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MULTI-VARIATE TIME SERIES BASED RECONSTRUCTION OF DYNAMICAL SYSTEMS

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Abstract. A multi-variate time series based reconstruction of dynamical systems is proposed. The components are given in the corrupted (by the observational noise) form of multivariate time series. They are obtained previously as a numerical solution of a corresponding system of autonomous ordinary differential equations with polynomial right sides, and solved with respect to the first derivatives.

1. Introduction

The problem of reconstructing multi-dimensional dynamical system from a multivariate time series is considered in the present paper both analytically and numerically. It is a continuation of a previous paper [1] of the authors, where the problem of reconstructing ordinary differential equation from a numerical solution in the form of single time series with observational (Gaussian or white) noise is considered. In order to *refine* the precision, the unknown dynamical system should be searched in canonical form, i.e. a system of first order differential equations solved with respect to their derivatives. In this form the reconstruction procedure of the equation right hand sides could be accomplished without differentiation and the precision would be good even in case that the noise dispersion is higher than 5%. Certainly, for this purpose a single component is not enough and a *complete* set of system components in the form of a corresponding multivariate time series is necessary. The question of *completeness* presents a problem for *refinement* of reconstruction procedure that should be solved by determining the *embedding* of the curve parameterized by the given set of system components.

The problem of *embedding* is particularly solved in the above-mentioned paper [1], by determining the phase space dimension from a single component. The corresponding *two theorems*, formulated there for this purpose are rigorously proven in another paper [2] of the authors. It is evident these theorems and corresponding geometrical algorithm for determining phase space dimension, can be used not only for the case of a given single component, but also for multivariate time series, by choosing to process one series. However, the knowledge of phase space dimension *is not enough* to solve the problem of embedding the curve parameterized by the given multivariate series. In this case we need a *direct* criteria showing the necessary and sufficient conditions of embedding. For this purpose appropriate *modified* formulations of the mentioned theorems are proposed here. By using them together with the wellknown *trajectory type classifications,* we will be able to define simple geometrical criteria for determining the *embedding* of the curve parameterized by the given multivariate time series. Then the corresponding geometrical part of the reconstruction algorithm will be described step by step. As for the computational part of the algorithm, it will be shown that in case of given chaotic components, the corresponding procedure is simpler and more reliable than that applied to analogous case in the previous paper [2].

The well-known dynamical systems of Rossler, Duffing and Lorenz [3] are often used as samples for testing various approaches and methods in reconstructing dynamics. For example, in the cases of given single and bivariate time series presenting solutions of Rossler system and Duffing equation respectively, the reconstructive approach developed in ([1], [2]) is successfully applied [4],[5]. However, the Lorenz system is just of a type that is impossible to be reduced to a single third-order differential equation (having polynomial right side with positive integer powers), in order to be reconstructed from a corresponding monovariate time series, analogously to Rossler one. That is why, in this paper the Lorenz system is considered as a main example of applying a new algorithm for reconstructing three-dimensional system from a given triple time series.

2. Mathematical statement of the problem

We suppose that the multivariate time series is noisy, i.e. let the real numbers in the left hand sides of

$$\begin{aligned} Y_1(t_i) &= x_1(t_i) + \nu_1(t_i), \\ Y_2(t_i) &= x_2(t_i) + \nu_2(t_i), \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \\ Y_m(t_i) &= x_m(t_i) + \nu_m(t_i). \end{aligned} \tag{2.1}$$

be a given time series, defined for $\forall t_i \in (t_1, t_N)$ on the real axis, where $t_i = i.\Delta t$ for every $i = 1,2,...,N$. We consider N is a sufficiently large integer, and Δt is a sufficiently small real number for our purposes. Moreover, let $\nu_1(t_i), \nu_2(t_i), \ldots, \nu_m(t_i)$ be a Gaussian (white) noises, unable by definition to produce statistically systematical errors. At the end, let $x_1(t_i), x_2(t_i), \ldots, x_m(t_i)$ be a discrete approximation of some *m*-dimensional curve $\vec{c}_m : [x_1(t), x_2(t), \ldots, x_m(t)]$, for which we don't know whether or not it is a solution of the unknown system of differential equations in the form

$$\begin{aligned} \frac{dx_1}{dt} &= P_{k_1}(x_1, x_2, ..., x_m), \\ \frac{dx_2}{dt} &= P_{k_2}(x_1, x_2, ..., x_m), \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \\ \frac{dx_m}{dt} &= P_{k_m}(x_1, x_2, ..., x_m). \end{aligned} \tag{2.2}$$

Here P_{k_i} $(i = 1,2,...,m)$ are polynomials of unknown power $0 < k_i \le 3$, and $m \le 3$ is the unknown phase space dimension of a dynamical system approximately presented by the multivariate time series (2.1).

