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Efficiency Evaluation with Convex Pairs

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Abstract

In this paper, we introduce a new approach to modeling technologies in productivity analysis. The approach uses pairs of associated input and output sets. It allows for different degrees of convexity in the overall production possibility set. Using blocking and antiblocking theory from combinatorial optimization, we also develop the dual representation of the technology. We show how this modeling framework contains the classical FDH and BCC models together with a variety of new models, including FDH models with assurance regions and models with diseconomies of scope (specialization gains). In all cases, the resulting Farrell efficiency programs can be formulated as linear programming problems.

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1 Introduction

The mathematical programming approach to efficiency evaluation, most notably Data Envelopment Analysis (DEA), has proved useful in numerous applications. Part of its success is due to the wide class of production

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structures that can be approximated using linear programming. The original model by Banker, Charnes and Cooper [6], the so-called VRS model makes a (minimal extrapolation) approximation of an arbitrary production possibility set that is convex and satisfies free disposability of inputs and outputs.

From a theoretical as well as an applied point of view, however, even the convexity assumption can be questioned. In particular, convexity assumes away (global) economies of scale and scope which are essential in many microeconomic theories and which have been observed in many industries.

In this paper we introduce a new modeling approach, the so-called convex pairs approach. A convex pair is defined by a convex input possibility set and a convex output possibility set such that all input vectors in the input set can produce all output vectors in the output set. We represent the technology as a union of convex pairs.

This approach allows us to work with convexity around one or more observations without assuming convexity across all observations in input space, in output space or in the full production space. Thus, for example, it allows us to introduce assurance regions and other dual information into the FDH set-up by Tulkens [35], who fully dispenses with convexity, i.e. the set-up operates only with free disposability, such that comparison is only possible by domination. More generally, the new approach enables us to model a spectrum of technologies ranging from the fully convex BCC technology to the non-convex FDH technology. Modelling using convex pairs hereby extends previous attempts to dispense with convexity. In particular, the convex projection approach of Petersen [27], Bogetoft [7], and Bogetoft, Tama and Tind [11] all dispensed with the assumption of global convexity while at the same time presuming convexity of input and output sets. The convex pairs approach dispenses with the latter assumption by assuming that the input sets and output sets may themselves be non-convex unions of convex subsets.

Other attempts have been made to relax the standard convexity assumption in DEA. Chang [15] and Post [28] have considered the cases where the input isoquants or the output isoquants but not both are assumed convex. Additional related work has been done by Kuosmanen [23] and [24] in which convexity is replaced by so-called conditional convexity. Post [28] considers a convex transformation of a non-convex possibility set by means of so-called transconvex functions.

An advantage of the new approach here is that efficiency scores can still be calculated using linear programming. Likewise, realloaction analyses can be performed using linear programming. This is the case even though the underlying production possibility set is no longer polyhedral convex. We demonstrate this using results from disjunctive programming.

Last but not least, dual representations can be developed using the blocking and antiblocking theory by Fulkerson [20]. This is a nice duality theory that takes advantage of the fact that the sets belong to the nonnegative orthant. The blocking and antiblocking theory gives particularly nice support to the creation of intersections and convex unions of individual basic sets. These operations correspond to the most common approaches to extend the production technology from indiviual basic sets In the dual space, these operations are simply reversed. Hence, the modelling and analysis can be done in the space that are most convenient to work with. Moreover, variations in the dual sets lead to easily recognizable variations in the primal space. Dispensing with positivity, for example, will be equivalent to introducing weak free disposability instead of strong free disposablity.

We commence in Section 2 with the basic building blocks, the pairs of convex input and output sets. In Section 3 we discuss the use of convex pairs in productivity modeling. In Section 4, we develop dual representations. We also discuss the advantage of making convex unions in the primal space and intersections in the dual space. LP formulations in primal as well as dual spaces are provided in Section 5. Illustrating experiments are performed in Section 6 and final remarks are given in Section 7.

2 Convex Pairs

The idea of the new approach is to model the production possibilities as a union of pairs of convex input and output sets.

Consider a set $\mathcal{N}$ of units or pairs or subsets. $\mathcal{N}$ indexes the building blocks we use to construct the technology. In an application, a unit could be a decision making unit (DMU) or a group of decision making units.

For each $i \in \mathcal{N}$ let L_i be a set of r dimensional input vectors x and let P_i denote a set of s dimensional output vectors y. For the inputs we assume that L_i satisfies

$$(L_i + \mathbb{R}^r_+) \cap \mathbb{R}^r_+ = L_i. \tag{1}$$

This means that L_i is nonnegative and satisfies free disposability in the input space. Moreover L_i is assumed to be convex.

Similarly for the outputs we assume that P_i satisfies

$$(P_i - \mathbb{R}^s_+) \cap \mathbb{R}^s_+ = P_i. \tag{2}$$

i.e. P_i is also nonnegative and satisfies free disposability in the output space. Also P_i is additionally assumed to be convex.

The input and output sets are associated as a pair (L_i, P_i) (or equivalently $L_i \times P_i$) by the assumption that any input vector $x \in L_i$ can produce any output vector $y \in P_i$.

Finally, consider the full technology T i.e. $T = \{(x, y) \in \mathbb{R}_+^{r+s} | \; x$ can produce $y\}$. In the convex pair approach we model T as the union of the feasible pairs, i.e.

$$T = \cup_{i \in \mathcal{N}} (L_i, P_i).$$

The convex pairs approach suggested here is formally defined by the set of convex pairs, i.e. the set $\{(L_i, P_i) | i \in \mathcal{N}\}$, and the convention that the full production set is the union of these sets, $T = \cup_{i \in \mathcal{N}} (L_i, P_i)$.

In the convex pairs approach we do not ignore the possibility of local convexity. We do however dispense with convexity of the full production set or the full input and output isoquants.

The main reasons for allowing convexity, if not globally then locally, are that

Convex technologies play a significant role in large parts of micro-economic theory and operations research, and there has been a long discussion of the pros and cons of different types of more or restricted convexity in productivity analysis. Contributions to this discussion have already been mentioned in the introduction.

The purpose of the present paper is not to discuss the alternative convexity assumptions per se, but rather to present an approach that enrich our ability to handle different types of restricted convexities. At the risk of being sloppy, it suffices therefore to state some of the primary motivation for and arguments against convexity.

