AMO - Advanced Modeling and Optimization, Volume 7, Number 2, 2005

A Perishable Inventory System with Service Facilities and Negative Customers

B. Sivakumar[1] and G. Arivarignan[2*]
Department of Applied Mathematics and Statistics
Madurai Kamaraj University
Madurai, INDIA.
[1]sivabkumar@yahoo.com, [2]arivarignan@yahoo.com

Abstract

We consider a continuous review perishable (s, S) inventory system with a service facility consisting of a waiting line of finite capacity and a single server. Two types of customers, ordinary and negative, arrive according to a Markovian Arrival Process (MAP). An ordinary customer joins the queue and a negative customer removes some ordinary customers from the queue. Unlike the conventional method of removal rules considered in the literature, we consider the following removal rule : a negative customer at an arrival epoch removes one or more ordinary waiting customers and the number of removals is a random variable depending on the number of waiting customers in the system. The life time of each item, the service time and the lead time of the reorders are all assumed to have independent exponential distributions. The joint probability distribution of the number of customers in the system and the inventory level is obtained in both transient and steady state cases. Various stationary system performance measures are computed and the total expected cost rate is calculated. The results are illustrated numerically.

Keywords : Stochastic Inventory; Markovian Demands; Service Facility; Positive Lead time; Negative Customers.

1 Introduction

In most of the inventory models considered in the Literature, demanded items are directly delivered from the stock (if available). The demands

*The research was supported by National Board for Higher Mathematics, INDIA, research award 48/3/2004/R&D - II/2114.

occurring during the stock-out period are either lost (lost sales) or satisfied only after arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) occurring during the stock-out period are satisfied. See Nahmias (1982), Raafat (1991), Kalpakam and Arivarignan, (1990), Elango and Arivarignan (2003), Liu and Yang (1999) and Yadavalli et al. (2004) for a review.

However, in the case of inventories maintained with service facilities, a demanded item is delivered to the customer after random time. In this case the items are delivered *not* at the time of a demand but after a random time of service causing the formation of queues. This policy necessitates the study of both the inventory level and queue length (joint) distributions.

Recently Berman et al. (1993) have considered an inventory management system with a service facility using one item of inventory for each service. They assumed that both the demand and service are deterministic. Queues can occur only during stock-outs. They determined the optimal order quantity that minimizes the total cost rate.

Berman and Kim (1999) analyzed a similar problem in a stochastic environment where arrival time points of customers form a Poisson process and the service times are exponentially distributed. A logically related model has been studied by He et al. (1998), who analyzed a Markovian inventory - production system, where customer demands are processed by a single machine in a batch of size one. Berman and Sapna (2000) studied an inventory control problem at a service facility requiring one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They analyzed the system with finite waiting room. Under a specified cost structure, the optimal ordering quantity that minimizes the long-run expected cost rate has been derived.

Elango (2001) has considered a Markovian inventory system with instantaneous supply of reorders at a service facility. The service time is assumed to have an exponential distribution with parameter depending on the number of waiting customers. Arivarignan et al. (2002) have extended this model to include exponential lead time. Perumal and Arivarignan (2002) have considered a Markovian inventory system with infinite waiting room. Arivarignan and Sivakumar (2003) have considered an inventory system with arbitrarily distributed demand, exponential service time and exponential lead time.

In all these articles, the authors have considered that the arrival of customers to the service station should join the system unless it is full. But in some applications the arriving customers, instead of joining the system, they remove some of the waiting customers from the system. This type of customer is called negative or irregular customer.

Gelenbe (1991) introduced the concept of negative customers in the queueing network. When a negative customer arrives to the service station, it immediately removes one ordinary customer if present. Research on queueing systems with negative arrivals has been greatly motivated by some practical applications in such as computers, neural networks and communication networks etc.

A real life situation for this includes distribution centers/sales centers of specialized or sophisticated or expensive items such as cars, electronic devices, consumers products (Television sets, refrigerator) wherein the waiting customers may be wooed or taken away by new arriving customers (usually touts of the competitive sales organizations). Instead of finding new customers from a large population, many companies look for the prospective customers at others' sales centers.

For comprehensive analysis of queueing networks with negative arrivals, one may refer to , Gelenbe and Pujolle (1998) and Choa et al. (1999). A recent review can be found in Artalejo (2000). The dependence between positive arrival and negative arrival of customers has been introduced by Shin and Choi (2003).

In this paper we have considered a perishable inventory management at a service facility with negative customers and finite waiting hall for customers. In section 2, we present the assumptions and notations followed in the rest of the paper. The transient and steady state analysis of the model are presented in section 3. In section 4, we derive various measures of system performance in the steady state and the total expected cost rate is calculated in section 5. Section 6 presents the cost analysis of the model using numerical examples.