The problem is to prove whether or not the curve $\vec{c}_m : [x_1(t), x_2(t), \ldots, x_m(t)]$ is a solution of system of type (2.2), and to determine the polynomial powers k_i and the unknown coefficients of the polynomials P_{k_i}, such that satisfying eqs. (2.2).

3. Reducing noise and smoothing time series by moving polynomial averages and least squares method

The above statement of problem contains the premise that there is a low-dimensional *multi-component* $x_1(t), x_2(t), \ldots, x_m(t)$, which is only observed as a *multi-measurement* $Y_1(t_i), Y_2(t_i), \ldots, Y_m(t_m)$ corrupted by some observational *multi-noise*$\nu_1(t_i), \nu_2(t_i), \ldots, \nu_m(t_i)$. In order to reduce this noise as much as possible, an appropriate procedure must be applied. For similar purposes embedding phase space methods of nonlinear noise reduction have been developed [6], [7], [8], but we will apply another method here.

Note that after the noise reduction we obtain a new series of points $y_1(t_i), y_2(t_i), \ldots, y_m(t_i)$, satisfying the relations

(3.1) $$y_1(t_i) \approx x_1(t_i), y_2(t_i) \approx x_2(t_i), \ldots, y_m(t_i) \approx x_m(t_i).$$

However, the last approximate equalities *do not* lead with necessity to the validity of the relations

(3.2) $$\frac{dy_1(t_i)}{dt} \approx \frac{dx_1(t_i)}{dt}, \frac{dy_2(t_i)}{dt} \approx \frac{dx_2(t_i)}{dt}, \ldots, \frac{dy_m(t_i)}{dt} \approx \frac{dx_m(t_i)}{dt}.$$

We have further, to consider approximately the points $y_1(t_i), y_2(t_i), \ldots, y_m(t_i)$ as differentiable time series. The approximate equalities (3.2) are true only in case of applying a suitable procedure leading to the determination of a numerical functions $y_1(t_i), y_2(t_i), \ldots, y_m(t_i)$ being sufficiently smooth. In the following, we will propose an algorithm for obtaining shorter time series $\{y_k(t_i)\}_{i=n+1}^{N-n}, (k = 1,2,\ldots,m)$ in such a way, that the equalitiy (3.1) and (3.2) is valid for every moment t_i, $i = n+1, \ldots, N-n$.

In order to obtain the smoothed time series (3.1), we choose some odd number $2n+1 < N$ of successive values of $Y_k(t_i), (k = 1,2,\ldots,m)$, i.e. $Y_k(t_i), Y_k(t_{i+1}), \ldots, Y_k(t_{i+2n})$. Then the value t_{i+n} is taken as the center of the interval of succession. We can approximate this succession by the wellknown polynomial fitting of power *s* and using the least squares method [9]. As a result we obtain a polynomial $p_{ks}(t)$ whose value is best approximated in the center t_{i+n}. That is, $p_{ks}(t_{i+n})$ is maximally close to the true value $x_k(t_{i+n})$ of the solution component $x_k(t)$. We call $p_{ks}(t_{i+n})$ *polynomial average* (the index *s* means the polynomial power). We denote $p_{ks}(t_{i+n}) = y_k(t_{i+n})$. Thus the relation (3.1) is valid in a point t_{i+n}. It is clear that in such a way we can determine $y_k(t_i)$ for every $i = n+1, n+2, \ldots, N-n$. This procedure of successive determination of all values of a time series $\{y_k(t_i)\}_{i=n+1}^{N-n}$ in the **shorter** segment $[n+1, N-n]$ is called *moving* *polynomial average* [9]. As a result of applying this procedure to the given time series $\{Y_{ki}\}_{i=1}^{N}$, we obtain the smoothed time series $\{y_k(t_i)\}_{i=n+1}^{N-n}$.

Further, the same procedure of shortening could be applied to the numerical derivatives of $\{y_k(t_i)\}_{i=n+1}^{N-n}$ and we obtain corresponding time series of the derivatives for which the relations (3.2) are valid. In this way we partially avoid the wellknown difficulty with differentiation of noisy data, by shortening consecutively the time series of every higher derivative as a result of applying the described algorithm. Thus the worth of avoiding this difficulty is that we obtain **shorter** time series for the derivatives.

