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Duality in MIP by Branch-and-Cut

Generating Dual Price Functions Using Branch-and-Cut

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Abstract

This paper treats duality in Mixed Integer Programming (MIP in short) using a combination of the branch-and-bound and the cutting plane methods. A dual of a MIP problem includes a dual price function F, that plays the same role as the dual variables in Linear Programming (LP in the following).

The price function is generated while solving the primal problem. However, different to the LP dual variables (apart from degeneracy), the characteristics of the dual price function depend on the algorithmic approach used to solve the MIP problem. Thus, the cutting plane approach provides nondecreasing and superadditive price functions while branch-and-bound algorithm generates piecewise linear, nondecreasing and convex price functions.

Here a hybrid algorithm based on branch-and-cut is investigated, and a price function for that algorithm is established. This price function presents a generalization of the dual price functions obtained by either the cutting plane or the branch-and-bound method.

1 Introduction

While duality is a well treated subject in linear programming (LP) (e.g. see Gass (1985)) and pure integer programming (IP) (Wolsey (1981), there are only few results concerning mixed integer programming (MIP)[2]. The characterization of a dual of an MIP problem

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[2]Nemhauser and Wolsey (1988) have stated the dual of MIP for superadditive dual function. Nemhauser and Wolsey (1985) have investigated duality for 0-1 MIP problems.

depends on the algorithm used to solve the primal problem. One of the most popular algorithms for MIP problems is the branch-and-cut method. This text provides a result about the price function of a general MIP problem when branch-and-cut algorithm is applied. Since the branch-and-cut method is a hybrid method, that uses the branch-and-bound and the cutting plane approaches simultaneously, the obtained price function will also be a general dual price function for the two algorithms.

Duality in mathematical programming is used in a variety of applications. Apart from conceptual interest it provides interesting economic interpretations of the problem. Moreover, using dual information usually improves the performance of an algorithm. While algorithms for LP produce unique dual programs (apart from degenerating programs), that are relatively easy to obtain, IP algorithms generate a dual function whose characteristics depend on the method used to solve the primal IP problem. Wolsey (1981) characterized this function for the branch-and-bound and the cutting plane methods.

Since the formulation of an MIP dual also contains a dual price function as in the case of IP problems, these results are used as inspiration to creating a dual price function for the general MIP problem, when the branch-and-cut algorithm is used.

2 MIP Problems

MIP deals with models, where a linear objective function has to be maximized (or minimized) subject to a set of linear inequality or equality constraints, and where some of the variables are integer.
A classical mixed integer program can be written as:

$$\begin{array}{lll} & \max & cx + dy \\ (P_{MIP}) & s.t. & Ax + By \leq b \\ & & x \in \mathbb{Z}^n_+, y \in \mathbb{R}^m_+ \end{array} \tag{1}$$

Here, x represents the integer variables while y represents the continuous variables. $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ are the objective coefficients for x and y respectively. $A \in \mathbb{R}^{k \times n}$ is a $k \times n$ coefficient matrix for integer variables x and analogously $B \in \mathbb{R}^{k \times m}$ is a $k \times m$ coefficient matrix for continuous variables y. $b \in \mathbb{R}^k$ is the right hand side vector of the constraints. A review on MIP can be found in Nemhauser and Wolsey (1988).

3 Mixed Integer Duality

Consider the MIP problem (P_{MIP}) given by (1). Let $\mathfrak{F}$ be the set of nondecreasing functions $F : \mathbb{R}^k \to \mathbb{R}$. The dual of the problem can be written as

$$\begin{array}{lll} \min & F(b) & \\ s.t. & F(Ax + By) & \geq cx + dy \quad \forall x \in \mathbb{Z}^n_+ \quad \& \quad \forall y \in \mathbb{R}^m_+ \\ & F \in \mathfrak{F} & \end{array} \tag{2}$$

Here F is the dual function, or the so called price function. It plays the same role as shadow prices[3] in the LP dual. Let, e.g. b be the available resources, $cx + dy$ be the profit from a production and $Ax + By$ be the production function. Then the original MIP problem (P_{MIP}) given by (1) can be interpreted as maximizing profit from production, given some constraints on available resources. One interpretation of the dual price function in the dual program (2) tells, how much extra resources are worth. In particular, if one constraint in the primal problem (1) represents a constraint on one single resource, then one extra unit of resource i is worth $F(e_i)$ units of payment, where e_i is the i'th unit vector.

The structure of an optimal price function F and its properties depend on the algorithmic approach used to solve the original MIP problem, and thus to generate F (if it is possible). A review on the pure integer programming case can be found in L. A. Wolsey (1981).

