AMO- Advanced Modeling and Optimization, Volume 6, Number 2, 2004

A New Fast Algorithm Based on Karmarkar's Gradient Projected Method for Solving Linear Programming Problems

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Abstract

The main aim of this work is to provide a new variant of Karmarkar's algorithm to give a high-performance algorithm for solving linear programming problems. Modification of the new interior point algorithm presents the parameter that has a great role in faster convergence of the new method comparing to methods presented by Karmarkar 1984 and Scherijver 1986. Successful convergence for problems of different sizes is obtained. Numerical results show that when we use a new parameter in the classical Karmarkar's algorithm, number of iterations is less than the number of iterations for two mentioned methods.

Keywords: Linear programming, Karmarkar's algorithm, Interior point methods

1 Introduction

The simplex method had no serious competition until 1984 when N. Karmarkar proposed a new-polynomial-time algorithm for linear programming problems [Karmarkar 1984]. Since proposition 1984 of Karmarkar's algorithm, a number of variant techniques have been arisen (see [Gonzaga 1987], [Hertog and Roos 1991], [Adler et al. 1989]) and some efforts have been made to unify the view of these techniques ([Shanno and Bagchi 1988]). Karmarkar in his method introduced a parameter α_k such that the potential function value in each iterations decreases about

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0.2. Later Scherijver [Scherijver 1986] presented another parameter α_s in which aforementioned decrease is at least 0.30685. Here in this paper we present a new parameter α_n which produces more potential decreases in each iteration than classical Karmarkar's algorithms.

2 Karmarkar's Projective Algorithm

This algorithm addresses LP problem of the form:

$$\begin{aligned} &Min && \mathbf{CX} \\ &s.t: && \mathbf{AX}=\mathbf{0} \\ & && \sum_{i=1}^{n} x_i = 1 \\ & && \mathbf{X} \geq \mathbf{0} \end{aligned} \tag{1}$$

where $\boldsymbol{A}$ is a $m \times n$ matrix of rank m, $\boldsymbol{C}$ is $1 \times n$ vector, $\mathbf{X} = (x_1, ..., x_n)$, $n \geq 2$ and $\boldsymbol{A}$ and $\boldsymbol{C}$ are all integers. The following assumptions hold:
i) The point $(1/n, \ldots, 1/n)$ is feasible in problem (1).
ii) The optimal objective value of problem (1) is zero.

3 Main Steps of Karmarkar's Algorithm

Step 0: Compute $r = \sqrt{\frac{1}{n(n-1)}}$ and ($\alpha = \frac{n-1}{3n}$ or $\alpha = \frac{1}{1+r}$),
and let $\mathbf{X}_0 = (1/n, ..., 1/n)$, put *k=0*.
Step 1: If $\boldsymbol{CX} < \varepsilon$ use the optimal rounding routine (section 3.1) to determine an optimal solution, and stop. Otherwise define $\mathbf{D}_k = diag\{x_{k1}, ..., x_{kn}\}$,

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{AD_k} \\ \boldsymbol{1} \end{bmatrix} \;;\; \mathbf{Y}_0 = (1/n, ..., 1/n) \;;\; \overline{\boldsymbol{C}} = \boldsymbol{CD_k}$$

and compute $\mathbf{C_p} = \left[\mathbf{I} - \mathbf{P^t(PP^t)P}\right]\overline{\mathbf{C}}^{\mathbf{t}}$ where $\boldsymbol{Y_{new}} = \boldsymbol{Y_0} - \alpha r \frac{\boldsymbol{C_p}}{\|\boldsymbol{C_p}\|}$.

Hence obtain $\boldsymbol{X_k} = \frac{\boldsymbol{D_k Y_{new}}}{\boldsymbol{1 D_k Y_{new}}}$. Increment k by one and repeat the step 1([Bazaraa, 1984 and Hamdy 1992]).