The main reasons for allowing convexity, if not globally then locally, includes:

- Convexity occurs naturally in some contexts. In particular, this may be the case if several (linear) production processes are available and the organization can freely decide how much time and other resources to allocate among them. From a theoretical perspective, this corresponds to a justification of convexity from a set of more elementary and intuitive axioms, viz divisibility and additivity as in for example Arrow and Hahn [4].

- Convexity provides a reasonable approximation in some contexts. In particular, if the data available on a given DMU is an aggregation of the processes used in different subunits or time intervals, convex combinations can approximate alternative but non-observed aggregations as suggested above.

- Convexity is sometimes an operationally convenient, but result wise harmless assumption. Thus if we are interested in cost efficiency, revenue efficiency or profit efficiency, we can often introduce convex input requirement sets, output requirement set or full input-output possibility sets without affecting the results. This presume of course that prices are fixed (independent of quantities, cf. below).

On the other hand, there are numerous theoretical as well as practical reasons for dispensing with global convexity assumption in productivity analysis, e.g.

- Convexity requires divisibility (since a convex combination is basically an addition of downscaled plans) which may not be possible, e.g. when different investments are considered or set-up times and switching costs are present. Farrell [19] early noted the problems of divisibility.
- Convexity assumes away economies of scale and scope (specialization) which are experienced in many industries.
- Prices may depend on the quantities such that the introduction of convexity is not a harmless operational convenience. This was early observed in relation with the FDH technology by Wunsch [37] and more recently Cherchye et al. [18] and Kuosmanen and Post [26] have pointed towards imperfect competition (i.e., price-making rather than price-taking) and risk aversion under price uncertainty as circumstances under which the organizations objectives become non-linear and hence non-convexities of the technology influence the optimal production plan

Now, given the model of the technology, an efficiency index can be introduced in the usual way. Let $(x_0, y_0) \in \mathbb{R}^{r \times s}$ be the input-output vector we want to evaluate. Traditionally DEA operates with an input oriented efficiency score as well as an output oriented efficiency score by means of a Farrell index. In the current setting the input oriented efficiency score is the optimal value of the program

$$\min\{\theta \in \mathbb{R} \mid \ (\theta x_0, y_0) \in T\}. \tag{3}$$

By the structure of T this is equivalent to

$$\min\{\theta \mid \exists i \in \mathcal{N} \text{ where } (\theta x_0, y_0) \in (L_i, P_i)\}$$

or simply

$$\min_{i \in \mathcal{N}} \min\{\theta \mid (\theta x_0, y_0) \in (L_i, P_i)\}.$$

Hence, the efficiency score can be determined by solving $|\mathcal{N}|$ optimization problems, one for each of the pairs of sets.

In the next section, we will discuss how to construct the input-output sets in some practical cases. Then, in Section 4, we develop alternative primal and dual representations of the technology, and use these representations to develop alternative LP formulations of the efficiency measurement problem above.

3 Technologies using Convex Pairs

Many different production structures can be modeled using the convex pairs framework. To emphasize this we shall now give a series of examples.

3.1 FDH model

Initially, we note that the *free disposability hull* (FDH) *model* is a special case of the present approach. Consider a set I of decision making units, DMU's, and for each $i \in I$ let x_i denote the nonnegative input vector used and y_i the nonnegative output vector produced by i'th DMU. Free disposability now implies that the set of inputs

$$L_i = \{x \mid x \geq x_i\}$$

can also produce the set of outputs

$$P_i = \{y \geq 0 \mid y \leq y_i\}$$

for all $i \in I$. So, geometrically L_i is a translation of the nonnegative orthant of the input space. In the output space P_i is a box, which has full dimension if and only if all elements in y_i are strictly positive. See Figure 1 for an illustration.

Figure 1: Feasible input-output combinations with free disposablity.

If free disposability is the only assumption made, the resulting technology

$$T = \cup_{i \in I}(L_i, P_i)$$

is simply the FDH model introduced by Tulkens [35].

3.2 FDH model with prices

Next, let us consider an FDH model with partial price information. The total lack of substitution possibilities presumed in the FDH model is rather extreme. Often, some substitution among the inputs and among the outputs is possible around a given production plan. To model this, we may to every DMU $i \in I$ attach a set of relative prices on the input side, $U_i \subseteq \mathbb{R}_0^r$, and a set of relative prices on the output side $V_i \subseteq \mathbb{R}_0^s$ and let the input set be

$$L_i = \{x | ux \geq ux_i \ \forall u \in U_i\}$$

and the output set be

$$P_i = \{y | vy \leq vy_i \ \forall v \in V_i\}.$$

This formulation can be used in a number of different contexts depending on the price information at hand. In the DEA literature, the so-called *assurance regions* suggested by Thompson et al. [33] has been widely used. In this set-up, upper and lower bounds are imposed on the relative prices, i.e.

$$\begin{aligned} U_i &= \{u \in \mathbb{R}_0^r | \alpha_{ihk}^u \leq \frac{u_h}{u_k} \leq \beta_{ihk}^u \quad h,k = 1,..,r, h < k\} \\ V_i &= \{v \in \mathbb{R}_0^s | \alpha_{ihk}^v \leq \frac{v_h}{v_k} \leq \beta_{ihk}^v \quad h,k = 1,..,s, h < k\}. \end{aligned}$$

A specific example of this is given in Figure 2 below. We assume here that two types of outputs correspond to monetary amounts in each of two periods. The difference between the hyperplanes spanning a given P_i in this case may reflect the difference between the borrowing and the lending interest rates.

Other specifications of the sets U_i and V_i are possible. Thus, Kuosmanen and Post [25] consider price domains specified by polyhedral convex cones with zero vertex.