The two most widespread algorithmic approaches to solve MIP problems are branch-and-bound and cutting plane approaches. A cutting plane algorithm for MIP was first proposed by Gomory (1960). However, the procedure appeared to be slow at first. Moreover, a finite cutting plane algorithm for MIP is still not known. If the classical Gomory cuts are used, Salkin (1989) mentions an example of a MIP problem by White (1961), that cannot be solved using the cutting plane method. Therefore the research was more concentrated on the branch-and-bound method proposed by Little (1963).

However, the cutting planes algorithms have been reconsidered in the early 90's with some impressive results. Thus, a cutting plane based lift-and-project algorithm was proposed (see Balas et al. (1993) and Lovasz and Schrijver (1991)). Moreover, one of the most widespread algorithm, branch-and-cut[4] is a mixture of both approaches where a cutting plane approach is added to the branch-and-bound framework.

[3]Dual variables

[4]Padberg and Rinaldi (1987) for pure integer programming and Crowder et al. (1983) for 0-1 MIP.

Two sets of functions will be useful when describing MIP problems. Let $\mathfrak{F}$ be the set of nondecreasing functions $F : \mathbb{R}^k \to \mathbb{R}$, as defined above. Thus

$$\mathfrak{F} = \{(F : \mathbb{R}^k \to \mathbb{R}) : F(a) \leq F(b) \forall a, b \in \mathbb{R}^k, a \leqq b\}.$$

Finally let $\mathfrak{H}$ be the set of nondecreasing and superadditive functions satisfying the following conditions:

1. $(F : \mathbb{R}^k \to \mathbb{R}) \in \mathfrak{H}$ is superadditive, i.e. $F(q_1) + F(q_2) \leq F(q_1 + q_2), \forall q_1, q_2 \in \mathbb{R}^k$,
2. $F(\mathbf{0}) = 0$,
3. $F \in \mathfrak{H}$ is nondecreasing, i.e. $F \in \mathfrak{F}$, and
4. $\overline{F}(q) = \lim_{\epsilon \searrow 0} \frac{F(\epsilon q)}{\epsilon}$ exists and is finite for all q.

In the cutting plane approach we will deal with functions in $\mathfrak{H}$, while dual price functions for branch-and-bound approach will be nondecreasing, polyhedral and convex.

3.1 Cutting Plane Framework

Algorithms based on the cutting plane setting are less common. This may be because no cutting plane based finite algorithm is known for the general MIP problem. See Marchand et al. (1999) for a review on cutting plane based algorithms for MIP problems. The original Gomory's cutting plane algorithm is sure to terminate only if the optimal objective function is integer valued. Other MIP algorithms restrict the variables to the 0-1 case.

Again consider the MIP problem (P_{MIP}) given by (1). The Gomory's strong cutting plane algorithm for MIP problems solves a family of problems (P^r):

$$\begin{aligned} \max \quad & z^r = cx + dy \\ s.t \quad & Ax + By \leq b \\ & Cx + \overline{C}y \leq C_b \\ & x, y \geq \mathbf{0} \end{aligned}$$

Here an element in the last set of constraints has the form $\sum_{j=1}^{n} G_r(A_{.j})x_j +$ $\sum_{j=1}^{m} \overline{G}_r(B_{.j})y_j \leq G_r(b)$ where the function $G_r(q) : \mathbb{R}^k \to \mathbb{R}$ represents a Gomory cut. r is the index representing the number of the cut in focus.

The form of the function $G_r(q)$ can be obtained from results in Nemhauser and Wolsey (1988). Let $\lfloor a \rfloor$ be the integral part, and f_a be the fractional part of $a \in \mathbb{R}$. That is, $a = \lfloor a \rfloor + f_a$ and $0 \leq f_a \leq 1$. For an α, $0 \leq \alpha < 1$, define $F_\alpha(a) : \mathbb{R} \to \mathbb{R}$ by

$$F_\alpha(a) = \lfloor a \rfloor + \max(0, \tfrac{f_a - \alpha}{1-\alpha}).$$

Let v be the row element of the inverse basis matrix corresponding to the source row in the constructed simplex tableau. For simplicity consider the first cut. Then $v = \{v_1, ..., v_k\}$, since the dimension of the basis is k. Let $V = \{1, ..., k\}$, $V^+ = \{i \in V | v_i \geq 0\}$ and $V^- = \{i \in V | v_i < 0\}$. Moreover, let α be the fractional part of the nonintegral value of the basic variable in the source row.