3.1 Optimal Rounding Routine

Starting with the final iterate $\boldsymbol{X_k}$ obtained with objective value $\boldsymbol{CX} < \varepsilon$, this procedure finds an extreme point solution $\overline{\boldsymbol{X}}$ with at least as good an objective value using the following method known as a purification scheme: If *n* linearly independent constraints are binding at $\boldsymbol{X_k}$, then it is already a basic feasible solution. Otherwise there exists a direction $d \neq 0$ lying in the null space of the binding constraints. The method now

moves the current iterate along the direction d if $\boldsymbol{C}d < 0$ and along the direction $-d$ otherwise until some constraint block any further motion by feasibility consideration. This must happen since the feasible region is bounded. Note that at the new solution the objective value is no more than $\boldsymbol{CX} < \varepsilon$ and at least one additional linearly independent constraint is binding. Proceeding in this fashion, a basic feasible solution $\overline{\boldsymbol{X}}$ to problem (1) can be obtained with objective value strictly less than ε [Bazaraa, 1984].

4 Karmarkar's Potential Function

Karmarkar in the analysis of his method used the following potential function:

$$\Phi_x(\mathbf{X}) = nLn(\mathbf{C}^t\mathbf{X}) - \sum_{i=1}^{n} Ln(x_i)$$

where $\boldsymbol{C}^t$ is referred to transpose of $\boldsymbol{C}$. At first, karmarkar presented $\alpha_k = (n-1)/3n$ and proved that each iteration decreases the potential function value by about 0.2 (for example see [Bazaraa, 1984], [Karmarkar, 1984]). Since then, Scherijver presented and proved the following lemma and theorem [Roos, 2001]:

Lemma 1: Taking $\alpha_s = \frac{1}{1+r}$; each iteration of the projective algorithm in [Scherijver, 1986] decreases the potential function value by at least 0.30685.

Theorem 1: After no more than $\frac{n}{\Psi(1)} Ln(\frac{\mathbf{C}^t\mathbf{X}_0}{\varepsilon})$ iteration, the algorithm stops with a feasible point $\boldsymbol{X}$ such that $\boldsymbol{C}^t\boldsymbol{X} \le \varepsilon$, where $\Psi : (-1, \infty) \to R^+$ and

$$\Psi(t) = t - Ln(1+t) \tag{2}$$

Remark: Parameter α_s presented by Scherijver is better than the Karmarkar's parameter α_k, however we present new parameter α_n that in each iteration decreases the potential function value more than the two former parameters. In the next section we present this new parameter α_n. Then we will prove that in each iteration more decreases of potential function will happen when we use this new α_n.

5 Presentation of new parameter

Lemma 2: Taking $\alpha_n = 1 - \frac{1}{n^4(1+\sqrt{n(n-1)})}$, decreases the potential function value in each iteration more than other classical Karmarkar's algorithms defined in [Karmarkar, 1984 and Roos, 2001]. Moreover when $n \to \infty$ the parameter α_s is almost equal to α_n.

Proof: In general, decrease of the objective function value in each iteration is computed as follows [Karmarkar, 1984]:

$$\mathbf{C}^{\mathrm{t}}\mathbf{X}^{i+1} = (1-\frac{\alpha_k}{n-1})\mathbf{C}^{\mathrm{t}}\mathbf{X}^{i}$$

where $\alpha_k \in (0,1)$, $\boldsymbol{X}^{\boldsymbol{i}}$ and $\boldsymbol{X}^{\boldsymbol{i+1}}$ are the two successive iteration solutions. Therefore, the maximum value of decrease for objective function value occurs when $1-\frac{\alpha_k}{n-1}$ tends to zero, that is, reduction factor $\frac{\alpha_k}{n-1}$ and consequently α_k is chosen close to one. Taking $\alpha_s < \alpha_n < 1$, we will have

$$\alpha_S = \frac{1}{1+r} < \alpha_n = 1-\varepsilon_n < 1$$

where $\varepsilon_n \in (0,1)$. Thus we can write the following inequality with respect to *n* the dimension of the problem (1):

$$\varepsilon_n < \frac{1}{1+\sqrt{n(n-1)}}$$

For $\varepsilon_n = \frac{1}{1+\sqrt{n(n-1)}}$ this is exactly the Scherijver's parameter α_S. Now for

$$\varepsilon_n = \frac{1}{n^4(1+\sqrt{n(n-1)})} << \frac{1}{1+\sqrt{n(n-1)}}$$

α_S is less than α_n and therefore this new parameter α_n has more objective function values decrease in two successive iterations. Now, when *n* is large enough, we have $\varepsilon_n \to 0$ and α_S and α_n tend to one with different rate of convergence. Since we assumed $\alpha_s < \alpha_n$ so, for the large amount of *n*, $\alpha_S \to \alpha_n$.