4. On the existence of dynamical system having as a solution a noise reduced multivariate time series

Firstly we have a given multi-time series $\{y_k(t_i)\}_{i=n+1}^{N-n}, (k=1,2,...,m)$, obtained possibly as a result of purifying a given time series $\{Y_k(t_i)\}_{i=1}^{N}, (k=1,2,...,m)$ contaminated with white noise. In view of (3.1) and (3.2), secondly we have the time series $\{x_k(t_i)\}_{i=n+1}^{N-n}$, which is a discrete approximation of a possible solution $x_k(t), (k=1,2,...,m)$. At the end, the last functions $x_k(t)$ are given in this section.

It is of great interest to answer the question: Whether or not there exists an *m*-th order dynamical system of type (2.2), having as a solution the functions $x_k(t)$?

The following theorems are important to answer that question (the first one is proven in a previous author's and co-authors paper [2], and the second can be easily demonstrated by analogy to a similar one in the paper cited):

Theorem 1. *Let* $x(t)$ *be a real-valued analytic function defined on interval* (t_1, t_N). *Moreover let for* $\forall t \in (t_1, t_N)$ *the inequality* $\frac{dx(t)}{dt} \neq 0$ *holds. Then there exists a unique real-valued analytic function* $F(x)$ *defined on the range of* $x(t)$, *such that* $x(t)$ *is a solution of*

$$\frac{dx}{dt} = F(x). \tag{4.1}$$

Theorem 2. *Let* $x_k(t), (k=1,2,...,m)$ *be real-valued and analytic functions defined on interval* (t_1, t_N), *such that* $\forall t \in (t_1, t_N)$ *the curve* $\vec{c}_m(t) = (x_1(t), x_2(t), ..., x_m(t))$, *m>1, is simple and regular. Then there exist real-valued analytic functions* $F_k(x_1, x_2, ..., x_m), (k=1,2,...,m)$, *such that* $\vec{c}_m(t)$ *is a solution of the system*

$$
(4.2)\qquad
\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, x_2, ..., x_m),\\
\frac{dx_2}{dt} &= F_2(x_1, x_2, ..., x_m),\\
&. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ ,\\
\frac{dx_m}{dt} &= F_m(x_1, x_2, ..., x_m).
\end{aligned}
$$

We say a curve $\vec{c}_m(t)$ is simple if there are no points of transversal or tangential self-intersection on it, i.e. if every point of the curve is simple (it is not a point of transversal or tangential self-intersection). On the other hand a point of self-intersection is a transversal one if there are more than one tangent in the point. When the tangent in the point of self-intersection is only one, it is called point of tangential self-intersection. Moreover, the curve is regular if $\frac{d\vec{c}_m(t)}{dt} \neq 0 \ \ \forall t \in (t_1, t_N)$, i.e. if every point of the curve is regular. So, the theorem 2 gives us sufficient conditions for the existence of system of type (4.2) having the given functions $x_k(t)$ as a solution. It is evident that the polynomial dynamical system (2.2) is a particular case of (4.2).

On the other hand the well-known theorems of trajectory type classifications give us the necessary conditions for the existence of above mentioned *m*-th order dynamical system. They claim:

Trajectory type classification 1. *"Let $F(x)$ in the differential equation (4.1) be an analytic function. Then every solution of equation (4.1) is either constant or a monotonous function.* [10]

Trajectory type classification 2. *"Let $F_k(x_1, x_2, ..., x_m)$ in the dynamical system (4.2) be analytic functions. Then every solution of the equation (4.2) is such that the curve $\vec{c}_m(t) = (x_1(t), x_2(t), ..., x_m(t))$ is either a fixed point, or a closed trajectory or an embedding (elementary and regular) one.* [11]

The curve $\vec{c}_m(t)$ is called elementary if it is a one-to-one function and one-to-one continuous. Every point of an elementary curve is called elementary point. The curve is embedding if it is both elementary and regular.