3.3 Local Substitution

Of course the substitution allowance in the FDH model with partial price information can easily be combined with the lack of substitution allowance in the original FDH model by using pairs of convex sets given by for example

$$\begin{aligned} L_i &= \{x | u_i x \geq u_i x_i \ \forall u_i \in U_i \text{ and } x \geq (1-\Delta_i)x_i\} \\ P_i &= \{y | v_i y \leq v_i y_i \ \ \forall v_i \in V_i \text{ and } y \leq (1+\delta_i)y_i\}. \end{aligned}$$

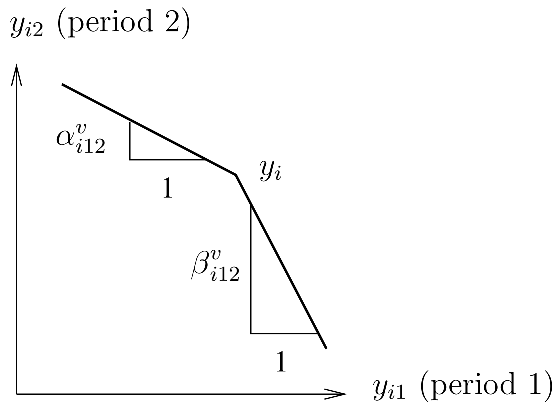

Figure 2: Feasible output combination with an (imperfect) capital market.

The idea here is that the substitution possibilities are only valid locally - when we try to lower any input more than the fraction $\Delta_i \in [0,1]$ from its original level or increases any output more than the fraction δ_i from its original level, the local substitution possibilities are no longer valid and we return to the extreme case of no further substitution.

In the previous formulations of convex pairs, we assumed that dual partial price information is available and that the convex sets can be constructed from these. In many applications, however, the information at hand may instead be available in primal form. To model *a context with primal substitution information*, we may to every DMU $i \in I$ attach a set of possible input vectors, $X_i \subseteq \mathbb{R}^r_+$, and a set of possible output vectors, $Y_i \subseteq \mathbb{R}^s_+$ and let the input set be

$$L_i = conv(X_i) + \mathbb{R}^r_+$$

and the output set be

$$P_i = (conv(Y_i) - \mathbb{R}^s_+) \cap \mathbb{R}^s_+.$$

Thus for example, in an agricultural context, different feeding plans can be used to span the input set in milk production and different land use plans can generate the output set of a crop farmer.

3.4 Convex Projections

A fourth instance of the convex pairs set-up is derived by assuming free disposability and convex projection in the input and output space, cf. Petersen [27], Bogetoft [7] and Bogetoft et al. [11]. Consider again a set I of DMUs, and for each $i \in I$ let x_i denote the nonnegative input vector used and y_i the nonnegative output vector produced. Free disposability now implies that the set of inputs $L_i = \{x \mid x \geq x_i\}$ can also produce the set of outputs $P_i = \{y \geq 0 \mid y \leq y_i\}$ for all $i \in I$. A basic question is: What is the minimal possibility set T containing data and satisfying free disposability such that the input projections $L(y) = \{x|(x, y) \in T\}$ and output projections $P(x) = \{y|(x, y) \in T\}$ are convex for all $(x, y) \in T$? To emphasize the symmetry of the framework let $\uplus$ denote the convex union operation, i.e. $A \uplus B$ is the smallest convex set $conv(A \cup B)$ containing A and B, and define the following two pairs of sets for arbitrary indices $i, j \in I$:

$$(L_n, P_n) = (L_i \cap L_j, P_i \uplus P_j) \tag{4}$$

and

$$(L_m, P_m) = (L_i \uplus L_j, P_i \cap P_j). \tag{5}$$

Due to free disposability any $x \in L_i \cap L_j$ and $y \in P_i \cup P_j$ constitute a feasible input-output combination. Since the production set $P(x)$ is required to be convex this is also true for any pair $(x, y) \in (L_n, P_n)$. Similarly any pair $(x, y) \in (L_m, P_m)$ is a feasible combination, since the consumption set $L(y)$ is convex.

Hence, (4) and (5) generate pairs of sets consisting of feasible input-output combinations in addition to the original pairs (L_i, P_i) for $i \in I$. This approach is continued in an iterative procedure to be outlined below.

If $L_i \subseteq L_j$ and $P_i \subseteq P_j$ for some $i, j \in I$ then the pair (L_j, P_j) is said to *dominate* the pair (L_i, P_i). In this case all input-output combinations of the i' th pair are included in the j'th pair and index j may be removed.

This may be formalized by the following

Procedure

Start: Let $N_1 = I$ and $l = 1$.

Step 1: Create by (4) and (5) all new pairs (L_n, P_n) and (L_m, P_m) based on existing pairs with indices $i, j \in N_l$.

Step 2: Let N_{l+1} consist of all pairs in N_l together with the new ones created in Step 1.

Step 3: Remove from N_{l+1} any pair which is dominated by another pair in N_{l+1}.

Step 4: If $N_{l+1} = N_l$ no more non-dominated pairs can be generated and the procedure terminates. Otherwise let $l := l + 1$ and go to Step 1.

The procedure creates a series of indices. This series, to be denoted by $\mathcal{N}$, may be finite or infinite dependent on possible fulfillment of the termination criterion in Step 4. Now the resulting possibility set T defined by

$$T = \cup_{i \in \mathcal{N}} (L_i, P_i)$$

is by construction the smallest possible possibility set satisfying the desired properties: It contains the input-output points of the DMU's, satisfies free disposability and has convex projections. More details including conditions for finiteness of the procedure may be found in Bogetoft et al. [11].

3.5 Specialization and Diseconomies of Scope

In general, the use of convex pairs allows us to model a series of phenomena, that are relevant in applications. In particular, we can have a model with gains from *specialization and diseconomies of scope.* This conflicts with the idea of convex isoquants, but not with the idea of having isoquants that are unions of convex subsets, as illustrated in figure 3 below, where technology (L_1, P_1) has a relative advantage specialization in the use of x_1 and production of y_1 compared to technology (L_2, P_2).