2 Model Description

We consider a service facility with a maximum inventory of S units, single server and finite waiting hall of size N including the one receiving the service. Thus a customer who finds N customers in the waiting hall does not join the queue. Two types of customers, ordinary and negative, are assumed to arrive according to a MAP. An ordinary customer, on arrival, joins the queue and the negative customer dose not join the queue and also removes one or more positive customers in the system, if any present; otherwise the negative customer does not affect the system behavior. We assume that the number of ordinary customers removed at an arrival epoch is a random variable Y with following probability function depends on the number of waiting customers, $p_{ik} = Pr\{Y = k\},\ \ k = 1, 2, \ldots, i; \sum_{j=1}^{i} p_{ij} = 1;\ i = 1, 2, \ldots, N$. The waiting customers receive their service one by one and they demand single item. The service time has exponential distribution with parameter $\mu(> 0)$. An order for $Q(= S - s\ >\ s + 1)$ items is placed whenever the inventory level drops to s and the items are received only after a random time which has exponential distribution with parameter $\beta(> 0)$. The life time of each item has negative exponential distribution with parameter $\gamma(> 0)$.

For the description of the arrival process, we use the description of MAP as given in Lucantoni et al. (1990). Consider a continuous-time Markov chain on the state space $1, 2, \ldots, M$. The arrival process is constructively defined as follows. When the chain enters a state $i,\ 1 \leq i \leq M$, it stays for an exponential time with parameter θ_i. At the end of sojourn time, there are three possible transitions: with probabilities $a_{ij},\ 1 \leq j \leq M$, the chain enters the state j when an ordinary customer arrives; with probabilities $b_{ij},\ 1 \leq j \leq M$, the chain enters the state j when a negative customer arrives; with probabilities $c_{ij},\ 1 \leq j \leq M, i \neq j$ the transitions corresponds to no arrival and the state of the chain is j. Note that the Markov chain can go from state i to state i only through an arrival. We define the square matrices $D_k,\ k = -1, 0, 1$, of size M by $[D_0]_{ii} = -\theta_i$ and $[D_0]_{ij} = \theta_i c_{ij},\ i \neq j$, $[D_1]_{ij} = \theta_i a_{ij}$ and $[D_{-1}]_{ij} = \theta_i b_{ij},\ 1 \leq i, j \leq M$. It is easily seen that $D = D_{-1} + D_0 + D_1$ is an infinitesimal generator of a continuous-time Markov chain. We assume that D is irreducible and $D_0\mathbf{e} \neq \mathbf{0}$, where $\mathbf{e}$ is the column vector whose components are all 1's with appropriate dimension.

Let ζ be the stationary probability vector of the continuous-time Markov chain with generator D. That is, ζ is the unique probability vector satisfying

$$\zeta D = \mathbf{0},\ \zeta\mathbf{e} = 1.$$

Let η be the initial probability vector of the underlying Markov chain governing the MAP. Then, by choosing η appropriately we can model the time origin to be

1. an arbitrary arrival point;
2. the end of an interval during which there are at least k arrivals;
3. the point at which the system is in specific state such as the busy period ends or busy period begins;

The important case is the one where we get the stationary version of the MAP by $\eta = \zeta$. The constant $\lambda = \zeta(D_{-1} + D_1)\mathbf{e}$, referred to as the

fundamental rate gives the expected number of arrivals per unit of time in the stationary version of the MAP. The quantity $\lambda_1 = \zeta D_1 \mathbf{e}$ and $\lambda_{-1} = \zeta D_{-1}\mathbf{e}$, gives the arrival rate of ordinary and negative customers respectively. Note that $\lambda = \lambda_1 + \lambda_{-1}$.

For further details on MAP and their usefulness in Stochastic modelling, the reader may refer to Chapter 5 in Neuts (1989), Ramaswami (1981), Lucantoni (1991, 1993), Latouche and Ramaswami (1999), Li and Li (1994), Lee and Jeon (2000), Chakravarthy and Dudin (2003) and references therein for a detailed introduction of the MAPs. Some recent reviews can be found in Neuts (1995) and Chakravarthy (1999).

Notations

$[A]_{ij}$	: $(i,j)-$th element of the matrix A.
A^*_α	: Laplace transform of any arbitrary matrix $A(t)$, for $Re\ \alpha > 0$.
$\mathbf{0}$	: zero matrix.
I	: an identity matrix.
$H(x)$	$= \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$
E_1	$= \{0, 1, \ldots, S\}$.
E_2	$= \{0, 1, \ldots, N\}$.
E_3	$= \{1, 2, \ldots, M\}$.
E	$= E_1 \times E_2 \times E_3$.

3 Analysis

Let $L(t), X(t)$ and $J(t)$, respectively, denote the inventory level, the number of customers (waiting and being served) in the system and phase of the arrival process at time t. From the assumption made on the input and output process it can be shown that the process $\{\ (L(t), X(t), J(t)); t \geq 0\}$, on the state space E, is a Markov process.