The practical procedure for determining the *elementarity* of a curve is the following: Firstly, we plot the plane projections

$$\vec{c}_{12}(t): \{x_1(t), x_2(t)\}, \vec{c}_{23}(t): \{x_2(t), x_3(t)\}, \dots, \vec{c}_{m-1,m}(t): \{x_{m-1}(t), x_m(t)\}$$

of the parameterized curve $\vec{c}_m(t): \{x_1(t), x_2(t), ..., x_m(t)\}$. Then, for every point of selfintersection (x_1^0, x_2^0) of the projection $\vec{x}_{12}(t)$, we verify whether or not there are *correspondant* points $(x_2^0, x_3^0), \ . \ . \ . \ , (x_{m-1}^0, x_m^0)$ on the projections $\vec{c}_{23}(t), \dots, \vec{c}_{m-1,m}(t)$ respectively. The points are *correspondant* ones, if the first coordinate of every consequent point is equal to the second coordinate of the previous one. In the notations we use this equality is valid, thus the point $(x_1^0, x_2^0, ..., x_m^0)$ is such of selfintersection of the curve $\vec{c}_m(t)$. The abscence of similar equality is a criterion (necessary and sufficient condition) for elementarity of the curve $\vec{c}_m(t)$, and it is also sufficient condition for its simplicity.

The practical procedure for establishing the *regularity* of the curve $\vec{c}_m(t)$ is also simple: We plot the graphs of all given functions $x_1(t), x_2(t), ..., x_m(t)$ and verify whether or not there is some point, where the functions simultaneously vanish. If no, the curve $\vec{c}_m(t)$ is regular. Otherwise, it is not.

5. Polynomial approximation of the right sides of the unknown dynamical system

In this section of our paper, we want to approximate an unknown dynamical system from its purified numerical solution, taken to be identical (see (3.1)) to the exact one $x_1(t), x_2(t), ..., x_m(t)$, which noise has been reduced sufficiently well by procedure proposed in the previous sections.

In this paper we propose procedure for determining unknown polynomial right sides of the differential equations. The procedure is based on the *least square method* and the fact that we know with sufficient precision the components of $\vec{c}_m(t)$ and its derivative. Further, for convenience the dot over variables is assumed to mean differentiation. In order to define the right hand sides of the differential equations, we construct the squared deviation S_k (between the left and right sides of the equations) in the form

$$
\begin{aligned}
S_k = \sum_{i=n+1}^{N-n} (\dot{x}_{ki} & - \xi_{k0} - \xi_{k1}x_{1i} - \xi_{k2}x_{2i} - \xi_{k3}x_{3i} - \xi_{k4}x_{1i}^2 - \xi_{k5}x_{2i}^2 - \xi_{k6}x_{3i}^2 - \\
& - \xi_{k7}x_{1i}x_{2i} - \xi_{k8}x_{1i}x_{3i} - \xi_{k9}x_{2i}x_{3i} - \xi_{k10}x_{1i}^3 - \xi_{k11}x_{2i}^3 - \\
& - \xi_{k12}x_{3i}^3 - \xi_{k13}x_{1i}^2x_{2i} - \xi_{k14}x_{1i}^2x_{3i} - \xi_{k15}x_{2i}^2x_{1i} - \\
& - \xi_{k16}x_{2i}^2x_{3i} - \xi_{k17}x_{3i}^2x_{1i} - \xi_{k18}x_{3i}^2x_{2i} - \xi_{k19}x_{1i}x_{2i}x_{3i})^2. \qquad (5.1)
\end{aligned}
$$

Here $k = 1,2,...,m.$, is a number of the variable x_k of the *m*-th dimensional dynamical system; $i = 1,2,..., N$ is a number of the value in the numerical time series $\{x_k(t_i)\}_{i=n+1}^{N-n}$. It is seen from (5.1) that for the case of polynomial of 3-rd degree with three variables (m=3), the number of unknown coefficients is 20. In order to define them, we should have 20 equations. These can be derived in the following way:

We differentiate (5.1) with respect to the unknown coefficients $\xi_{k0}, \xi_{k1}, \xi_{k2}, \ldots, \xi_{k19}$. After some algebraic transformations, as a result, for every $k = 1,2,..., m$, (m=3) we obtain 20 equations in the form

(5.2)

$$\xi_{k0} N + \xi_{k1}\sum_{i=1}^{N} x_{1i} + \quad . \quad . \quad . \quad + \xi_{k19}\sum_{i=1}^{N} x_{1i}x_{2i}x_{3i} = \sum_{i=1}^{N}\dot{x}_{ki},$$

$$\xi_{k0}\sum_{i=1}^{N} x_{1i} + \xi_{k1}\sum_{i=1}^{N} x_{1i}^{2} + \quad . \quad . \quad . \quad + \xi_{k19}\sum_{i=1}^{N} x_{1i}^{2}x_{2i}x_{3i} = \sum_{i=1}^{N} x_{1i}\dot{x}_{ki},$$