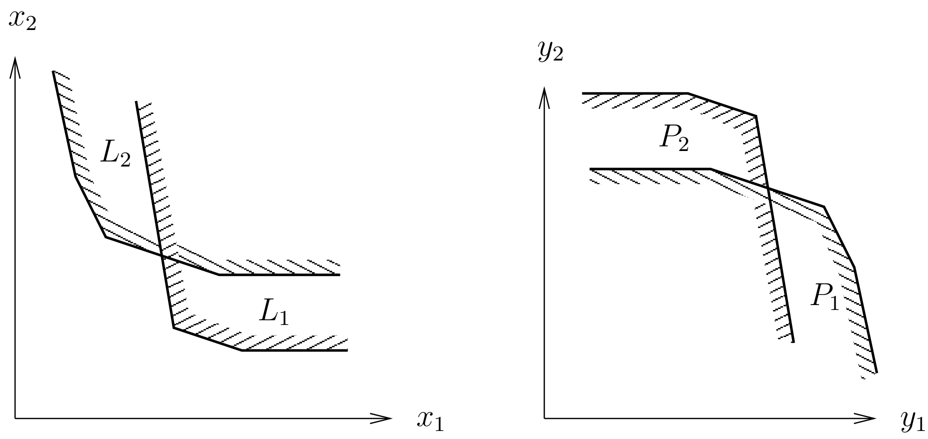

Figure 3: Gains from specialization and diseconomies of scope.

3.6 In-comparabilities

Also, we can model the occurrence of *in-comparabilities - or ordinal comparabilities only.* The different convex pairs can simply correspond to different values of a categorical or ordinal variable, e.g. *quality levels.* Convexification of unions of such subsets makes little sense since it would lead to average categorical or ordinal values which is meaningless by the definitions of categorial and ordinal variables, see Roberts [29]. Along the same line, we may think of different sets as belonging to different sub-technologies or investment types. Again, it may not be relevant to convexify across such *investments.* Thus for example, if two museums, one having invested mainly in modern art and another having invested mainly in impressionists, are both able to generate the same output in terms of revenue, visitors etc., it does not mean that an average art museum with half modernist and half impressionist art may be able to attract the same number of visitors or generate the same revenue. Figure 3 could illustrate this as well if we think of x_1 and x_2 as the amount of modern art and impressionism, respectively.

3.7 Further Perspectives

Using a series of examples, we have shown above how convex pairs can be used to approximate the information we have about the technology in many cases. We close this section with a few more methodological remarks.

From a conceptual perspective, we believe that the modeling via a union of convex pairs may be interesting. Like classical preference theory, cf. e.g. Bogetoft and Pruzan [9], classical production theory with convex production sets has come under attack in the productivity analysis literature. It is often felt that the classical paradigm is much too restrictive to aid real world evaluations. Indeed, it can be argued that one should not necessarily aim at making consistent and comprehensive comparisons at all costs. In particular, strong arguments can be made for *in-comparabilities* (like in the cases with categorical or ordinal properties). Also, *intransitivities* may exist in the sense that local comparisons and convexifications may be natural without global comparisons necessarily being relevant. Both deviations from the classical framework can be encompassed by the convex pairs approach.

From a practical perspective, a convex pairs model may be thought of as an initial, incomplete but nevertheless useful framework that may subsequently be refined. Of course, a slackening of the demands as to convexity of the technology leads to less forceful evaluations. Using a less involved technology assumption may result in a larger set of efficient units from which further discrimination must be undertaken using other, presumably less formal and more intuitive means. Still, this should not discourage us. An initial screening of the DMUs based on easily accepted substitution premises may be a good starting point - just like the notion of efficiency in itself. In fact this view of efficiency evaluation as a process of enriching the production and hereby the efficiency relation so as to aid the evaluation of units, conforms nicely with the idea of evaluations being a cyclic process involving additional information as we go along.

Also, incomplete technology models may be useful in an attempt to handle repetitive evaluations. Even a partial model may handle many evaluations, e.g. identify under-performance, and only in those cases, where the partial analysis can not identify inefficiencies, and where additional information can be convincingly verified, it may be necessary to work with the enriched technological description. In a hierarchical organization, it may for example be useful to endow lower levels with at least a partial description of technology since this may reduce the frequency of referrals that are called for.

Now, to enrich the production structure, we have already illustrated in the iterative procedure in Section 3.4 that two operations are important. One is to make convex unions $\uplus$, and the other is to make intersections $\cap$. We do not argue that the iterative procedure should always be continued to convergence nor that all pairs of intersections and convex unions should be formed even in a one shot procedure. We simply claim that for a selected set of pairs, the construction of convex intersections and unions may be natural, and that using the natural convexifications in input and output space may enrich the technology and hereby the ability to identify non-efficient units.

4 Non-negative polarity

In this section, we shall develop the relationship between primal and dual formulations of a convex pair (L, P). We have two reasons to develop these relationships. First, to model a specific situation it is sometimes most convenient to use a primal approach and sometimes most convenient to use a dual approach. Also, the operations we use to enrich the technology are more easily performed in one rather than the other space. Thus, to form convex unions, it is most convenient to have the sets described in primal terms, since the convex union of sets is then the convex hull of the union of activities spanning the original sets. On the other hand, to form convex intersections, a dual description is convenient because the intersection is defined by a set of constraints. It follows that to allow for easy expansions of the technology, we must know the relationship between primal and dual formulations.

As the following shows we are able to choose a description utilizing some polarity results from convex analysis, see for example Rockafellar [30]. However, some slight modification of the classical results is required due to our assumptions about free disposability and non-negativity.

4.1 Antiblockers

Let us first look at the output side. For simplicity omit the indices and consider an output set P satisfying (2). Assume additionally that P is closed and $0 \in P$. By (2) the last assumption is equivalent to saying that P is non-empty.

Polarity for polyhedra of this form has been studied by Fulkerson [21] for the investigation of a class of problems in combinatorial optimization. He introduced the notion of an antiblocker and we shall here see that the basic concept appears useful for an investigation of polarity properties in data envelopment analysis.

The *antiblocker* of P is defined by

$$\mathcal{A}(P) = \{y^* \in \mathbb{R}^p_+ \mid y^* y \leq 1 \text{ for all } y \in P\}. \tag{6}$$

Note that $\mathcal{A}(P)$ is also closed, contains 0 and satisfies (2). Hence the properties for P carry over to its antiblocker.

Antiblockers are similar to the notion of polar sets. For an arbitrary set $Q \in \mathbb{R}^p$ the *polar set* of Q is defined by

$$\{y^* \in \mathbb{R}^p \mid y^* y \leq 1 \text{ for all } y \in Q\}.$$

So the main distinction between the notion of a polar set and an antiblocker is that all sets in the last case are non-negative. Various duality properties exist for polar sets. So, subject to a minor modification due to non-negativity they can be transformed into similar properties for antiblockers, as demonstrated by following propositions.