The infinitesimal generator of this process is given by

$$[A]_{ij} = \begin{cases} A_i, & j = i, \qquad\quad i = 0, 1, \ldots, S, \\ B_i, & j = i - 1, \quad i = 1, 2, \ldots, S, \\ C, & j = i + Q, \quad i = 0, 1, \ldots, s, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

where

$$\text{For } i = 1, 2, \ldots, S,$$

$$[B_i]_{kl} = \begin{cases} i\gamma I, & l = k; \quad k = 0, 1, \ldots, N; \\ \mu I, & l = k-1; \quad k = 1, 2, \ldots, N; \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

$$[C]_{kl} = \begin{cases} \beta I, & l = k; \quad k = 0, 1, \ldots, N; \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

$$[A_0]_{kl} = \begin{cases} D_1, & l = k+1; & k = 0, 1, \ldots, N-1; \\ p_{k(l+1)} D_{-1}, & l = 0, 1, \ldots, k-1; & k = 1, 2, \ldots, N; \\ D_0 + D_{-1} - \beta I, & l = k; & k = 0; \\ D_0 - \beta I, & l = k; & k = 1, 2, \ldots, N-1; \\ D_0 + D_1 - \beta I, & l = k; & k = N; \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

$$\text{For } i = 1, 2, \ldots, S,$$

$$[A_i]_{kl} = \begin{cases} D_1, & l = k+1; & k = 0, 1, \ldots, N-1; \\ p_{k(l+1)} D_{-1}, & l = 0, 1, \ldots, k-1; & k = 1, 2, \ldots, N; \\ D_0 + D_{-1} - H(s-i)\beta I - \mu I, & l = k; & k = 0; \\ D_0 - H(s-i)\beta I - \mu I, & l = k; & k = 1, 2, \ldots, N-1; \\ D_0 + D_1 - H(s-i)\beta I - \mu I, & l = k; & k = N; \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It may be noted that all the sub matrices are square matrices of order $(N+1)M$.

3.1 Transient Analysis

Define $\psi((i,k,m),(j,l,n);t) =$

$$Pr\{L(t) = j, X(t) = l, J(t) = n | L(0) = i, X(0) = k, J(0) = m\},$$
$$(i,k,m),(j,l,n) \in E.$$

Consider the matrix

$$\Psi(t) = (\ (\ \psi((i,k,m),(j,l,n);t)\ )\ ).$$

The Kolmogorov backward differential equation satisfied by $\Psi(t)$, is given by

$$\Psi'(t) = \Psi(t)A$$

and the solution of the above differential equation is given by

$$\Psi(t) = e^{At}$$

where e^{At} represents $I + At + \frac{A^2t^2}{2!} + \cdots$.

Alternatively, if we use the notation Ψ^*_α to denote the Laplace transform of the function (or matrix) Ψ then we have

$$\Psi^*_\alpha = (\alpha I - A)^{-1}.$$

The matrix $\Psi = (\alpha I - \Theta)$ has the following block partitioned form

$$[\Psi]_{ij} \quad = \quad \begin{cases} -B_i, & j = i-1, \quad i = 1, 2, \ldots, S, \\ -C, & j = i+Q, \quad i = 0, 1, \ldots, s, \\ \alpha I - A_i, & j = i, \qquad\quad i = 0, 1, \ldots, S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It may be observed that $(\alpha I - \Theta)$ is an almost upper-triangular matrix in block partitioned form. That is

$$[\Psi]_{ij} \quad = \mathbf{0}, \quad j = 0, 1, \ldots, i-2; \quad i = 2, 3, \ldots, S.$$

In order to compute Ψ^{-1} we proceed as follows : Consider an upper-triangular matrix U and an almost upper-triangular matrix R having the form

$$U = \begin{pmatrix} U_{00} & U_{01} & U_{02} & \cdots & U_{0S} \\ \mathbf{0} & U_{11} & U_{12} & \cdots & U_{1S} \\ \mathbf{0} & \mathbf{0} & U_{22} & \cdots & U_{2S} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & U_{SS} \end{pmatrix}$$

with $U_{ii} = I, \quad i = 1, 2, \ldots, S$, and

$$R = \begin{pmatrix} R_{00} & R_{01} & R_{02} & \cdots & R_{0(S-1)} & R_{0S} \\ -B_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -B_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -B_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & -B_S & \mathbf{0} \end{pmatrix}.$$

such that $\Psi U = R$. We find the sub matrices $[U]_{ij}$ and $[R]_{0j}$ by computing the product ΨU and equating it to R. The (i,j)-th sub matrix of ΨU is given by

$$\begin{cases} -B_i U_{jj}, & i = 1,2,\ldots,S; \quad j = i-1, \\ F_i U_{ij} - B_i U_{(i-1)j}, & i = s+1,\ldots,S; \quad j = i, i+1,\ldots,S, \\ & \text{ or} \\ & i = 1,2,\ldots,s; \quad j = i, i+1,\ldots,Q-1, \\ F_0 U_{0j}, & i = 0; \quad j = 0,1,2,\ldots,Q-1, \\ -CU_{Qj} + F_0 U_{0j}, & i = 0; \quad j = Q, Q+1,\ldots,S, \\ -CU_{(Q+i)j} + F_i U_{ij} - B_i U_{(i-1)j}, & i = 1,2,\ldots,s; \quad j = Q+1, Q_1+2\ldots,S. \end{cases}$$