. . . .

$$\xi_{k0}\sum_{i=1}^{N} x_{1i}x_{2i}x_{3i} + \xi_{k1}\sum_{i=1}^{N} x_{1i}^{2}x_{2i}x_{3i} + \quad . \quad . \quad . \quad + \xi_{k19}\sum_{i=1}^{N} x_{1i}^{2}x_{2i}^{2}x_{3i}^{2} = \sum_{i=1}^{N} x_{1i}x_{2i}x_{3i}\dot{x}_{ki}.$$

This system of 20 linear algebraic equations can be solved with respect to the 20 unknowns $\xi_{k0}, \xi_{k1}, \xi_{k2}, \ldots, \xi_{k19}$ by the methods of matrix algebra. Certainly, we should write *m* systems of type (5.2) for every $k = 1,2,\ldots,m$. In order to solve every system of type (5.2) we introduce a *constructive matrix*

(5.3)
$$A = \begin{pmatrix} 1 & x_{11} & x_{21} & . \; . \; . \; . & x_{11}x_{21}x_{31} \\ 1 & x_{12} & x_{22} & . \; . \; . \; . & x_{12}x_{22}x_{32} \\ . & . & . & . \; . \; . \; . & . \\ 1 & x_{1N} & x_{2N} & . \; . \; . & x_{1N}x_{2N}x_{3N} \end{pmatrix}$$

for 3-rd power polynomial of 3 variables. This matrix has 20 columns and *N* rows. It is one and the same for every $k = 1,2,\ldots,m$. For second power polynomial of 3 variables the corresponding matrix will have 10 columns and the unknown coefficients will be 10, too.

We introduce also vectors of coefficients

(5.4) $$\vec{\xi}_k = (\xi_{k0}, \xi_{k1}, \ldots, \xi_{k19}), \quad k = 1,2,\ldots,m \quad (m = 3),$$

and vector of every time series presenting the numerical derivative of *k*-th variable

(5.5) $$\vec{\dot{x}}_k = (\dot{x}_{k1}, \dot{x}_{k2}, \dot{x}_{k3}, \ldots, \dot{x}_{kN}), \quad k = 1,2,\ldots,m \quad (m = 3).$$

Then the solution of *k*-th system (5.2) is given by the matrix formula

(5.6) $$\vec{\xi}_k = (A'A)^{-1} A' \vec{\dot{x}}_k, \quad k = 1,2,\ldots,m \quad (m = 3),$$

if the vector $\vec{\dot{x}}_k$ is entirely contained in the subspace spanned by the columns of A, what is definitively satisfied in our case. Here A' is a transposed matrix of A. The Moore-Penrose inverse used in (5.6) is known to produce unstable results if A is ill-conditioned and, as it will be demonstrated here, depend on uncertainties (for example - lack of solution uniqueness of (2.2) as it is claimed by theorem 2). Nevertheless, in many cases the last relationship is of exceptional convenience for practical computations by computer. For this purpose, it can be used a special software known as MatLab (Matrix Laboratory). In view of the fact that the number *N* may be practically choosen arbitrary large, a high precision of reconstruction can be achieved. Thus, we can expect that the solution of reconstructed dynamical system will be near the purified solution $x_1(t), x_2(t), \ldots, x_m(t)$. In all cases, the quadratic deviation (dispersion) between the corresponding components of a purified and reconstructed solution can be used as a optimal criterion of an appropriate search method. By applying similar method the precision of reconstruction can be improved additionally.

6. A new algorithm for reconstructing Lorenz system from a given triple time series with observational noise

We consider three components $x_1(t), x_2(t), x_3(t)$, (or triple time series) of numerical solution of the well-known Lorenz system [3]

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{aligned} \tag{6.0}$$

corrupted by noise with 6,75% dispersion. Here $\sigma = 10$ and $b = 2.6666$ are dimensionless constants which characterize the system, and $r = 28$ is a control parameter. The solution is taken in the time interval $t \in (0,2)$. We suppose the dimension of Lorenz system, its right hand sides and values of system parameters are not known. They can be found by applying the following *algorithm*:

Step 1. Purifying the components $x_1(t), x_2(t), x_3(t)$ by the moving average version of least square method described in the previous sections. As a result we obtain the three smoothed curves $x_1(t), x_2(t), x_3(t)$ presented in figure1. We must have in view that the smoothed curves in fig.1 are shorter than contaminated ones. The last following the smoothed, but not shown in figure 1 in view of the too large noise covering the purified curves.