Introduce an additional closed set O containing 0 and satisfying (2). We may then state the following

Proposition 1 $\mathcal{A}(P \uplus O) = \mathcal{A}(P) \cap \mathcal{A}(O)$.

The proof follows the same lines as for a similar statement for general polar sets. See Rockafellar [30, Corollary 16.5.2].

The next proposition states the involutory property for antiblockers. This was shown by Fulkerson [20].

Proposition 2 $\mathcal{A}(\mathcal{A}(P)) = P$.

As a counterpart to Proposition 1 we shall state the following dual version.

Proposition 3 $\mathcal{A}(P \cap O) = \text{cl } (\mathcal{A}(P) \uplus \mathcal{A}(O))$.

This may be proven similarly as in Rockafellar [30].

The following example shows the neccessity of the closure operation cl in Proposition 3 in general.

Example 1

Consider in the two dimensional output space the two polyhedra $P =$ conv $[(0,0),(0,0.5),(0.5,0)]$ and $O =$ conv $[(0,0),(1,0)]$. They each have a form that could be the output part of a feasible input-output pair. However O has no output in the second component. Now $\mathcal{A}(P) = \{(y_1^*, y_2^*) \mid (0,0) \leq (y_1^*, y_2^*) \leq (2,2)\}$ and $\mathcal{A}(O) = \{(y_1^*, y_2^*) \mid 0 \leq y_1^* \leq 1 \text{ and } y_2^* \geq 0\}$. In this case $\mathcal{A}(P) \uplus \mathcal{A}(O)$ is not closed. See Figure 4.

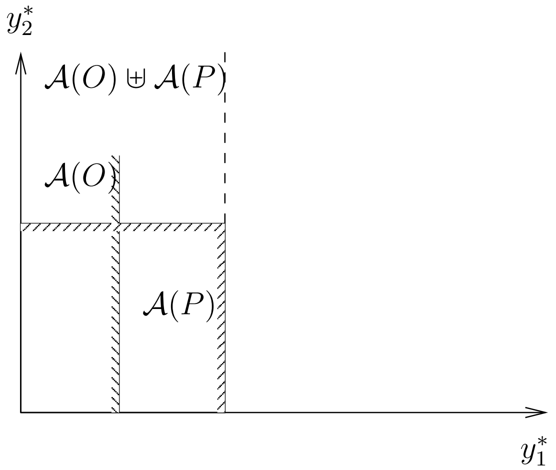

Figure 4: $\mathcal{A}(O) \uplus \mathcal{A}(P)$ is not closed.

The trouble in the above example is the lower dimension of O. Indeed, the closure operation can be removed if $\mathcal{A}(P)$ and $\mathcal{A}(O)$ have the same recession cone, see Rockafellar [30, Corollary 9.8.1]. This occurs if the cones generated by P and O have full dimension p in the output space, in which case $\mathcal{A}(P)$ and $\mathcal{A}(O)$ are bounded. These observations can be summarized into the following

Corollary 4 *If the output sets P and O are both full dimensional then $\mathcal{A}(P \cap O) = \mathcal{A}(P) \uplus \mathcal{A}(O)$.*

The above results make it possible to interchange the role of intersection and convex union as summarized in

Proposition 5

$$P \cap O = \mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) \tag{7}$$
$$\textit{and } \operatorname{cl}(P \uplus O) = \mathcal{A}(\mathcal{A}(P) \cap \mathcal{A}(O)). \tag{8}$$

Proof: $\mathcal{A}(P)$ and $\mathcal{A}(O)$ satisfy the properties (2). Hence by Proposition 1 we get $\mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) = \mathcal{A}\mathcal{A}(P) \cap \mathcal{A}\mathcal{A}(O)$. Finally, application of Proposition 2 on the last terms implies (7). (8) follows by Propositions 2 and 3.

4.2 Blockers

We shall here study the input sets in the framework of blockers. This notion is also due to Fulkerson [20] . Again for simplicity we drop the index and consider an input set L satisfying (1). Additionally assume that L is closed and $0 \notin L$. Together with (1) the last assumption is equivalent to saying that $L \neq \mathbb{R}^r_+$.

The definition of a blocker is similar to the definition (6) of an antiblocker, however with the inequality reversed. So, the blocker $\mathcal{B}(L)$ of L is going to be defined by

$$\mathcal{B}(L) = \{x^* \in \mathbb{R}^r_+ \mid x^*x \geq 1 \text{ for all } x \in L\}. \tag{9}$$

The blocker $\mathcal{B}(L)$ is closed, $0 \notin \mathcal{B}(L)$ and satisfies (1). Hence the assumed properties of B are alway valid for a blocker.

Similar structures have been studied as so-called aureoled sets by Weddepohl [36] and Ruys and Weddepohl [31] and as so-called reverse polar sets by Tind [34]. Based on these results we can obtain results of the same type as for antiblockers with due consideration to the properties (1).

Introduce an additional closed set K, $0 \notin K$ and satisfying (1). Similar to Proposition 1 we have

Proposition 6 $\mathcal{B}(L \uplus K) = \mathcal{B}(L) \cap \mathcal{B}(K)$.

Also we get the involutory property:

Proposition 7 $\mathcal{B}(\mathcal{B}(L)) = L$.

We additionally get

Proposition 8 $\mathcal{B}(L \cap K) = \mathcal{B}(L) \uplus \mathcal{B}(K)$.

In analogue with Proposition 5 we may finally get

Proposition 9

$$\begin{aligned} L \cap K &= \mathcal{B}(\mathcal{B}(L) \uplus \mathcal{B}(K)) \\ \text{and } L \uplus K &= \mathcal{B}(\mathcal{B}(L) \cap \mathcal{B}(K)). \end{aligned}$$

4.3 Polarity in DEA

Proposition 5 and Proposition 9 make it possible to substitute the convex union operation in (4) and (5) in the original spaces by an intersection operation in the dual spaces, respectively. Similarly the intersection operation in (4) and (5) may be substituted by a convex union operation in the dual spaces. In particular if the original output sets P_i for $i \in I$ are full dimensional then also all subsequent output sets generated by (4) and (5) become full dimensional. In this case we may use Corollary 4 and dismiss the closure operation in Proposition 5.