Holm and Tind (1988)[5] show that $G(q)$ defined by

$$G(q) = F_\alpha(vq) - \frac{1}{1-\alpha} \sum_{i \in V^-} v_i q_i$$

is superadditive and nondecreasing and that

$$\overline{G}(q) = \frac{1}{1-\alpha} \min(-\sum_{i \in V^-} v_i q_i, \sum_{i \in V^+} v_i q_i)$$

is concave and piecewise linear. Additionally $G(q)$ generates cuts in the Gomory strong MIP cutting plane algorithm.

The algorithm terminates if some problem (P^r) is found to be infeasible, or if a mixed integer solution is found. However, this Gomory cutting plane algorithm is finite for integral optimum objectives only. For a general MIP we are not sure to obtain a solution after a finite number of cuts.

3.1.1 MIP duality in Cutting Plane Framework

Gomory's strong mixed integer cutting plane algorithm generates nondecreasing superadditive optimal dual price functions. Suppose that p Gomory cuts are needed to find the optimal solution for the primal MIP problem. With the Gomory cuts given by the function

[5]Based on Nemhauser and Wolsey (1988)

$G(q)$ defined above, the optimal price function $F(q) : \mathbb{R}^k \to \mathbb{R}$ and its directional derivative are given by

$$F(q) = \sum_{i=1}^{k} u_i q_i + \sum_{i=k+1}^{k+p} u_i G_i(q) \tag{3}$$

and

$$\overline{F}(q) = \sum_{i=1}^{k} u_i q_i + \sum_{i=k+1}^{k+p} u_i \overline{G}_i(q)$$

respectively. Here $u_1, ..., u_k, u_{k+1}, ..., u_{k+p} \geq 0$ represent the dual variables obtained at termination. The first k variables correspond to the original MIP constraints, while the last p variables correspond to the additional Gomory cuts.

The superadditive dual of a MIP is then (see also Nemhauser and Wolsey (1988))

$$\begin{array}{rcll} \min_{F \in \mathfrak{H}} F(b) & & & \\ s.t.\ F(A_{.j}) & \geq & c_j & j = 1, ..., n \\ \overline{F}(B_{.j}) & \geq & d_j & j = 1, ..., m \end{array}$$

Here $F(q)$ is nondecreasing and superadditive and $\overline{F}(q)$ is concave and piecewise linear.

3.2 Branch-and-Bound Framework

The LP based branch-and-bound approach produced some effective algorithms like branch-and-price and branch-and-cut. A review on algorithms based on LP branch-and-bound approach can be found in Johnson et al. (2000).

Consider the mixed integer problem (P_{MIP}) given by (1). The classical branch-and-bound algorithm solves a family of subproblems (P_t), $t = 1, ..., r$:

$$\begin{array}{ll} \max & cx + dy \\ s.t. & Ax + By \leq b \\ & x \in X_t, y \in \mathbb{R}^m_+ \end{array} \tag{4}$$

where $\mathbb{Z}^n_+ \subseteq \bigcup_{t=1}^{r} X_t$. Assume in the following that $X_t = \{x \in \mathbb{R}^n : g_j^t \leq x_j \leq h_j^t, j = 1, ..., n, x \geq \mathbf{0}\}$ as it is done in Klamroth et al. (2002), where g_j^t and h_j^t are lower and upper integer bounds respectively. This assumption is satisfied by LP based branch-and-bound approaches and many other branch-and-bound algorithms. A branch-and-bound algorithm terminates if one of the following is true:

- All the generated subproblems (P_t), $t = 1, ..., r$, are shown to be infeasible or,
- The optimal solution to some subproblem P_{t^*}, (x^{t^*}, y^{t^*}), is found, such that x^{t^*} is integer valued, and for $z_{t^*} = cx^{t^*} + dy^{t^*}$ we have that $z_{t^*} \geq z_t$ for all $t \neq t^*$. Here z_t represents the objective value of the subproblem (P_t).

3.2.1 MIP Duality in Branch-and-Bound Framework

Using branch-and-bound algorithms the generated optimal price function is not necessarily superadditive. Although branch-and-bound algorithms have been widespread in solving MIP problems, there are no results concerning generation of optimal price functions based on branch-and-bound known to the author. A treatment for the pure integer programming problem can be found in Wolsey (1981).

Consider the original MIP problem (P_{MIP}) given by (1) and the subproblems (P_t) given by (4). The following lemma shows how to construct a dual feasible function for (P_{MIP}) given dual feasible functions for its subproblems.

Lemma 3.1: *If $F_t \in \mathfrak{F}$, $\quad t = 1, ..., r$, are dual feasible functions for the subproblems (P_t), $\quad t = 1, ..., r$ in the sense that*

$$F_t(Ax + By) \geq cx + dy \quad \forall x \in X_t, y \in \mathbb{R}^m_+$$

then

$$F(q) := \max_{t=1,...,r} F_t(q)$$

is a dual feasible function for the original MIP problem (P_{MIP}) in (1).