By equating the submatrices of ΨU to the corresponding elements of R, we get

$$[U]_{ij} = \begin{cases} (B_{i+1}^{-1}F_{i+1})(B_{i+2}^{-1}F_{i+2})\cdots(B_j^{-1}F_j), & i = 0,1,2,\ldots,j+1; \\ & j = 1,2,\ldots,Q, \\ & \text{or} \\ & i = j-Q, j-Q+1,\ldots,S; \\ & j = Q+1, Q+2,\ldots,S, \\ -B_{i+1}^{-1}CU_{(i+Q+1)j} + B_{i+1}^{-1}F_{i+1}U_{(i+1)j}, & i = 0,1,\ldots,j-Q-1; \\ & j = Q+1, Q+2,\ldots,S \end{cases}$$

and

$$[R]_{0j} = \begin{cases} F_0, & j = 0, \\ F_0 U_{0j}, & j = 1,2,\ldots,Q-1, \\ -CU_{Qj} + F_0 U_{0j}, & j = Q, Q+1,\ldots,S. \end{cases}$$

It should be noted that B_α^{-1} exists, since B_α is a lower-triangular matrix with zero elements except in the main and lower diagonals.

The equation $\Psi U = R$ implies

$$\begin{aligned} (\Psi U)^{-1} &= R^{-1} \\ U^{-1}\Psi^{-1} &= R^{-1} \\ \Psi^{-1} &= UR^{-1}. \end{aligned}$$

It can be verified that the inverse of R is given by,

$$R^{-1} = \begin{pmatrix} \mathbf{0} & -B_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -B_2^{-1} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B_{S_1}^{-1} \\ R_{0S}^{-1} & R_{0S}^{-1}R_{00}B_1^{-1} & R_{0S}^{-1}R_{01}B_2^{-1} & \cdots & R_{0S}^{-1}R_{0(S_1-2)}B_{S_1-1}^{-1} & R_{0S}^{-1}R_{0(S-1)}B_S^{-1} \end{pmatrix}$$

From $\Psi U = R$, we obtain

$$\begin{aligned} \det(\Psi U) &= \det(R) \\ \det(\Psi)\det(U) &= \det(R_{0S})\det(-B_1)\det(-B_2)\cdots\det(-B_S). \end{aligned}$$

Since U is an upper-triangular matrix and the B_i's are lower-triangular matrices, their determinant different from zero. Hence $\det(R_{0S}) \neq 0$. This proves the existence of R_{0S}^{-1}.

From $\Psi^{-1} = UR^{-1}$, we compute the $(i,j)-$ th submatrix (denoted by Ψ^{ij}) of $\Psi^{-1} = (\alpha I - \Theta)^{-1}$ we obtain

$$\Psi^{ij} = \begin{cases} U_{iS}R_{0S}^{-1}R_{0(j-1)}B_j^{-1}, & i = j, j+1, \ldots, S; \quad j = 1, 2, \ldots, S, \\ U_{iS}R_{0S}^{-1}, & i = 0, 1, 2, \ldots, S; \quad j = 0, \\ U_{iS}R_{0S}^{-1}R_{0(j-1)} - U_{i(j-1)}B_j^{-1}, & i = 0, 1, \ldots, S-1; \quad j = i+1, \ldots, S. \end{cases}$$

3.2 Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{(L(t), X(t), J(t)), t \geq 0\}$ on the finite state space E is irreducible. Hence the limiting distribution

$$\boldsymbol{\Phi} = (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \ldots, \boldsymbol{\phi^{S-1}}, \boldsymbol{\phi^S})$$

with $\boldsymbol{\phi^m} = \Big(\phi^{(m,0,1)}, \phi^{(m,0,2)}, \ldots, \phi^{(m,0,M)}, \phi^{(m,1,1)}, \phi^{(m,1,2)}, \ldots, \phi^{(m,1,M)}, \ldots\ldots,$ $\phi^{(m,N,1)}, \phi^{(m,N,2)}, \ldots, \phi^{(m,N,M)}\Big)$, $m = 0, 1, \ldots, S$, where $\phi^{(i,k,l)}$ denotes the steady state probability for the state (i,k,l) of the process, exists and is given by