Example 2

Consider in the two dimensional output space two DMU's indexed by 1 and 2 and with output vectors $(5, 4)$ and $(6, 1)$, respectively. We thus have
$P_1 = \{(y_1, y_2) \mid (0, 0) \leq (y_1, y_2) \leq (5, 4)\}$
$= \{(y_1, y_2) \geq (0, 0) \mid \frac{1}{5}y_1 \leq 1, \frac{1}{4}y_2 \leq 1\}$ and
$P_2 = \{(y_1, y_2) \mid (0, 0) \leq (y_1, y_2) \leq (6, 1)\}$
$= \{(y_1, y_2) \geq (0, 0) \mid \frac{1}{6}y_1 \leq 1, y_2 \leq 1\}$.
These sets are indicated on Figure 5. We obtain
$\mathcal{A}(P_1) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 5y_1^* + 4y_2^* \leq 1\}$
and $\mathcal{A}(P_2) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 6y_1^* + 1y_2^* \leq 1\}$.
Those sets are also indicated on Figure 5.
Then $\mathcal{A}(P_1) \cap \mathcal{A}(P_2) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 5y_1^* + 4y_2^* \leq 1 \text{ and } 6y_1^* + 1y_2^* \leq 1\}$
implying that
$\mathcal{A}(\mathcal{A}(P_1) \cap \mathcal{A}(P_2))$
$= \{(y_1, y_2) \geq (0, 0) \mid \frac{3}{19}y_1 + \frac{1}{19}y_2 \leq 1, \frac{1}{6}y_1 \leq 1 \text{ and } \frac{1}{4}y_2 \leq 1\}$
$= P_1 \uplus P_2$.
This shows that the antiblocker of $\mathcal{A}(P_1) \cap \mathcal{A}(P_2)$ is equal to $P_1 \uplus P_2$ illustrating (8). The two sets are shown on Figure 5 by thick borderlines.

By definition we have for any $y^* \in \mathcal{A}(P)$ that $\{y \mid yy^* \leq 1\} \supseteq P$. In other words for fixed $y^* \in \mathcal{A}(P)$ then $yy^* \leq 1$ is a valid inequality for P. The inequality has nonnegative coefficients and a positive right hand side, normalized to one. The inequality may be interpreted as a resource constraint in the usual sense. Special interest is devoted the binding valid inequalities. For those a $y \in P$ exists such that $y^*y = 1$. The coefficents y^* may be interpreted as a vector of marginal substitution possibilities, i.e. a price vector for trade offs. In general we have that $\mathcal{A}(P_1) \subseteq \mathcal{A}(P_2)$ if $P_2 \subseteq P_1$. This implies that $\mathcal{A}(P \uplus Q) \subseteq \mathcal{A}(P)$. This expresses that the vector of tradeoffs for the convex union of P and Q is included in the vectors of valid inequalities for any of the two sets. Due to normalization (all valid

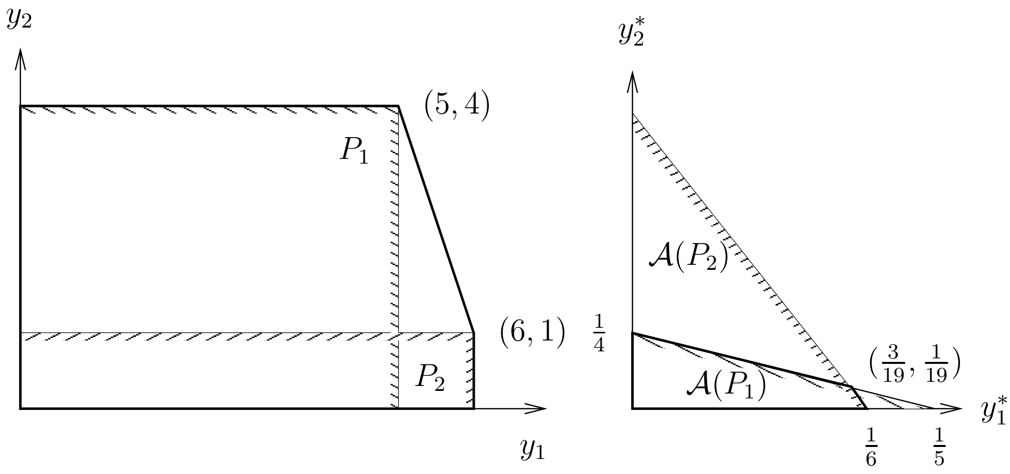

Figure 5: Example.

inequalities have right hand side equal to one) this diminishes the size of the trade off coefficients for $(P) \uplus (Q)$ in comparison with the coefficients for the smaller set P and Q, respectively. This is perhaps not too surprising. But by Proposition 1 we get additionally and more informative that the set of trade off vectors is equal to the intersection of trade off vectors for P and Q.

A similar observation holds for blockers.

5 Productivity index

This section discusses how a productivity index may be calculated in a convex pairs technology. Consider the technology T defined by the unions of the convex pairs (L_i, P_i) with $i \in \mathcal{N}$, i.e.

$$T = \cup_{i \in \mathcal{N}} (L_i, P_i).$$

Assume that all sets L_i and P_i , $i \in \mathcal{N}$ are polyhedral. By the blocking theory developed in Section 4.2 we may therefore rewrite an input set L_i as

$$L_i = \{x \geq 0 \mid B_i x \geq \mathbf{1}\} \tag{10}$$

where B_i is a nonnegative matrix and $\mathbf{1}$ is a vector of ones of conformable dimensions. Similarly by Section 4.1 we have

$$P_i = \{y \geq 0 \mid A_i y \leq \mathbf{1}\} \tag{11}$$

where A_i is a nonnegative matrix and $\mathbf{1}$ is a vector of ones of conformable dimensions.