Proof

Let $x \in \mathbb{Z}^n_+$, and $y \in \mathbb{R}^m_+$. Then because $\mathbb{Z}^n_+ \subseteq \bigcup_{t=1}^r X_t$, $x \in X_t$ for some $t = 1, ..., r$. Hence, since F_t is feasible for (P_t), $F_t(Ax + By) \geq cx + dy$. But due to the definition of F, $F(Ax + By) \geq F_t(Ax + By) \geq cx + dy$. Moreover, F is nondecreasing since F_t is nondecreasing for $t = 1, ..., r$. This implies that $F \in \mathfrak{F}$. Thus, all in all F is a dual feasible function for the original MIP problem (P_{MIP}).

$\square$

Next we show that a dual optimal function F for the original MIP problem in fact exists, provided the problem has a finite optimal solution. This result together with a way to construct F is established in the theorem below.

Theorem 3.1 *If the original MIP program (P_{MIP}) in (1) has a final optimal solution, and an LP based branch-and-bound algorithm terminates in a finite number of subproblems (P_t), $t = 1, ..., r$, then there exists a dual optimal price function $F \in \mathfrak{F}$ where*

$$F(q) := \max_{t=1,\ldots,r} (\pi^t q + \alpha^t), \quad \alpha^t \in \mathbb{R}, \pi^t \in \mathbb{R}^k, \pi^t \geq \mathbf{0}. \tag{5}$$

Proof

Let z^* be the optimum objective value of (P_{MIP}) and consider some arbitrarily chosen terminating subproblem (P_t), $t \in 1, ..., r$. (P_t) is either infeasible or has an optimal solution where integer variables have integer values.

a) If the linear program (P_t) has an appropriate optimal solution with corresponding optimal objective value z_t, then its dual LP is feasible. Let $(\pi^t, \underline{\pi}^t, \overline{\pi}^t) \geq \mathbf{0}$ be the optimal solution of the dual. Here, the variable π^t corresponds to the initial constraints in (P_{MIP}), while the variables $\underline{\pi}^t$ and $\overline{\pi}^t$ represent the extra integer $\geq$ and $\leq$ constraints respectively, that are generated by the branch-and-bound algorithm. Since $(\pi^t, \underline{\pi}^t, \overline{\pi}^t)$ is feasible for the dual LP

$$\pi^t A_{.j} - \sum_{j=1}^{n} \underline{\pi}_j^t + \sum_{j=1}^{n} \overline{\pi}_j^t \geq c_j \quad j = 1, ..., n$$

and

$$\pi_t B_{.i} \geq d_i \quad i = 1, ..., m.$$

Define a nondecreasing function F_t as

$F_t(q) := \pi^t q + \alpha^t$, where $\alpha^t = -\underline{\pi}^t g^t + \overline{\pi}^t h^t$.

F_t satisfies

$F_t(Ax + By) = \pi^t(Ax + By) + \alpha^t = \pi^t(Ax + By) - \underline{\pi}^t g^t + \overline{\pi}^t h^t \geq$
$\pi^t(Ax + By) - \underline{\pi}^t x + \overline{\pi}^t x = \pi^t Ax + \pi^t By - \underline{\pi}^t x + \overline{\pi}^t x \geq cx + dy \; \forall x \in X_t, y \in \mathbb{R}^m$.

Thus, F_t represents a dual feasible function for (P_t) in the sense of lemma 3.1. Moreover, by linear programming duality, $F_t(b) = \pi^t b - \underline{\pi}^t g^t + \overline{\pi}^t h^t = z_t$ for terminating (P_t). Here $z_t \leq z^*$.

b) If (P_t) on the other hand is infeasible, there exits a dual ray $(\omega^t, \underline{\omega}^t, \overline{\omega}^t) \geq \mathbf{0}$, that satisfies $\omega^t A_{.j} - \sum_{j=1}^n \underline{\omega}_j^t + \sum_{j=1}^n \overline{\omega}_j^t \geq c_j, \quad j = 1, ..., n,\ \omega_t B_{.i} \geq d_i, \quad i = 1, ..., m$ and $\omega^t b - \underline{\omega}^t g^t + \overline{\omega}^t h^t < 0$. The definitions of ω are analogous to the definitions of π above. Consider some dual feasible solution $(\pi^p, \underline{\pi}^p, \overline{\pi}^p) \geq \mathbf{0}$ of the dual of (P_t). This may be available from the parent node in the branch-and-bound tree. Combining it with the dual ray we obtain a vector $(\pi^t, \underline{\pi}^t, \overline{\pi}^t) := (\pi^p, \underline{\pi}^p, \overline{\pi}^p) + \mu(\omega^t, \underline{\omega}^t, \overline{\omega}^t)$, where $\mu \in \mathbb{R}_+$.