$$\boldsymbol{\Phi} A = \mathbf{0} \quad \text{and} \quad \sum\sum_{(i,k,m)}\sum \phi^{(i,k,m)} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

$$
\begin{aligned}
\boldsymbol{\phi^{i}} B_i + \boldsymbol{\phi^{i-1}} A_{i-1} &= \mathbf{0}, \qquad i = 1, 2, \ldots, Q, \\
\boldsymbol{\phi^{i}} B_i + \boldsymbol{\phi^{i-1}} A_{i-1} + \boldsymbol{\phi^{i-1-Q}} C &= \mathbf{0}, \qquad i = Q+1, \ldots, S, \\
\boldsymbol{\phi^{S}} A_S + \boldsymbol{\phi^{s}} C &= \mathbf{0}.
\end{aligned}
$$

After long simplifications, the above equations, except the last one, yields

$$
\begin{aligned}
\boldsymbol{\phi^{i}} &= (-1)^i \boldsymbol{\phi^{0}} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} && i = 1, 2, \ldots, Q, \\
&= (-1)^i \boldsymbol{\phi^{0}} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} - \boldsymbol{\phi^{0}} C B_i^{-1} && i = Q+1, \\
&= (-1)^i \boldsymbol{\phi^{0}} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} \\
&\quad + (-1)^{i-Q} \boldsymbol{\phi^{0}} C B_{Q+1}^{-1} A_{Q+1} \cdots A_{i-1} B_i^{-1} \\
&\quad + (-1)^{i-Q} \boldsymbol{\phi^{0}} \left\{ \sum_{l=1}^{i-Q-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \right. \\
&\qquad \left. \left(B_{Q+l+1}^{-1} A_{Q+l+1} \cdots B_i^{-1} \right) \right\} && i = Q+2, Q+3, \ldots, S,
\end{aligned}
$$

where $\boldsymbol{\phi^{0}}$ can be obtained by solving,

$\boldsymbol{\phi^{S}} A_S + \boldsymbol{\phi^{s}} C = \mathbf{0}$ and $\sum_{i=0}^{S} \boldsymbol{\phi^{i}} \mathbf{e} = 1$,

that is

$$
\begin{aligned}
&\boldsymbol{\phi^{0}} \Big[(-1)^S A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{S-1} B_S^{-1} A_S + (-1)^s C B_{Q+1}^{-1} A_{Q+1} \cdots A_{S-1} B_S^{-1} A_S \\
&+ (-1)^s \left\{ \sum_{l=1}^{s-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \left(B_{Q+l+1}^{-1} A_{Q+l+1} \cdots B_S^{-1} \right) \right\} A_S \\
&+ (-1)^s A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{s-1} B_s^{-1} C \Big] = \mathbf{0},
\end{aligned}
$$

and

$$
\begin{aligned}
&\boldsymbol{\phi^{0}} \Bigg[I + \sum_{i=1}^{Q} (-1)^i A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} + (-1)^{Q+1} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_Q B_{Q+1}^{-1} \\
&- C B_{Q+1}^{-1} + \sum_{i=Q+2}^{S} \Big\{ (-1)^i A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} + (-1)^{i-Q} C B_{Q+1}^{-1} A_{Q+1} \cdots A_{i-1} B_i^{-1} \\
&+ (-1)^{i-Q} \left\{ \sum_{l=1}^{i-Q-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \left[B_{Q+l+1}^{-1} A_{Q+l+1} \cdots B_i^{-1} \right] \right\} \Big\} \Bigg] \mathbf{e} = 1.
\end{aligned}
$$

4 System Performance Measures

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time is presented in the next section.

4.1 Expected Inventory Level

The expected inventory level ξ_I is given by,

$$\xi_I = \sum_{i=1}^{S} i \sum_{j=0}^{N} \phi^{(i,j)} \mathbf{e}.$$

4.2 Mean Reorder Rate

The expected reorder rate ξ_R is given by,

$$\xi_R = \mu \sum_{j=1}^{N} \phi^{(s+1,j)} \mathbf{e} + (s+1)\gamma \sum_{j=0}^{N} \phi^{(s+1,j)} \mathbf{e}.$$

4.3 Mean Perishable Rate

The mean perishable rate ξ_P is given by

$$\xi_P = \sum_{i=1}^{S} \sum_{j=0}^{N} i\gamma \phi^{(i,j)} \mathbf{e}.$$

4.4 Mean Balking Rate

Let ξ_B denote the mean balking rate of the customers is given by

$$\xi_B = \frac{1}{\lambda_1} \sum_{i=0}^{S} \phi^{(i,\ N)} D_1 \mathbf{e}.$$

4.5 Mean Rate of Loss due to Negative Arrival

Let ξ_{NE} denote the mean rate of loss due to negative arrival is given by

$$\xi_{NE} = \frac{1}{\lambda_{-1}} \sum_{i=0}^{S} \sum_{j=1}^{N} \left[\sum_{k=1}^{j} p_{ik} \phi^{(i,j)} D_{-1} \mathbf{e} \right].$$