For a given index i such that $y_0 \in P_i$ the computation of the Farrell index in (3) can be transformed into the following linear programming problem.

$$\begin{array}{ll} \min\limits_{\theta} & \theta \\ \text{s.t.} & B_i x_0 \theta \geq \mathbf{1}. \end{array}$$

The objective is to select an index i giving the minimal value of the above program. This can be formulated as a disjunctive programming problem leading further to a linear programming formulation, see Balas [5]. Using this technique we introduce the additional variables $z_i \in \mathbb{R}_+$ and $\theta_i \in \mathbb{R}$ for $i \in \mathcal{N}$ and consider the linear programming problem.

$$\begin{array}{ll} \min\limits_{\theta_i, z_i} & \sum\limits_{i \in \mathcal{N}} \theta_i \\ \text{s.t.} & B_i x_0 \theta_i - z_i \mathbf{1} \geq 0 \text{ for all } i \in \mathcal{N} \\ & (A_i y_0 - \mathbf{1}) z_i \leq 0 \text{ for all } i \in \mathcal{N} \\ & \sum\limits_{i \in \mathcal{N}} z_i = 1 \\ & z_i \geq 0. \end{array}$$

This program is linearly homogeneous in the z_i variables, $0 \leq z_i \leq 1$. Hence an optimal solution may be found by putting a single variable z_i equal to 1 and the remaining ones to 0. The selected value will correspond to the input-output pair (L_i, P_i) in (3) giving the minimal value of θ.

As an alternative to the closed half-space characterization used in (10) and (11) the input and output sets may be characterized by their extreme points. For this purpose let R_i denote the index set of all extreme points of the input set L_i and let e_{ij} denote an extreme point, $j \in R_i$. Similarly, for the corresponding output set P_i let f_{ij} denote an extreme point together with the index set S_i. In the case of output sets some extreme points may be removed as they may be dominated by other extreme points with larger elements. This is due to free disposability in the output space where dominated extreme points may occur on the axes of coordinates. By introduction of the variables λ_{ij} and μ_{ij} we have

$$\begin{aligned} L_i &= \{x \geq 0 \mid x \geq \sum_{j \in R_i} e_{ij} \lambda_{ij}, \sum_{j \in R_i} \lambda_{ij} = 1, \lambda_{ij} \geq 0\} \text{ and} \\ P_i &= \{y \geq 0 \mid y \leq \sum_{j \in S_i} f_{ij} \mu_{ij}, \sum_{j \in S_i} \mu_{ij} = 1, \mu_{ij} \geq 0\}. \end{aligned}$$

(Strictly speaking the non-negativity condition in L_i is not required as all extreme points e_{ij} are nonnegative in our case). In this framework the

Farrell index in (3) may be calculated by the following linear program.

$$\begin{aligned}
\min \quad & \sum_{i \in \mathcal{N}} \theta_i \\
\text{s.t.} \quad & \sum_{j \in R_i} \lambda_{ij} e_{ij} \leq x_0 \theta_i \text{ for all } i \\
& \sum_{j \in S_i} \mu_{ij} f_{ij} \geq y_0 z_i \text{ for all } i \\
& \sum_{j \in R_i} \lambda_{ij} = z_i \text{ for all } i \\
& \sum_{j \in S_i} \mu_{ij} = z_i \text{ for all } i \\
& \sum_{i \in \mathcal{N}} z_i = 1 \\
& \lambda_{ij}, \mu_{ij} \geq 0 \text{ for all } i, j \\
& z_i \geq 0 \text{ for all } i.
\end{aligned}$$

It should be noted that the extreme points of L_i are all non-negative normals of facets for the antiblocker $\mathcal{A}(L_i)$. By the involutory correspondence stated in Proposition 2 we symmetrically have that the extreme points of $\mathcal{A}(L_i)$ correspond to the non-negative facets of L_i, which again is the minimal set of rows in B_i required to define L_i by (10). For details see Fulkerson [20].

We shall see that the above linear programming model is a generalization of some classical models as well.

With only a single input-output pair, i.e. when $|\mathcal{N}| = 1$, we may delete index i and denote the single pair by (L, P). Assume additionally that the number of non-dominated extreme points are the same in the two sets L and P to be indexed by R. Furthermore, let the vector λ be equal to μ. In this setting we get the classical varying return to scale model studied in [6], in which the extreme points correspond to the decision making units. With the current notation we obtain the usual standard form:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \sum_{j \in R} \lambda_j e_j \leq x_0 \theta \\
& \sum_{j \in R} \lambda_j f_j \geq y_0 \\
& \sum_{j \in R} \lambda_j = 1 \\
& \lambda_j \geq 0.
\end{aligned}
$$

Alternatively, we may assume that each pair (L_i, P_i) are given by the original DMU's illustrated by Figure 1. Then each set of the pair has only one non-dominated extreme point. We may thus remove the index j and additionally assume that $\lambda_i = \mu_i = z_i$. In this setting the model reduces to the FDH model studied in [35] and here stated as a linear programming problem,

$$
\begin{aligned}
\min \quad & \sum_{i \in \mathcal{N}} \theta_i \\
\text{s.t.} \quad & z_i e_i \leq x_0 \theta_i \text{ for all } i \\
& z_i f_i \geq y_0 z_i \text{ for all } i \\
& \sum_{i \in \mathcal{N}} z_i = 1 \\
& z_i \geq 0 \text{ for all } i
\end{aligned}
$$

in which $\mathcal{N}$ is the index set for the DMU's.

In all of the above models we have for simplicity excluded the introduction of slacks to indicate the cases in which an input-output vector is efficient according to the index, i. e. $\theta = 1$, but nevertheless is dominated. Those slacks may however easily be introduced in a traditional manner, see for example Charnes et al. [16].

A similar analysis as above can be done in connection with the establishment of an output oriented efficiency score.

6 Illustrative experiments

The procedure in Section 3.4 consists of iterations numbered by index l. We shall run a couple of experiments with the input index while completing the first two iterations.

In the initialization, we form the n pairs of recession cones generated by free disposability. In the first iteration, we should generate new polyhedra and add these to the ones from the initialization. This, however, is easy. Convex union is simply formed by merging extreme points and intersection is formed by taking maximum and minimum coordinates, respectively, for input and output polyhedra. For the original polyhedra, both input and output polyhedra have exactly one extreme point. This can for our purposes be seen as a degenerate version of polyhedra with two extreme points, the points being equal. The polyhedra generated by convex hull formation in the first iteration will have exactly two extreme points each, while those generated by intersection will have one - again, we can for our purposes see this as a degenerate version of polyhedra with two extreme points.

