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The three-dimensional Fermat-Weber problem with Tchebychev distances

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Abstract: The three-dimensional Fermat-Weber facility location problem with Tchebychev distances is investigated. Expanding on previous research for the analogous two-dimensional problem, this study presents a new algorithm for solving the elusive three-dimensional case. The algorithm presented herein finds near optimal solutions in practical computational times. Some experimental results are also conveyed.

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1. INTRODUCTION

Single facility location problems in continuous space were among the first location problems to be investigated [Wesolowsky, 1993]. Location problems occur often in real life whether it is locating a machine in a machine shop or locating a new distribution center on a supply chain network with respect to existing destinations. There exist several well-known and often exact techniques to solve various instantiations of the single facility minisum location problem. To wit, however, very few have studied location problems with Tchebychev distances. Herein, after a brief literature review of related works, we present an algorithm that finds near optimal solutions for the three-dimensional Fermat-Weber location problem with Tchebychev distances.

One of the basic parameters in continuous location modeling is the distance metric. The most common distance metrics in continuous space are those known as the class of l_p distance metrics as shown in equation 1 for n dimensional space:

$$l_p = \left(\sum_{i=1}^{n} \left(\left| X_i - X_a \right|^p \right) \right)^{1/p} \qquad \ldots(1)$$

where the X_i's are the coordinates of the existing facilities and X_a is the n dimensional vector location of the new facility. Note for $p = 1$, l_1 represents the rectilinear, or Manhattan, distance metric, for $p = 2$, l_2 is the Euclidean, or straight-line, distance metric, and for $p = \infty$, l_∞ is known as the Tchebychev distance metric. The Tchebychev distance metric in three dimensions can be written as:

$$l_\infty = Max\left(\left| x_i - x_a \right|, \left| y_i - y_a \right|, \left| z_i - z_a \right| \right) \qquad \ldots(2)$$

As noted above, the Tchebychev distance metric is not widely used in location problems. However, there exist some specific applications in automated warehouses for which it proves to be very useful. This rest of this treatise is organized as follows. First, a literature review specifically on the use of Tchebychev distances in location modeling and closely related is presented. Then, a formal problem definition and our algorithm are both delineated. A modest conclusion section ends this research note.

2. LITERATURE REVIEW AND BACKGROUND

Tchebychev distances have wide ranging and varied applications in the fields of material handling systems operations, CNC tool path planning, and manufacturing in general. Given below are some citations of how the Tchebychev metric has been studied in different situations.

Francis, McGinnis and White [Francis, McGinnis and White, 1992] depict a technique in which a two-dimensional Tchebychev space can be converted to rectilinear space. Therein, the authors solve the two-dimensional minimax location problem with Tchebychev distances. They detail the relationship between rectilinear and Tchebychev distances, with the help of the diamond-covering problem with its center at the origin. This diamond is rotated about the z-axis by 45 degrees to form a square with its center at the origin. Thus the corresponding points in are obtained using the following conversion equation.

$$Q(x, y) = (x, y)\left[\frac{1\ -1}{1\ \ 1}\right] = (x + y, -x + y) \qquad \ldots(3)$$

The inverse transformation can be found by:

$$Q(u^{'}, v^{'}) = (u, v)\left[\frac{1/2\ \ 1/2}{-1/2\ 1/2}\right] = \frac{1}{2}(u - v, u + v) \qquad \ldots(4)$$

By using these transformations, it was found that the rectilinear distance between any two vertices of the diamond was the same as the Tchebychev distance between any two vertices of the square. These same transformations *extended to three dimensions* play an integral role in our algorithm below.

Hwang and Lim [1993] described the problem of selecting a dwell point location for a server when it is idle in an automated storage and retrieval system. This problem is converted to one of locating a single facility with Tchebychev distances and the Tchebychev minimax location problem. Two known models are altered and converted to algorithms.

Gass and Witzgall [2004] used linear programming techniques to approximate the Tchebychev Minimax Criterion that is utilized in finding a circle that is the closest to a given set of points. This is useful in problems in location theory and also in the quality control of shapes such as drilled holes, spheres, etc. Laporte, Lopes, and Soumis [1998] developed tool and strip sequencing policies for manufacturing applications using both the Manhattan and the Tchebychev metrics.

Bozer, Schorn and Sharp [1990] analyze the performance of different geometric techniques that are used to solve the Chebychev Traveling Salesman Problem (CTSP) and also introduce a new heuristic called the band insertion heuristic for the same. The band insertion technique is also compared with existing techniques. The CTSP has a large number of applications in material handling systems.

Gaboune, Laporte, and Soumis [1993] derived expected distances between uniformly distributed points between pairs of co-planar rectangles as well as between pairs of rectangular parallelepipeds using three distance metrics: the rectilinear, the Euclidean, and the Tchebychev metric. Their claim of being the first to report l_∞ expected distances in two dimensions is countered by Bozer and White [Bozer and White, 1984]. However, their three-dimensional work on expected distances in l_∞ is ingenious. Peters, Smith, and Hale [1996] determined where the server for an automated storage and retrieval (AS/RS) system should remain when idle such that the time taken for the next operation is minimized. Their two-dimensional location model uses a continuous approximation of discrete locations and incorporates the Tchebychev distance metric.