4.6 Mean Waiting Time

Let $\overline{W}$ denote the mean waiting time of the customers. Then by Little's formula

$$\overline{W} = \frac{L_1}{\lambda_e}$$

where

$$L_1 = \sum_{i=0}^{S}\sum_{j=0}^{N} j\phi^{(i,j)}\mathbf{e},$$

and the effective arrival rate (Ross [30]), λ_e is given by

$$\lambda_e = \frac{1}{\lambda_1}\sum_{i=0}^{S}\sum_{j=0}^{N-1}\phi^{(i,j)}D_1\mathbf{e}.$$

5 Cost Analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S,s,N) = c_s\xi_R + c_h\xi_I + c_p\xi_P + c_b\xi_B + c_{ne}\xi_{NE} + c_w\overline{W},$$

where

c_s : Setup cost per order.
c_h : The inventory carrying cost per unit item per unit time.
c_p : Perishable cost per unit item per unit time.
c_b : Balking cost per customer per unit time.
c_{ne} : Loss per unit time due to arrival of a negative customer.
c_w : Waiting time cost of a customer per unit time.

Substituting ξ's and $\overline{W}$ we get

$$TC(S,s,N) = c_s\left(\mu\sum_{j=1}^{N}\phi^{(s+1,j)}\mathbf{e} + (s+1)\gamma\sum_{j=0}^{N}\phi^{(s+1,j)}\mathbf{e}\right)$$

$$+c_h \sum_{i=1}^{S} i \sum_{j=0}^{N} \phi^{(i,j)}\mathbf{e} + c_p \sum_{i=1}^{S} \sum_{j=0}^{N} i\gamma \phi^{(i,j)}\mathbf{e}$$
$$+c_b \frac{1}{\lambda_1} \sum_{i=0}^{S} \phi^{(i,\ N)} D_1 \mathbf{e} + c_{ne} \frac{1}{\lambda_{-1}} \sum_{i=0}^{S} \sum_{j=1}^{N} \left[\sum_{k=1}^{j} p_{ik} \phi^{(i,j)} D_{-1} \mathbf{e} \right]$$
$$+c_w \frac{\sum\limits_{i=0}^{S} \sum\limits_{j=0}^{N} j\phi^{(i,j)}\mathbf{e}}{\frac{1}{\lambda_1} \sum\limits_{i=0}^{S} \sum\limits_{j=0}^{N-1} \phi^{(i,j)} D_1 \mathbf{e}}.$$

Since the computation of the ϕ's are recursive, it is quite difficult to show analytically the convexity of the total expected cost rate. However we present the following examples to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

6 Numerical Illustrations

In this section we discuss some numerical examples that reveal the convexity of the total expected cost rate. For all the examples we assume that the arrival process is Hyperexponential.

The table 1 gives the total expected cost rate for various combinations of S and s when fixed values for other parameters and costs are assumed. They are

$$D_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 7.2 & 0.8 \\ 0.72 & 0.08 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1.8 & 0.2 \\ 0.18 & 0.02 \end{pmatrix}.$$

$N = 4, \gamma = 0.5, \mu = 10, \beta = 0.8, c_s = 45, c_h = 0.1, c_b = 5, c_p = 1.2, c_w = 40, c_n = 2, p_{ij} = 1/i$, where $i = 1, 2, \ldots, 4,\ j = 0, 1, \ldots, i-1$.

Table 1. Total Expected Cost Rate as a Function of S and s

	Total Expected Cost Rate					
s / S	5	6	7	8	9	10
50	59.1765	59.0462	<u>59.0435</u>	59.1441	59.3303	59.5889
51	59.1653	59.0334	<u>59.0277</u>	59.1238	59.3041	59.5554
52	**59.1605**	**59.0274**	<u>59.0191</u>	59.1112	59.2862	59.5309
53	59.1616	59.0277	<u>**59.0172**</u>	**59.1059**	59.2761	59.5148
54	59.1685	59.0340	<u>59.0217</u>	59.1074	**59.2733**	59.5066
55	59.1806	59.0459	<u>59.0322</u>	59.1151	59.2773	**59.5058**
56	59.1978	59.0631	<u>59.0482</u>	59.1289	59.2877	59.5117

In the above table, the minimum cost rate for each row is underlined and the minimum cost rate for each column is shown in bold.

The table 2 gives the total expected cost rate for various combinations of S and N. We have assumed constant values for other parameters and costs. Namely

$$D_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 4.5 & 0.5 \\ 0.45 & 0.05 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 4.5 & 0.5 \\ 0.45 & 0.05 \end{pmatrix}.$$

$s = 3, \gamma = 0.5, \mu = 10, \beta = 0.8, c_s = 5, c_h = 0.1, c_b = 75, c_p = 1.2,$ $c_w = 50, c_n = 1.5, p_{ij} = 1/i$, where $i = 1, 2, \ldots, N,\ j = 0, 1, \ldots, i-1$.