Define $F_t \in \mathfrak{F}$ for (P_t) by $F_t(q) = \pi^t q + \alpha^t$, $\alpha^t := -\underline{\pi}^t g^t + \overline{\pi}^t h^t$. Then we have that:

$F_t(Ax + By) = \pi^t(Ax + By) - \underline{\pi}^t g^t + \overline{\pi}^t h^t =$
$(\pi^p + \mu\omega^t)(Ax + By) - (\underline{\pi}^p + \mu\underline{\omega}^t)g^t + (\overline{\pi}^p + \mu\overline{\omega}^t)h^t =$
$\pi^p(Ax + By) + \mu\omega^t(Ax + By) - \underline{\pi}^p g^t - \mu\underline{\omega}^t g^t + \overline{\pi}^p h^t + \mu\overline{\omega}^t h^t \geq cx + dy,\ \forall x \in X_t, y \in \mathbb{R}_+^m$

Thus again we are dealing with a dual feasible function F_t for (P_t). Moreover, we see that

$$\lim_{\mu \to \infty} F_t(b) = \lim_{\mu \to \infty} (\pi^t b + \alpha^t) = \lim_{\mu \to \infty} ((\pi^p + \mu\omega^t)b - (\underline{\pi}^p + \mu\underline{\omega}^t)g^t + (\overline{\pi}^p + \mu\overline{\omega}^t)h^t) =$$
$$\lim_{\mu \to \infty} (\pi^p b - \underline{\pi}^p g^t + \overline{\pi}^p h^t + \mu(\omega^t b - \underline{\omega}^t g^t + \overline{\omega}^t h^t)) = -\infty.$$

Thus we always can choose μ so $F_t(b) < z^*$.

Summarizing, $F_t(q) = \pi^t q + \alpha^t$ is a dual feasible function for all terminating (P_t), $t = 1, ..., r$. Thus, using lemma 3.1, the price function F given by (5), is dual feasible for (P_{MIP}). We assumed that (P_{MIP}) has an finite optimal mixed integer solution, let it be (x^*, y^*). But then there exists a $t^* \in \{1, ..., r\}$ such that (x^*, y^*) is the optimum solution for (P_{t^*}) and hence $z^* = cx^* + dy^* = F_{t^*}(b)$. Since $F_t(b) \leq z^*$ for all $t = 1, ..., r$, $F(b) = \max_{t=1,...,r} F_t(b) = F_{t^*}(b) = z^*$ and thus is dual optimal for (P_{MIP}). All in all, the constructed optimal dual price function F exists and is dual optimal for (P_{MIP}).

□

The theorem shows that a standard LP based branch-and-bound algorithm generates a price function that is piecewise linear, nondecreasing and convex, as it was the case with pure IP problems (see Wolsey (1981)). We also see that F in general is not superadditive.

There are several versions of the branch-and-bound algorithms, depending on which variable to branch on, if several integer variables have non-integer values in an optimal solution of a LP relaxation. Each version produces one optimal dual price function. Thus, the generated price function is only one possible solution out of many and depends on the version of the algorithm.

For a special kind of MIP problems, however, an interpretation involving a superadditive price function can be obtained using branch-and-bound algorithms. An analogous result for the pure integer programming case can be found in Wolsey (1981). Consider the following bounded MIP problem $(\overline{P})$:

$$\begin{array}{ll} \max & cx + dy \\ s.t. & Ax + By \leq b \\ & \quad -x \leq -g \\ & \quad x \leq h \\ & x \in \mathbb{Z}^n_+, y \in \mathbb{R}^m_+ \end{array}$$

Let again $X_t = \{x \in \mathbb{R}^n : g^t_j \leq x_j \leq h^t_j, j = 1, ..., n, x \geq \mathbf{0}\}$ and $\{x : \mathbf{0} \leq g \leq x \leq h, x \geq \mathbf{0}$ and integer$\} \subseteq \bigcup_{t=1}^{r} X_t$. The dual of $(\overline{P})$ is

$$\begin{array}{llll} \min & F(b, -g, h) & & \\ s.t. & F(A_{.j}, -e_j, e_j) & \geq & c_j \\ & \overline{F}(B_{.j}, 0, 0) & \geq & d_j \\ & F \in \mathfrak{H} & & \end{array}$$

where e_j is the j'th unit vector.