Thus, the result of the first iteration will be a list of, say, m combinations of input and output polyhedra, each polyhedron given by two extreme points (which might be equal). The actual computation in the first iteration can be done in a spreadsheet and seen as a preprocessing step.

The result of the first iteration is a list of m times 4 extreme points, of those 2 input extreme points x_{i1}, x_{i2} and 2 output extreme points y_{i1}, y_{i2}. In order to find θ we look at

$$\begin{array}{ll} \min & \theta \\ \text{s.t.} & \lambda_{i1}x_{i1} + \lambda_{i2}x_{i2} \leq x_0\theta \\ & \lambda_{j1}x_{j1} + \lambda_{j2}x_{j2} \leq x_0\theta \\ & \lambda_{i1} + \lambda_{i2} = 1 \\ & \lambda_{j1} + \lambda_{j2} = 1 \\ & \mu_{i1} + \mu_{i2} + \mu_{j1} + \mu_{j2} = 1 \\ & \mu_{i1}y_{i1} + \mu_{i2}y_{i2} + \mu_{j1}y_{j1} + \mu_{j2}y_{j2} \geq y_0 \\ & \lambda_{i1}, \lambda_{i2}, \lambda_{j1}, \lambda_{j2}, \mu_{i1}, \mu_{i2}, \mu_{j1}, \mu_{j2} \geq 0. \end{array}$$

This program calculates a productivity index by intersecting two input polyhedra and forming the convex hull of the corresponding output polyhedra. By looping over this for all combinations of i and j, where $i, j = 1, ..., m$, we can calculate the lowest θ value. A similar program is run intersecting two output polyhedra while the convex hull is generated by the corresponding input polyhedra. Finally, the minimal θ value is taken of all values. We have done computation on test data taken from the airline industry. There are $n = 15$ DMU's with 2 inputs and 2 outputs in the selected set. The result was very close to the results obtained by the FDH model, as only a single DMU changed from being efficient to getting the value $\theta = 0.988$.

For a constructed example of how the second iteration can have an effect, we look at Table 1 with 4 DMUs. Clearly DMU 4 is inefficient, even under

DMU i	x_{i1}	x_{i2}	y_{i1}	y_{i2}
1	4	8	5	9
2	5	7	13	5
3	11	3	8	7
4	13	10	7	6

Table 1: Input and output values.

FDH. We find the θ values listed in Table 2. By making a diagram, it is

DMU	FDH	ours	VRS
1	1	1	1
2	1	1	1
3	1	1	1
4	0.846	0.584	0.554

Table 2: θ values.

easily seen that the effect of the second iteration is due to the following combination

$$(H_1 \cap H_2) \uplus H_3 \text{ and } (K_1 \uplus K_2) \cap K_3 \tag{12}$$

which cannot, of course, be reached using a single iteration.

7 Conclusion

This paper is concerned with a new and flexible way to model technologies, the convex pairs approach, and its pratical implementation using primal or dual representations.

We have argued that many interesting instances can be modeled using this approach. In particular, the framework can encompass free disposability hulls with and without partial substitution information like in assurance regions, the convex projection model, as well as the occurrence of gains from specialization and diseconomies of scope and the occurrence of incomparabilities due to investments, categorical variables and ordinal properties. Moreover, we have argued that dual pairs may be particularly useful as building blocks in an iterative evaluation process when the production- and hereby the efficiency- relation is enriched as we go along.

We have shown also how the Farrell index in all instances can be represented as the solution to a linear program.

Extensions are possible. For example one may in a similar way analyze the free replicability model based on Tulkens [35]. This is an integer programming model, for which the duality theory for integer programming should be used, see for example Schrijver [32]. This implies that a polarity analysis may be done in a dual space consisting of Chvatal functions. Work along this line has been done in Agrell and Tind [1]. Indeed the above analysis may be performed on any optimization model in DEA, for which an appropriate duality theory exists with no duality gap.

Before closing, we briefly comment on two possible modifications of the dual pairs approach.

Firstly, we have used the Farrell efficiency as a prototype efficiency calculation. This is natural since the vast majority of productivity analysis papers rely on it. On the other hand, there are many other measures that can be used. Also, once a production model like the dual pairs model has been established, there are many other questions which can be analyzed in addition to the mere efficiency evaluations. An important class of such problem is the reallocation problems where the potential gains from reallocating production with a given technological structure is examined, cf eg. Andersen and Bogetoft [2], Bogetoft, Strange and Thorsen [10], Bogetoft and Wang [12], Brännlund, Färe and Grosskopf [14], Brännlund, Chung, Färe and Grosskopf [13], and Korhonen and Syrjänen [22]. It is worthwhile to observe that the linearization of the resulting proram, e.g. a generalized productivity analysis problem like

$$\max_{(x,y)}\{F(x,y) \mid \ (x,y) \in T = \cup_{i\in I}(L_i, P_i)\}$$

is often possible using the same principles as demonstrated in this paper.

Secondly, it should be noted that the dual pair approach, despite its flexibility, is restrictive in terms of local convexifications in the full input - output space. It is, for example, easy to work with assurance regions on the input side or on the output side, but not on the input and outputs sides simultanously. To introduce such variations, we could instead introduce a straightforward disjunctive convex approach with

$$T = \cup_{i\in I} T_i$$

where $T_i \subseteq R_+^{r\times s}$ is a traditional input-output production set, assumed for example to be convex and free disposable. In this settiung, the usual duality theory could be used on the individual subsets. In Andersen and Bogetoft [3], for example, we have used this approach in an efficiency evaluation and re-allocation context to determine the posssible gains from re-allocating fishery quotas among vessel and assuming that it is possible to vary the scale of operation with constant return to scale as long as the variations are restricted to for example ±30% of the present scale. Similarly, in Bogetoft, Eeckaut and Fried [8], we have used this approah together with an FDH assumption to extend the production possibility set of credit unions. The reason we have relied in this paper on the more restrictive formulation $T = \cup_{i \in I}(L_i, P_i)$ is that it allow us to dispense with convexiifications accross inputs and outputs simultanously, and that it allowed us to introduce into the productivity analysis literature the idea of duality based on blockers and anti-blockers.

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