Table 2. Total Expected Cost Rate as a Function of S and N

	Total Expected Cost Rate					
N / S	7	8	9	10	11	12
20	25.75106	<u>25.74979</u>	25.75125	25.75285	25.75402	25.75476
21	**25.74849**	<u>**25.74724**</u>	**25.74865**	**25.75019**	**25.75132**	**25.75203**
22	25.76205	<u>25.76083</u>	25.76220	25.76370	25.76479	25.76548
23	25.78941	<u>25.78821</u>	25.78955	25.79102	25.79208	25.79275
24	25.82865	<u>25.82747</u>	25.82879	25.83022	25.83126	25.83191
25	25.87817	<u>25.87702</u>	25.87832	25.87972	25.88074	25.88137

The table 3 gives the total expected cost rate for various combinations of s and N, by assuming fixed values for other parameters and costs. Namely

$$D_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 6.3 & 0.7 \\ 0.63 & 0.07 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 2.7 & 0.3 \\ 0.27 & 0.03 \end{pmatrix}.$$

$S = 25, \gamma = 0.2, \mu = 30, \beta = 10, c_s = 1, c_h = 0.01, c_b = 1.7, c_p = 0.1,$ $c_w = 10, c_n = 1.8, p_{ij} = 1/i$, where $i = 1, 2, \ldots, N,\ j = 0, 1, \ldots, i-1$.

Table 3. Total Expected Cost Rate as a Function of s and N

	Total Expected Cost Rate				
N / s	1	2	3	4	5
1	2.307515	<u>2.294122</u>	2.308558	2.320529	2.324296
2	**2.290509**	**<u>2.289324</u>**	**2.308187**	**2.318326**	**2.321373**
3	2.338927	<u>2.310476</u>	2.332163	2.341358	2.344080
4	2.375302	<u>2.345016</u>	2.365690	2.374449	2.377012
5	2.414083	<u>2.383362</u>	2.403645	2.412215	2.414704

Since, in each table, the value which is both underlined and in bold is smaller than the row minima and column minima, we have obtained a (local) optima for the associated cost function of the table. The numerical values in each table also exhibit the convexity of the cost function in the domain of its arguments.

7 Conclusions

In this paper we have described a perishable inventory management at service facilities with negative customers. This model is suitable for the cases where the negative customer an arrival removes a random number of ordinary customers from the system. We have derived the joint probability distribution of the inventory level and the number of customers in the transient and steady state cases. We have also derived various stationary measures of system performances and computed total expected cost rate. We have provided numerical examples to illustrate the convexity of the total expected cost rate in steady state.

References

[1] Arivarignan, G., Elango, C. and Arumugam, N., (2002) *A Continuous Review Perishable Inventory Control System at Service Facilities*, Advances in Stochastic Modelling, in: J. R.Artalejo, A. Krishnamoorthy,(ED.)., Notable Publications, Inc., New Jersey, USA, pp.19 - 40.

[2] Arivarignan, G. and Sivakumar, B., (2003) *Inventory System with Renewal Demands at Service Facilities,* Stochastic Point Processes, in: S.K. Srinivasan and A. Vijayakumar (ED.), Narosa Publishing House, New Delhi, India, pp.108 - 123.

[3] Artalejo, J. R., (2000) *G-networks : A Versatile Approach for Work Removal in Queueing Networks*, European J. Oper. Res. vol.126, pp.233 -249.

[4] Boucherie, R. J. and Boxma, O. J., (1996) *The Workload in the M/G/1 Queue with Work Removal*, Probability in the Engineering and Informational Sciences, vol. 10, pp.261 - 277.

[5] Berman, O. Kaplan, E. H. and Shimshak, D. G., (1993) *Deterministic Approximations for Inventory Management at Service Facilities*, IIE Transactions, vol. 25, pp.98-104.

[6] Berman, O. and Kim, E., (1999) *Stochastic Inventory Policies for Inventory Management of Service Facilities,* Stochastic Models, vol. 15, pp.695-718.

[7] Berman, O. and Sapna, K. P., (2000) *Inventory Management at Service Facilities for Systems with Arbitrarily Distributed Service Times,* Stochastic Models, vol. 16, pp.343 - 360.

[8] Chakravarthy, S., (2001) *The Batch Markovian Arrival Process: A Review and Future Work*, Advances in Probability and Stochastic Processes, A. Krishnamoorthy et al., (ED.), Notable publication, Inc., New Jersey, USA, pp.21 - 49.