Step 2. Applying theorems 1 and 2 from the previous sections to the components presented in figure 1 and phase plots presented in figure 2 and 3. In view of the absence of simultaneous triple self-intersections in figure 1 we conclude the three-dimensional curve $\vec{c}_3(t): [x_1(t), x_2(t), x_3(t)]$ is *regular.* From the lack of coincidence between the second and first coordinates of the points of self-intersection in figure 2 and 3 respectively, it follows the curve $\vec{c}_3(t)$ is *elementary.* (It is evident from figure 1, that the given components are not monotonous functions. Moreover, the presence of self-intersections in figures 2 and 3 means the corresponding two-dimensional curves are not elementary). On this basis and the mentioned theorem 2 we conclude that there exists a three-dimensional dynamical system in the form

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2, x_3), \\ \dot{x}_2(t) &= f_2(x_1, x_2, x_3), \\ \dot{x}_3(t) &= f_3(x_1, x_2, x_3). \end{aligned} \tag{6.1}$$

Step 3. Applying the specific version of least square method described in section 1 to determine the right hand sides of the dynamical system (6.1). As a result we obtain that the corresponding functions have the forms

$$
(6.2)\quad
\begin{aligned}
f_1(x_1,x_2,x_3) &= w_1 + w_2x_1 + w_3x_2 + w_4x_3, \\
f_2(x_1,x_2,x_3) &= w_5 + w_6x_1 + w_7x_2 + w_8x_3 + w_9x_1^2 + w_{10}x_2^2 + w_{11}x_3^2 + \\
&\quad + w_{12}x_1x_2 + w_{13}x_1x_3 + w_{14}x_2x_3, \\
f_3(x_1,x_2,x_3) &= w_{15} + w_{16}x_1 + w_{17}x_2 + w_{18}x_3 + w_{19}x_1^2 + w_{20}x_2^2 + w_{21}x_3^2 + \\
&\quad + w_{22}x_1x_2 + w_{23}x_1x_3 + w_{24}x_2x_3.
\end{aligned}
$$

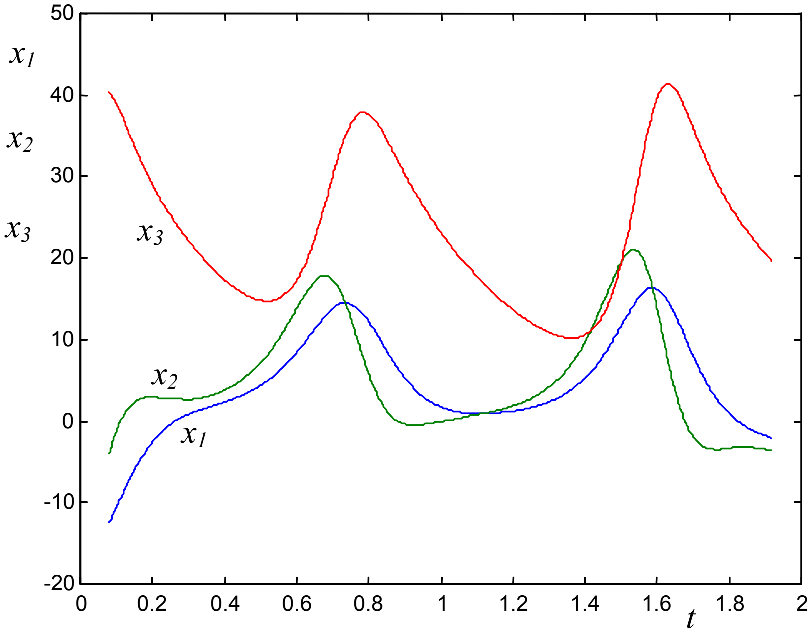

Figure 1. Smoothed (purified) components (dispersion 0.12%) of the corrupted solution (dispersion 6.75%).