Now, we present the following theorem to show the lower bound of potential function decreasing values in each iteration:

Theorem 1: Taking $\alpha_n = 1-\frac{1}{n^4(1+\sqrt{n(n-1)})}$; each iteration of the new projective algorithm decreases the potential function values by at least:

$$0.30685 + (\alpha_n - \alpha_S)r^2 + \frac{r^2}{2}(\alpha_n{}^2 - \alpha_S{}^2)(\frac{n}{R^k}-1).$$

Proof: Let us to denote Δ_s and Δ_n as the potential function decreasing in each iteration corresponding to α_s and α_n respectively. From lemma 1 of [Scherijver, 1986] for $0 < \alpha_s < 1$ we can write:

$$\Delta_S \geq \alpha_S r^2 + n\Psi(-\alpha_S \frac{r}{R}) - \Psi(-\alpha_S r) \qquad (3)$$

where $R = \sqrt{\frac{n-1}{3n}}$. Thus for $0 < \alpha_n < 1$ we will have:

$$\Delta_n \geq \alpha_n r^2 + n\Psi(-\alpha_n \frac{r}{R}) - \Psi(-\alpha_n r). \quad (4)$$

From (2) and (4) we obtain:

$$\Delta_n \geq \alpha_n r^2 + \alpha_n r + Ln(1-\alpha_n r) - n\alpha_n \frac{r}{R} - nLn(1-\alpha_n \frac{r}{R}) \quad (5)$$

Expand functions $Ln(1-\alpha_n r)$ and $Ln(1-\alpha_n \frac{r}{R})$ and substituting then into (5), thus

$$\Delta_n \geq \alpha_n r^2 + \frac{r^2}{2}{\alpha_n}^2(\frac{n}{R^2}-1) + \cdots + \frac{r^k}{k}{\alpha_n}^k(\frac{n}{R^k}-1) + \cdots \quad (6)$$

In the similar way for the Scherijver's algorithm we can write:

$$\Delta_S \geq \alpha_S r^2 + \frac{r^2}{2}{\alpha_S}^2(\frac{n}{R^2}-1) + \cdots + \frac{r^k}{k}{\alpha_S}^k(\frac{n}{R^k}-1) + \cdots \quad (7)$$

From inequalities (6) and (7) we have:

(8)

$$\Delta_n - \Delta_S \geq (\alpha_n - \alpha_S) r^2 + \frac{r^2}{2}({\alpha_n}^2 - {\alpha_S}^2)(\frac{n}{R^2}-1) + \cdots + \frac{r^k}{k}({\alpha_n}^k - {\alpha_S}^k)(\frac{n}{R^k}-1) + \cdots$$

Since for all k, $(\frac{n}{R^k}-1) \geq 0$, in the inequality (8) all of the right hand side terms are

positive and consequently we have:

$$\Delta_n - \Delta_S \geq (\alpha_n - \alpha_s) r^2 + \frac{r^2}{2}({\alpha_n}^2 - {\alpha_S}^2)(\frac{n}{R^2}-1).$$

We know that for $t = 1$, $\Delta_s \geq \Psi(t) = 1 - Ln(2) = 0.30685$ [Scherijver, 1986]. This means that:

$$\Delta_n \geq 0.30685 + (\alpha_n - \alpha_S) r^2 + \frac{r^2}{2}({\alpha_n}^2 - {\alpha_S}^2)(\frac{n}{R^2}-1)$$

and then the proof is complete.

Theorem 2: After no more than

$$\frac{n}{\Psi(1)+(\alpha_n-\alpha_S)r^2+\frac{r^2}{2}(\alpha_n{}^2-\alpha_S{}^2)(\frac{n}{R^2}-1)}Ln(\frac{\mathbf{C}^t\mathbf{X_0}}{\varepsilon})$$

number of iterations, the algorithm will stop with a feasible point solution $\boldsymbol{X}$, where

$\boldsymbol{C}^t\boldsymbol{X} \le \varepsilon$.

Proof: Proof is similar to the proof of theorem 1, if we put:

$$\Psi(1)+(\alpha_n-\alpha_S)r^2+\frac{r^2}{2}(\alpha_n{}^2-\alpha_S{}^2)(\frac{n}{R^2}-1)$$

Instead of $\Psi(1)$.