Theorem 3.2 *If the bounded MIP program $(\overline{P})$ has a final optimal solution, and solving $(\overline{P})$ with an LP based branch-and-bound algorithm results in a finite number of terminating subproblems (P_t), $t = 1, ..., r$, then there exists a dual feasible price function $F \in \mathfrak{H}$ of the form*

$$F(q) := \min_{t=1,...,r} u^t q, \quad u^t \in \mathbb{R}^{k+2n}, u^t \geq \mathbf{0}.$$

Proof

Set $u^t = (\pi^t, \underline{\pi}^t, \overline{\pi}^t)$, as in the proof of theorem 3.1. Thus, u^t is the dual variables of some subproblem $(\overline{P}_t)$ in case a) and a combination of a feasible solution and a dual ray in case b). Since $\pi^t A_{.j} - \underline{\pi}^t_j + \overline{\pi}^t_j \geq c_j$ for all $t = 1, ..., r$, $F(A_{.j}, -e_j, e_j) = \min_{t=1,\dots,r}(\pi^t A_{.j} - \underline{\pi}^t_j + \overline{\pi}^t_j) \geq c_j$. Moreover, since $\pi^t B_{.j} \geq d_j$ for all $t = 1, ..., r$, $\overline{F}(B_{.j}, 0, 0) = \min_{t=1,\dots,r}(\pi^t B_j) \geq d_j$.

F is clearly superadditive and nondecreasing and $F(\mathbf{0}) = 0$. Finally finite $\overline{F}(q)$ exists for all q. Thus, $F \in \mathfrak{H}$. All in all, F is dual feasible for $(\overline{P})$.

□

The generated price function is a weak dual function and serves as an upper bound for the value function of the primal problem $(\overline{P})$.

3.3 MIP Duality in Branch-and-Cut Framework

The branch-and-cut algorithm is a hybrid algorithm, that combines the branch-and-bound and the cutting planes approaches. Thus, at each node we try to find a violated cut first. If it is not available within a reasonable amount of time we branch. A description of the algorithm can among others be found in Cordier et al. (1999). This algorithm turned out to be quite effective for solving MIP problems.

The following theorem states a result about the dual optimal function of the MIP problem, if an branch-and-cut algorithm is applied. Such a dual function exists provided that the primal problem has a finite optimal solution, and the number of terminating subproblems is finite. Moreover, the theorem shows a way to find an optimal dual price function.

Theorem 3.3 *If the original MIP program (P_{MIP}) in (1) has a final optimal solution, and solving (P_{MIP}) with a branch-and-cut algorithm results in a finite number of terminating subproblems $(\tilde{P}_t)$, $t = 1, ..., r$, then there exists a dual optimal price function $F \in \mathfrak{F}$ where*

$$F(q) := \max_{t=1,\dots,r} (\pi^t q + \alpha^t + \sum_{s=1}^{\delta(t)} \tilde{\pi}^t_s G^t_s(q)), \quad \alpha^t \in \mathbb{R}, \pi^t \in \mathbb{R}^k_+, \tilde{\pi}^t \in \mathbb{R}^{\delta(t)}_+, G^t_s \in \mathfrak{H}.$$

Here $\delta(t) \geq 0$ is the number of Gomory cuts G^t_s in subproblem $(\tilde{P}_t)$.

Proof

As in the proof of theorem 3.1 let z^* be the optimum objective value of (P_{MIP}). Consider some arbitrarily chosen terminating subproblem $(\tilde{P}_t)$:

$$
\begin{array}{rl}
\max & cx + dy \\
s.t. & Ax + By \leq b \\
 & Cx + \overline{C}y \leq C_b \\
 & x \in X_t, y \in \mathbb{R}^m_+
\end{array}
$$

where an element in the last constraints has the form $\sum_{j=1}^{n} G_s^t(A_{.j})x_j + \sum_{j=1}^{m} \overline{G}_s^t(B_{.j})y_j \leq G_s^t(b)$, and $G_s^t(q)$ represents the s'th Gomory cut in problem $(\tilde{P}_t)$. If some cuts are present in a parent node subproblem, then these cuts will also be present in its child node subproblem, if such a child node exists.

case a) Suppose that the LP relaxation of $(\tilde{P}_t)$ has an optimal mixed integer solution with objective value z_t. Let $(\pi^t, \overline{\pi}^t, \underline{\pi}^t, \tilde{\pi}^t) \geq \mathbf{0}$ be the optimal dual solution. $\pi^t, \overline{\pi}^t, \underline{\pi}^t$ are as defined in proof for theorem 3.1, and $\tilde{\pi}^t$ corresponds to the Gomory cuts constraints. Since we have $\delta(t)$ cuts in problem $(\tilde{P}_t)$, $\tilde{\pi}^t$ has dimension $\delta(t)$. We see that

$$\pi^t A_{.j} - \sum_{j=1}^{n} \underline{\pi}_j^t + \sum_{j=1}^{n} \overline{\pi}_j^t + \sum_{s=1}^{\delta(t)} G_s^t(A_{.j})\tilde{\pi}_s^t \geq c_j \quad j = 1, ..., n$$

and

$$\pi^t B_{.i} + \sum_{s=1}^{\delta(t)} \overline{G}_s^t(B_{.i})\tilde{\pi}_s^t \geq d_i \quad i = 1, ..., m.$$