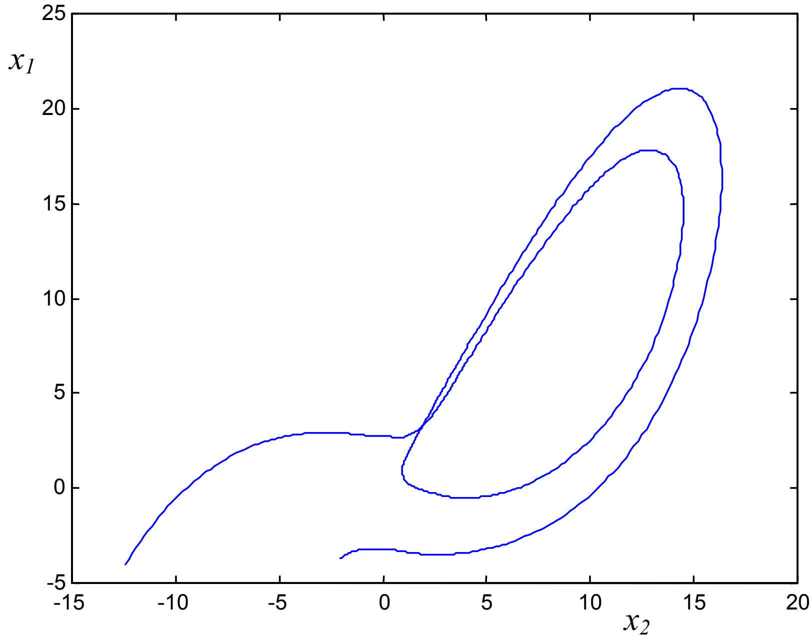

Figure 2. Phase plot projection of the smoothed solution on the plane (x_1, x_2).

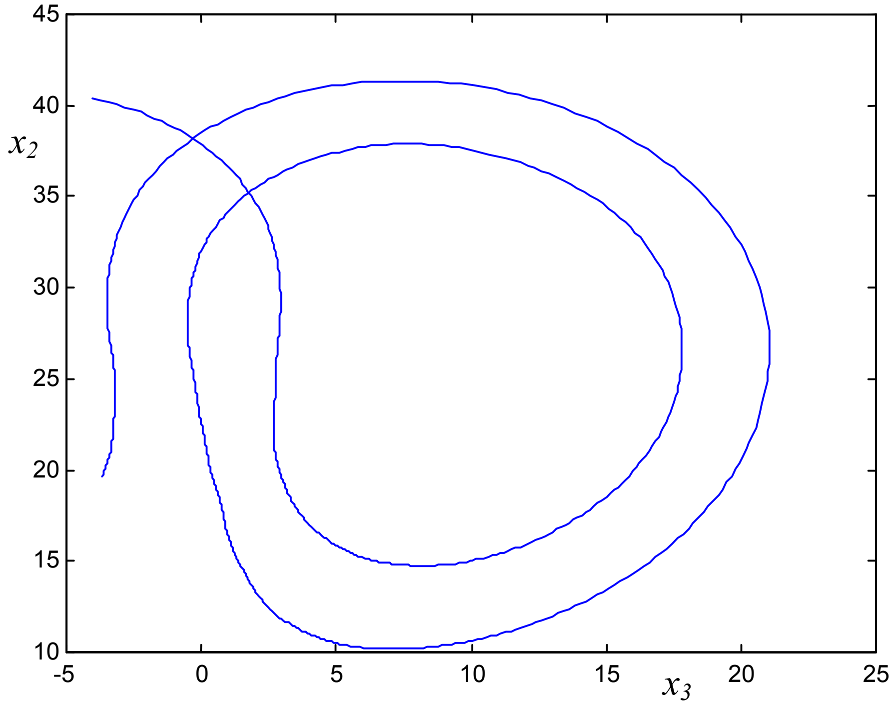

Figure 3. Phase plot projection of the smoothed solution on the plane (x_2, x_3).

where

(6.3)
$$\begin{aligned}&(w_1, w_2, w_3, w_4) \equiv \vec{\xi}_1,\\ &(w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}) \equiv \vec{\xi}_2,\\ &(w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, w_{20}, w_{21}, w_{22}, w_{23}, w_{24}) \equiv \vec{\xi}_3,\end{aligned}$$

are vectors of coefficients which components have the following concrete values

(6.4)
$\vec{\xi}_1$=(-0.0124 ; -9.9975 ;10.0113 ;0.0070),
$\vec{\xi}_2$=(-4.0487;27.868;-0.7211;0.3251;0.0117;-0.0182;-0.0062;0.0246;-0.9983;-0.0122),
$\vec{\xi}_3$=(-2.7679;-0.3280; 0.1837; -2.3359;0.0392;0.0056;-0.0091;0.9786;0.0075;-0.0110).

By replacing these values of coefficients from (6.3) to (6.2) and then to (6.1), we obtain a dynamical system with polynomial right hand sides, that can be solved at initial conditions taken from the purified components shown in figure 1. As a result we obtain reconstructed solution presented in figure 4 together with the smoothed one. It is seen that the coincidence is rather well not only in the interval (0,2) *of reconstruction*, but also in the next interval (2,4) *of prediction*.