6 Numerical Results

Various types of the linear programming problems with different range of sizes are considered. In the following plots (Fig. 1-4) we show that for the more accurate solutions we need to increase the number of iterations. The rate of increases in three different algorithms (Karmarkar, Scherijver and new method) varies, and it is dependent to α_k, α_s and α_n respectively. Our observations show that convergence of the new algorithm is faster than the two former algorithms. As the size of the problem increases, the solutions of Scherijver's algorithm tend to the solutions of our new algorithm. For the large scale problems, the solutions of Scherijver's algorithm and new algorithm is the same (Fig.4).

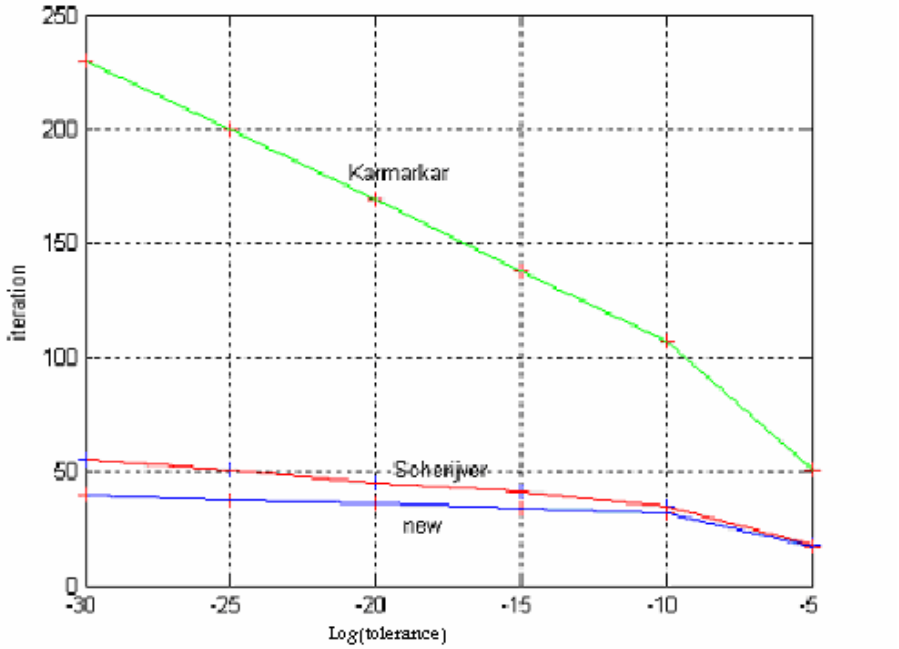

Figure 1-Plots of the tolerance in the approximation to the solutions of the linear programming problems of 2×2 dimensions.

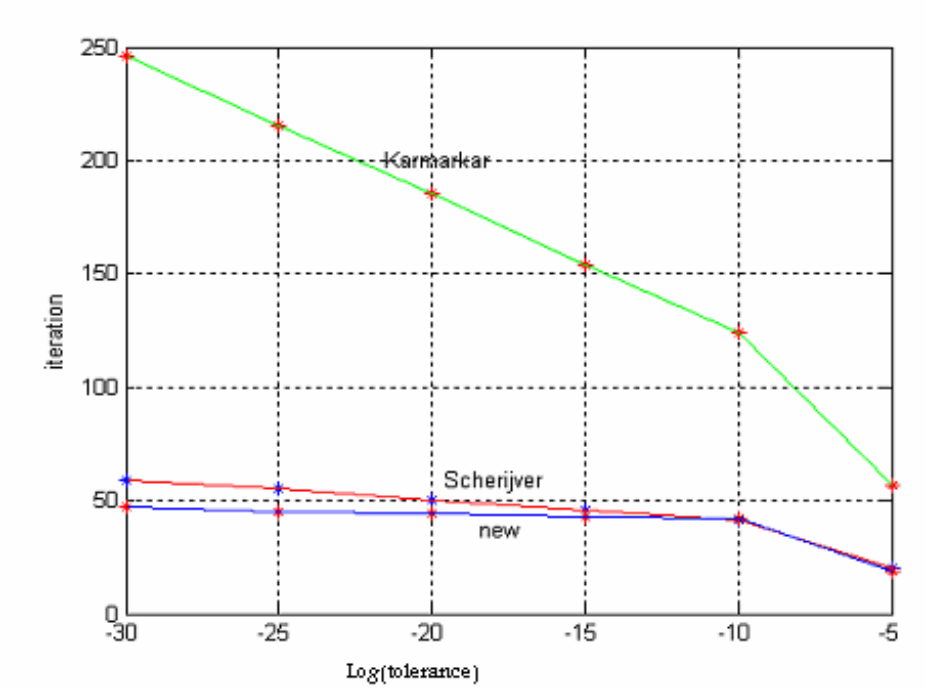

Figure 2-Plots of the tolerance in the approximation to the solutions of the linear programming problems of 3×3 dimensions.