Define the nondecreasing function F_t by

$F_t(q) := \pi^t q + \alpha^t + \sum_{s=1}^{\delta(t)} G_s^t(q)\tilde{\pi}_s^t$ with $\alpha^t = -\underline{\pi}^t g^t + \overline{\pi}^t h^t, t = 1, ..., r.$

Since G_s^t is a Gomory cut it is superadditive. But then $\sum_{s=1}^{\delta(t)} G_s^t(Ax + By)\tilde{\pi}_s^t \geq \sum_{s=1}^{\delta(t)} G_s^t(Ax)\tilde{\pi}_s^t + \sum_{s=1}^{\delta(t)} G_s^t(By)\tilde{\pi}_s^t$. Moreover,this also implies that $\sum_{s=1}^{\delta(t)} G_s^t(Ax)\tilde{\pi}_s^t \geq \sum_{s=1}^{\delta(t)} G_s^t(A)x\tilde{\pi}_s^t$, and analogously $\sum_{s=1}^{\delta(t)} G_s^t(By)\tilde{\pi}_s^t \geq \sum_{s=1}^{\delta(t)} G_s^t(B)y\tilde{\pi}_s^t$. Due to the definition of G_s^t given in section 3.1 $G_s^t(B) \geq \overline{G}_s^t(B)$, $s = 1, ..., \delta(t)$. Finally, since G_s^t is a Gomory cut $G_s^t(0) = 0$.

All this implies that, for all $t = 1, ..., r$, F_t satisfies

$F_t(Ax + By) = \pi^t(Ax + By) + \overline{\pi}^t h^t - \underline{\pi}^t g^t + \sum_{s=1}^{\delta(t)} G_s^t(Ax + By)\tilde{\pi}_s^t \geq$
$\pi^t Ax + \pi^t By + \overline{\pi}^t x - \underline{\pi}^t x + \sum_{s=1}^{\delta(t)} G_s^t(Ax)\tilde{\pi}_s^t + \sum_{s=1}^{\delta(t)} G_s^t(By)\tilde{\pi}_s^t \geq$

$\pi^t Ax + \pi^t By + \overline{\pi}^t x - \underline{\pi}^t x + \sum_{s=1}^{\delta(t)} G_s^t(A)x\tilde{\pi}_s^t + \sum_{s=1}^{\delta(t)} G_s^t(B)y\tilde{\pi}_s^t \geq$
$\pi^t Ax + \pi^t By + \overline{\pi}^t x - \underline{\pi}^t x + \sum_{s=1}^{\delta(t)} G_s^t(A)x\tilde{\pi}_s^t + \sum_{s=1}^{\delta(t)} \overline{G}_s^t(B)y\tilde{\pi}_s^t \geq$
$cx + dy \quad \forall x \in X_t, y \in \mathbb{R}^m.$

Thus, the function F_t represents a dual feasible function for $(\tilde{P}_t)$. Moreover, by linear programming duality, $F_t(b) = \pi^t b - \underline{\pi}^t g^t + \overline{\pi}^t h^t + \sum_{s=1}^{\delta(t)} G_s^t(b)\tilde{\pi}_s^t = z_t$ for terminating $(\tilde{P}_t)$ and $z_t \leq z^*$.

case b) If $(\tilde{P}_t)$ is infeasible then there exists a dual ray $(\omega^t, \underline{\omega}^t, \overline{\omega}^t, \tilde{\omega}^t) \geq \mathbf{0}$, such that $\omega^t A_{.j} - \sum_{j=1}^n \underline{\omega}_j^t + \sum_{j=1}^n \overline{\omega}_j^t + \sum_{s=1}^{\delta(t)} G_s^t(A_{.j})\tilde{\omega}_s^t \geq c_j, j = 1, ..., n$, $\pi^t B_{.i} + \sum_{s=1}^{\delta(t)} \overline{G}_s^t(B_{.i})\tilde{\omega}_s^t \geq d_i, i = 1, ..., m$, and $\omega^t b - \underline{\omega}^t g^t + \overline{\omega}^t h^t + \sum_{s=1}^{\delta(t)} G_s^t(b)\tilde{\omega}_s^t < 0$. Analogous to the proof for theorem 3.1 define

$(\pi^t, \underline{\pi}^t, \overline{\pi}^t, \tilde{\pi}^t) := (\pi^p, \underline{\pi}^p, \overline{\pi}^p, \tilde{\pi}^p) + \mu(\omega^t, \underline{\omega}^t, \overline{\omega}^t, \tilde{\omega}^t)$, where $\mu \in \mathbb{R}_+$.