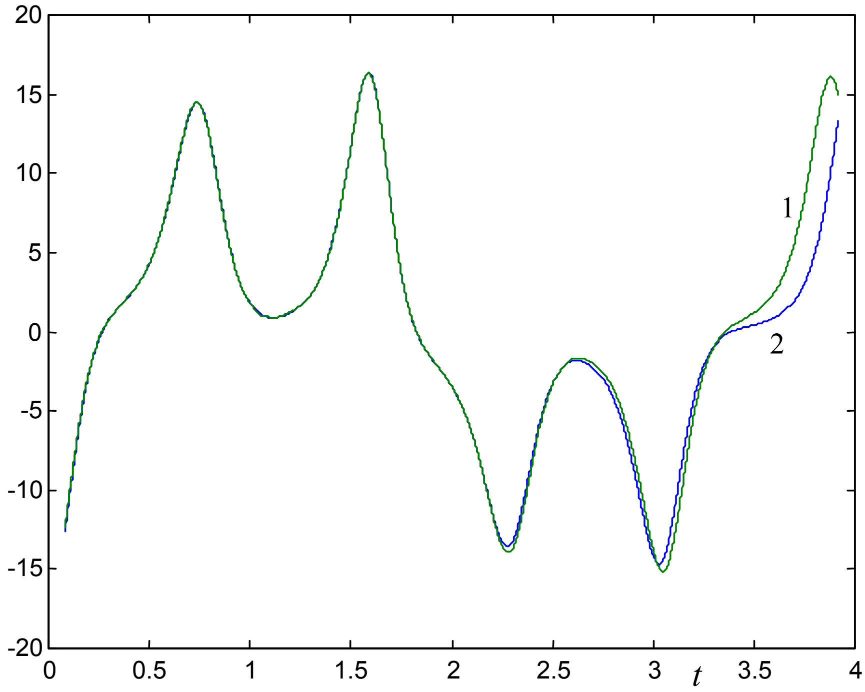

Figure 4. Satisfactory fitting between the first components of smoothed (1) and reconstructed (2) solutions

7. Concluding discussion

At the end, we present the *original* coefficients of the Lorenz system we have used in the beginning of section 2, to produce the given solution in figure 1. These are

$$\vec{\xi}_1=(0;\ -10;\ 10;\ 0),$$
$$\vec{\xi}_2=(0;\ 28;\ -1;\ 0;\ 0;\ 0;\ 0;\ 0;\ -1;\ 0), \qquad (7.1)$$
$$\vec{\xi}_3=(0;\ 0;\ 0;\ -2.6666;\ 0;\ 0;\ 0;\ 1;\ 0;\ 0).$$

It is seen there are essential differences only between the first components of the vectors $\vec{\xi}_2$ and $\vec{\xi}_3$, in (6.4) and (7.1) presenting *reconstructed* and *original* coefficients respectively. This means some instability occurs in the reconstruction procedure of these components that may be related to the lack of uniqueness of reconstructed dynamical system under given solution. This *subtle* point needs the following explanations:

We should pay attention that when reconstructing Lorenz system the algorithm requires the application of both theorem 1 and 2 as it is said in Step 2. However the theorem 2 treats only the question of existence but not of the uniqueness one. (In the theorem 1 both existence and uniqueness are treated). Therefore, the question of uniqueness remains open in a multi-dimensional case. Indeed, as it is proved by Erugin [12] in the three-dimensional case (including Lorenz system) the reconstruction of dynamical system is not unique. Thus, there is no contradiction between analytical theory and numerical realization of the proposed algorithm, as it would seem at first sight. Moreover, from all 24 coefficients only 3 are not precisely determined. These 3 ones do not figure in the Lorenz system but are expediently introduced by the very algorithm namely in order to verify the question for possible non-uniqueness of the reconstruction process. That means the obtained numerical discrepancy is not artifact to be removed, but a valuable indication for existence of fundamental problem needing additional investigation on this topic. Similar conclusion seems to support the assertion of L. A. Aguire and Ch. Letellier in their paper [13], that "in some cases, to reconstruct the dynamics using more than one observable could be worse than to reconstruct using a scalar measurement". Other authors, however, claim about multivariate series analysis "advantages over the use of only a single-variable series" [14]. Thus, there is a discussion in the literature testifying the point is really *subtle*.

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