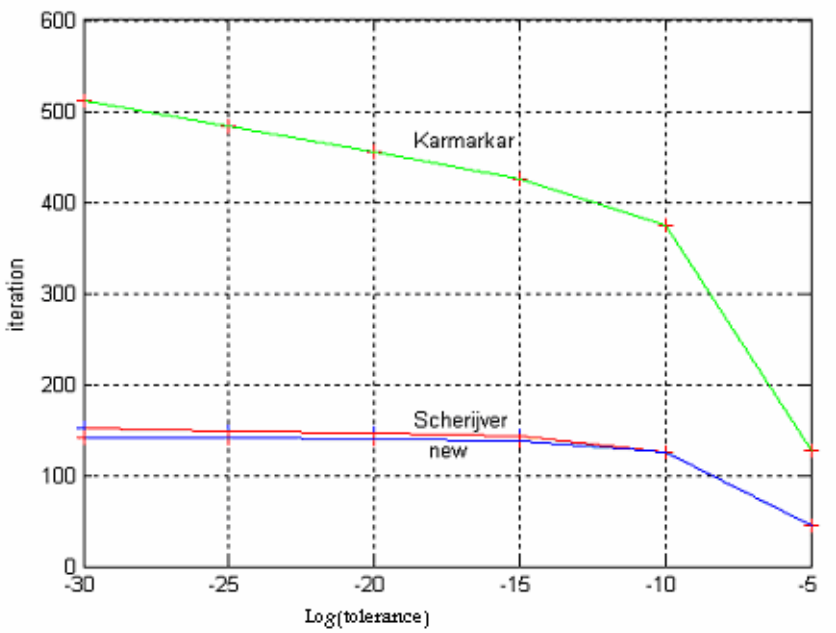

Figure 3- Plots of the tolerance in the approximation to the solutions of the linear programming problems of 50×50 dimensions.

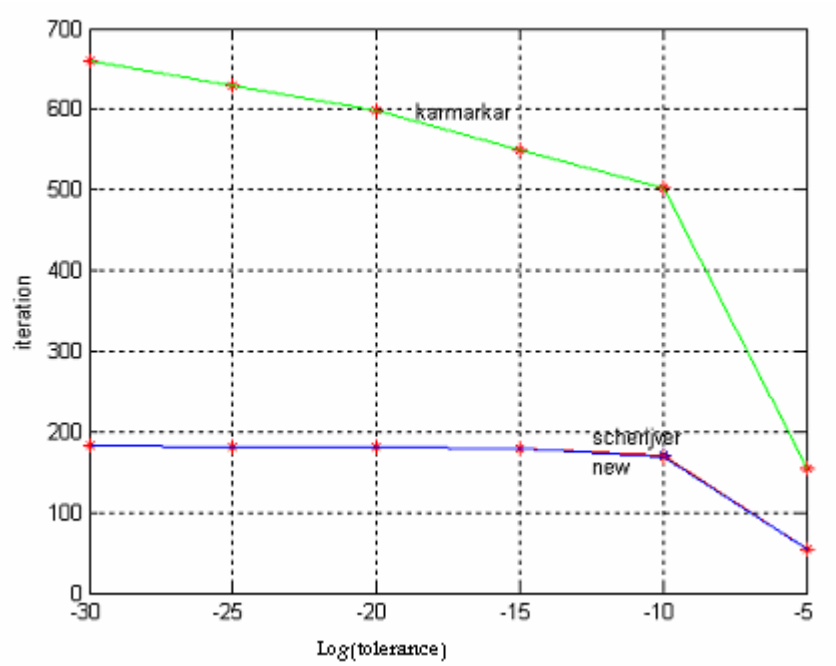

Figure 4- Plots of the tolerance in the approximation to the solutions of the linear programming problems of 100×100 dimensions.

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