_3 + 0.5015x_4 + 0.5025x_5 \ge 0.04998 \\ &\sum_{j=1}^{5} x_j = 1 \\ &0 \le x_j \le 0.5, \quad j = 1, \cdots, 5 \end{aligned}$$

Its solution is (0, 0, 0.112821, 0.387179, 0.50), the optimal value is 0.0731128. Thus the optimal portfolio is (0, 0, 0.112821, 0.387179, 0.50), the optimal possibility return is 0.0731128.

Let β =0.03, from (4) we obtain

$$\begin{aligned} &\max_x \ 0.055x_1 + 0.059x_2 + 0.068x_3 + 0.067x_4 + 0.078x_5 \\ &\text{s.t} \quad 0.0403x_1 + 0.0406x_2 + 0.0486x_3 + 0.5045x_4 + 0.5075x_5 \ge 0.04994 \\ &\sum_{j=1}^{5} x_j = 1 \\ &0 \le x_j \le 0.5, \quad j = 1, \cdots, 5 \end{aligned}$$

Its solution is (0, 0, 0.356756, 0.143244, 0.50), the optimal value is 0.0728568. Thus the optimal portfolio is (0, 0, 0.356756, 0.143244, 0.50), the optimal possibility return is 0.0728568.

Let β =0.05.

From (4) we obtain:

$$\begin{aligned} &\max_x \ 0.055x_1 + 0.059x_2 + 0.068x_3 + 0.067x_4 + 0.078x_5 \\ &\text{s.t} \quad 0.0405x_1 + 0.041x_2 + 0.04995x_3 + 0.5075x_4 + 0.5125x_5 \ge 0.0499 \\ &\sum_{j=1}^{5} x_j = 1 \\ &0 \le x_j \le 0.5, \quad j = 1, \cdots, 5 \end{aligned}$$

Its solution is (0, 0, 0.50, 0, 0.50), optimal value is 0.073. Thus the optimal portfolio is (0, 0, 0.50, 0, 0.50), the optimal possibility return is 0.073.

The example shows that optimal possibility return is increasing with increasing of β . Through choosing the values of the parameter β according to the investor's frame of mind, the investor may get a favorite investment strategy.

5. Conclusion

In this paper, we consider trapezoidal possibility distribution as the possibility distribution of the rates of returns on the securities and propose a possibilistic mean VaR portfolio selection model. A possibilistic programming approach based on fuzzy VaR has been proposed. The possibilistic programming problem can be solved by transforming it into a linear programming problem based on the possibilistic theory. A numerical example is given to illustrate the proposed method can be used efficiently to solve portfolio selection problem.

References

[1] Arnott, R.D. and Wanger, W.H., The measurement and control of trading costs, *Financial Analysts Journal*, 46(6) (1990), 73-80.

[2] Bellman, R. and Zadeh, L.A., Decision making in a fuzzy environment, *Management Science*, 17(1970), 141-164.

[3] Carlsson, C. and Fuller, R., On possibilistic mean value and variance of fuzzy numbers, *Fuzzy sets and systems*, 122 (2001), 315-326.

[4] Carlsson, C., Fuller, R. and Majlender, P., A possibilistic approach to selecting portfolios with highest utilty score, *Fuzzy sets and systems*, 131(2002), 13-21.

[5] Dubois, D. and Prade, H., *Possibility theory*, Plenum press, New York 1998.

[6] Inuiguchi, M. and Ramik, J., Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, *Fuzzy Sets and Systems*, 111(2000), 3-28.

[7] Leibowitz, M.L. and Kogelman, S., Asset Allocation under Shortfall Constraints. *Journal of Portfolio Management*, Winter (1991), 18-23.

[8] Leon, T., Liern, V. and Vercher, E., Viability of infeasible portfolio selection problem: a fuzzy approach, *European Journal of Operational Research*, 139(2002), 178-189.

[9] Liu, B. and Iwamura, K., Chance constrained programming with fuzzy parameters, *Fuzzy sets and systems*, 94(1998), 227-237.

[10] Markowitz, H., Portfolio Selection, *Journal of Finance*, 7 (1952), 77-91.

[11] Ostermask, R., A fuzzy control model (FCM) for dynamic portfolio management, *Fuzzy sets and Systems*, 78(1998), 243-254.

[12] Ramaswamy, S., Portfolio selection using fuzzy decision theory, working paper of bank for international settlements, No. 59, 1998.

[13] Tanaka, H., Guo, P. and Türksen, I.B., Portfolio selection based on fuzzy probabilities and possibility distributions, Fuzzy Sets and Systems, Vol. 111, pp. 387-397, 2000.

[14] Watada, J., Fuzzy portfolio model for decision making in investment, In Y. Yoshida editror, *Dynamical Aspects in fuzzy decision making*, Physica Verlag, Heidelberg, 2001, 141-162.

[15] Yoshimoto, A., The mean-variance approach to portfolio optimization subject to transaction costs, *Journal of the Operational Research Society of Japan*, 39 (1996), 99-117.

[16] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy sets and systems*, 1(1978), 3-28.