[9] Chakravarthy, S. and Dudin, A., (2003), *Analysis of a Retrial Queueing Model with MAP Arrivals and Two types of Customers*, Mathematical and Computer Modelling, vol. 37, pp.343 - 363.

[10] Chao, X., Miyazawa,M. and Pinedo, M., (1999) *Queueing Networks : Customers, Signal and Product Form Solutions*, Wiley, Chichester.

[11] Elango, C., (2001) *A Continuous Review Perishable inventory System at Service Facilities*, Ph. D., Thesis, Madurai Kamaraj University, Madurai.

[12] Elango, C. and Arivarignan, G., (2003) *A Continuous Review Perishable Inventory Systems with Poisson Demand and Partial Backlogging,*

Statistical Methods and Practice : Recent Advances, in : N. Balakrishnan, N. Kannan and M.R.Srinivasan, (ED.), Narosa Publishing House, New Delhi, pp.343 - 355.

[13] Gelenbe, E., (1991) *Production-form Queueing Networks with Negative and Positive Customers*, J.Appl.Prob. vol.30, pp.742 - 748, 1991.

[14] Gelenbe, E. and Pujolle, G., (1998) *Introduction to Queueing Networks*, Second Edition,Wiley Chichester.

[15] He, Q-M, Jewkes, E. M., and Buzacott, J., (1998), *An Efficient Algorithm for Computing the Optimal Replenishment Policy for an Inventory-Production System*, in: A. Alfa and S. Chakravarthy (ED.), Advances in Matrix Analytic Methods for Stochastic Models, Notable Publications, New Jersey, USA, pp.381 - 402.

[16] Kalpakam, S. and Arivarignan, G., (1990), *Inventory System with Random Supply Quantity,* OR Spektrum, vol. 12, pp.139-145.

[17] Latouche, G. and Ramaswami, V., (1999) *Introdution to Matrix Analytic Methods in Stochastic Modelling*, SIAM, Philadelphia.

[18] Lee, G. and Jeon, J., (2000), *A New Approach to an $N/G/1$ Queue*, Queueig Systems, vol. 35, pp.317 - 322.

[19] Li, Q. L. and Li, J. J., (1994), *An Application of Markov-modulated Poisson Process to Two-unit Series Repairable System*, Journal of Engereeing Mathematics, vol. 11, pp.56 - 66.

[20] Liu, L. and Yang, T., (1999), *An (s, S) Random Lifetime Inventory Model with a Positive Lead Time,* European Journal of Operational Research, vol. 113, pp.52-63.

[21] Lucantoni, D. M., (1991), *New Results on the Single Server Queue with a Batch Markovian Arrival Process*, Stochastic Models, vol. 7, pp.1 - 46.

[22] Lucantoni, D. M., (1993), *The $BMAP/G/1$ Queue : a Tutorial*, Models and Techniques for Performance Evaluation of Computer and Communications Systems, L. Donatiello and R. Nelson (ED.), Springer-Verlag, New York, pp.330 - 358.

[23] Lucantoni, D.M., Meier-Hellstern, K.S., and Neuts, M.F., (1990) *A Single Server Queue with Server Vacations and a Class of Non-renewal Arrival Processes,* Advances in Applied Probability, 22, pp.676 - 705.

[24] Nahmias, S., (1982), *Perishable Inventory Theory: A Review,* Operations Research, vol. 30, pp.680-708.

[25] Neuts, M. F., (1989), *Structured Stochastic Matrices in $M/G/1$ Type and Their Applications*, Marcel Dekker.

[26] Neuts, M. F., (1995), *Matrix-analytic Methods on the Theory of Queues* in Advances in Queueing : Theory, Methods and Open Problems, J.H. Dshalalow (ED.), CRC, pp.265 - 292.

[27] Perumal, V. and Arivarignan, G., (2002), *A Continuous Review Perishable Inventory System at Infinite Capacity Service Facilities,* ANJAC Journal of Science, vol. 1, pp.37 - 45.

[28] Raafat, F., (1991), *A Survey of Literature on Continuously Deteriorating Inventory Models*, Journal of Operational Research Society, vol. 42, pp.27 - 37.

[29] Ramaswami, V., (1981), *The $N/G/1$ Queue and Its Detailed Analysis*, Adv. Appl. Prob., vol. 12, pp.222 - 261.

[30] Ross, S. M., (2000), *Introduction to Probability Models*, Harcourt Asia PTE Ltd, Singapore.

[31] Shin, Y.W. and Choi, B.D., (2003) *A Queue with Positive and Negative Arrivals Governed by a Markov Chain,* Probability in the Engineering and Informational Sciences, 17, pp.487 - 501.

[32] Yadavalli V. S. S., CdeW Van Schoor, Strashein, J. J., Udayabakaran, S., (2004), *A Single Product Perishing Inventory Model with Demand Interaction,* ORiON, vol. 20, pp.109 - 124.