Here, $(\pi^p, \underline{\pi}^p, \overline{\pi}^p, \tilde{\pi}^p)$ represents a dual feasible solution for $(\tilde{P}_t)$. Then let F_t be defined by $F_t(q) := \pi^t q + \alpha^t + \sum_{s=1}^{\delta(t)} G_s^t(q)\tilde{\pi}_s^t$, $\alpha^t := -\underline{\pi}^t g^t + \overline{\pi}^t h^t$. Analogous to case a) F_t satisfies $F_t(Ax + By) \geq cx + dy, \forall x \in X_t, y \in \mathbb{R}_+^m$. Thus F_t represents a dual feasible function for $(\tilde{P}_t)$.

Moreover, since

$$\lim_{\mu \to \infty} F_t(b) = \lim_{\mu \to \infty} (\pi^t b + \alpha^t + \sum_{s=1}^{\delta(t)} G_s^t(q)\tilde{\pi}_s^t) =$$

$$\lim_{\mu \to \infty} (\pi^p b - \underline{\pi}^p g^t + \overline{\pi}^p h^t + \sum_{s=1}^{\delta(t)} G_s^t(b)\tilde{\pi}_s^p + \mu(\omega^t b - \underline{\omega}^t g^t + \overline{\omega}^t h^t + \sum_{s=1}^{\delta(t)} G_s^t(b)\tilde{\omega}_s^t)) = -\infty.$$

we always can choose μ so $F_t(b) < z^*$.

Summarizing, $F_t(q) = \pi^t q + \alpha^t + \sum_{s=1}^{\delta(t)} G_s^t(q)\tilde{\pi}_s^t$ is a dual feasible function for all terminating $(\tilde{P}_t), t = 1, ..., r$. Thus, $F(q) := \max_{t=1,...,r} F_t(q)$ is a dual feasible function for (P_{MIP}), if branch-and-cut algorithm is used due to lemma 3.1. Since we assumed that there is a finite mixed integer solution (x^*, y^*) there exists $t^* \in \{1, ..., r\}$ such that (x^*, y^*) is the optimal solution for $(\tilde{P}_{t^*})$. Hence $z^* = cx^* + dy^* = F_{t^*}(b)$. Since $F_t(b) \leq z^*$ for all $t = 1, ..., r$, $F(b) = F_{t^*} = z^*$ and this $F(q)$ is optimal for (P_{MIP}).

□

The constructed function is not only the dual optimal function for the branch-and-cut algorithm. It also represents a general form of such a dual optimal function if either a cutting plane or branch-and-bound based algorithm is used. In case of a cutting plane algorithm, we only deal with one node, the root node. Moreover, the variables α^t disappear. Thus we end up with the same formulation as (3). For a pure branch-and-bound algorithm, we do not have any constraints representing Gomory cuts, $\delta(t) = 0$, for all $t = 1, ..., r$. In this case we are back to the same formulation of the dual price function (5) in theorem 3.1.

4 Summary

The presented paper gave a short presentation of the MIP problem and three of its solution methods. Additionally, some duality results were shown. In particular, the formulation of a dual of a MIP problem contains a dual price function F. The characteristics of this function, however, depend on the algorithm used to generate it. Applying the cutting plane algorithm we obtain a nondecreasing and superadditive price function. Using a branch-and-bound algorithm, on the other hand, provides a piecewise linear, nondecreasing and convex price function, which in general is not superadditive. However, section 3.2.1 presented a superadditive weak dual price function for the bounded MIP problem, if branch-and-bound approach is applied.

Section 3.3 presents a general dual function for the branch-and-bound and the cutting plane approach. This dual function is additionally the price function for the branch-and-cut algorithm. The branch-and-cut algorithm is now very popular when solving MIP problems. One important brick in the algorithm is the generation of cuts. In this chapter we used the classical Gomory cut. However, there exist other cuts, e.g. the lift-and-project cut (see Balas et al. (1993), Balas et al. (1996)) or the mixed integer rounding cut (see Marchand and Wolsey (2001)). One idea for further research would be to show similar duality results for these cuts.

Apart from the conceptual interest, the result can be useful in economic interpretations of MIP models as well as sensitivity analysis.

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