3. PROBLEM DEFINITION

The problem under investigation is the Fermat-Weber single facility location problem in $\Re^3$ with Tchebychev distances. One technique to solve location problems with Tchebychev distances is to search the entire feasible region formed by the extreme points of the existing facilities. This search technique is obviously a brute force technique, which involves iteratively evaluating all the integer coordinates within the feasible region as possible candidates for the optimal location of the new facility. Furthermore, in three-dimensional Tchebychev space, due to the maximum function inherent to the metric, the feasible region is the minimum rectangular parallelepiped that encloses the maximum *coordinate-wise* distances between pairs of the existing facilities, which enlarges the solution space significantly. We will refer to the dimensions of this rectangular parallelepiped as the *coordinate-wise extents* of the feasible region.

Consider five coordinates in Cartesian space: (12114, 5122, 8981), (3226, 18789, 10455), (1840, 2564, 19314), (9656, 22742, 1257), and (20346, 4910, 1127). The coordinate-wise extents for this points are

$\{1840 \leq x \leq 20346\}$, $\{2564 \leq y \leq 22742\}$, and $\{1127 \leq z \leq 19314\}$ yielding (18506 * 20178 * 18187) = 6,791,281,654,716 candidate integer locations. Obviously, brute force techniques have limits.

Hence, based on the two dimensional work by Francis, McGinnis and White [Francis, McGinnis, and White, 1992], a three dimensional rotational algorithm was developed which finds near optimal (or, in some cases, provably optimal) locations quickly. The locations found by this new algorithm are often sub-optimal, but the savings with respect to solution time makes it a practical solution. Furthermore, the relative difference between the solutions obtained herein and the optimal solutions are small (all less than 5% and most less than 2% for the various example problems below).

4. METHODOLOGY AND EXAMPLES

The algorithm delineated below is an extension of the technique presented by Francis, McGinnis and White [1992] to accommodate the analogous three-dimensional minisum single facility location problem. The three-dimensional rotational algorithm is presented below.

A rotational algorithm for the 3-D Fermat-Weber location problem

1. Each of the coordinate locations representing the existing facilities is rotated 45 degrees about all three Cartesian axes using all possible combinations. There are total of fifteen unique possible combinations/orders to rotate the existing facility locations (see the rotation matrices in the Appendix) into rectilinear space.
2. Each of these fifteen sets of new points are then used to find the optimal location of the new facility using rectilinear distances for which a simple and exact heuristic is known. This exercise results in fifteen candidate locations.
3. These fifteen candidate locations are then rotated back into the original Tchebychev space via the corresponding inverse matrix in the Appendix.
4. Objective function values for the fifteen candidate locations are then calculated. The location(s) that result in the smallest objective function value is(are) chosen.

Both the brute force technique and the three-dimensional rotational algorithm described herein were coded into VisualBasic© for evaluation and comparison. The program reads in values from an accompanying Excel© sheet and then executes the brute force technique as well as the rotational algorithm methodology described herein. It also tracks the time needed to solve each method as part of a comparison of how effective the rotational algorithm is in finding practical solutions.

Given below in tabular form are three synthetic examples, each with varying number of existing facilities and varying ranges of coordinate wise extents.

Table 1. Example Problems (P4, 1.80 GHz, 752 MB RAM)

	Example 1	Example 2	Example 3
No. of existing facilities:	60	120	700
Extent of coordinates:	1 to 101	1 to 300	1 to 300
Brute force time:	82 seconds	4534 seconds	30065 seconds
Rotational algorithm time:	15 seconds	12 seconds	86 seconds
Brute force location:	(52, 53, 52)	(159, 154, 148)	(153, 148, 150)
Rotational algorithm location:	(52, 54, 52)	(156.5, 148.9, 141.8)	(153.1, 145.4, 148.4)
Percent error (x%, y%, z%):	(0, 1.89, 0)	(1.56, 3.30, 4.13)	(0.10, 1.70, 0.10)

A fourth example problem was tried with 1000 existing facility locations and coordinate-wise extents of 500 in all three dimensions. The brute force program ran for two days without finding a solution. The rotational algorithm found a solution within four minutes.

5. CONCLUSIONS

An algorithm to solve the three dimensional Fermat-Weber problem with Tchebychev distances was developed utilizing coordinate system conversions. Example problems were then compared with searching the entire solution space. The algorithm presented herein produced significant savings in computational time with near optimal solutions.

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APPENDIX

Rotation Matrices

Rotate about X

1.0000	0.0000	0.0000
0.0000	0.7071	-0.7071
0.0000	0.7071	0.7071

Inverse rotate about X

1.0000	0.0000	0.0000
0.0000	0.7071	0.7071
0.0000	-0.7071	0.7071

Rotate about Y

0.7071	0.0000	0.7071
0.0000	1.0000	0.0000
-0.7071	0.0000	0.7071

Inverse rotate about Y

0.7071	0.0000	-0.7071
0.0000	1.0000	0.0000
0.7071	0.0000	0.7071

Rotate about Z

0.7071	-0.7071	0.0000
0.7071	0.7071	0.0000
0.0000	0.0000	1.0000

Inverse rotate about Z

0.7071	0.7071	0.0000
-0.7071	0.7071	0.0000
0.0000	0.0000	1.0000