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A STOCK-REVIEW EOQ MODEL WITH STOCK-DEPENDENT DEMAND, QUADRATIC DETERIORATION RATE

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Abstract.

A stock-review inventory model is developed for perishable items with uniform replenishment rate and stock-dependent demand. The deterioration function per unit time is a quadratic function of time. The associative cost function under some constraints is optimized due to the limitation of storage capacity.
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1. Introduction

The (S, Q) model with lost sales was first discussed by Hadley and Whitin [1]. They derived an exact formulation of the average inventory cost for an (S, Q) policy with poisson demand and constant deterministic lead times. They also presented an easy approximation of the average cost and developed an iterative procedure to optimize the policy parameters which has become the standard text book approach [2, 3]. Thereafter, Johansen and Thorstenson[4] formulated and solved the same model as a semi-Markov decision model. The first contribution in a continuous review inventory model was made by Nahamias and Demmy[5]. They analysed an (S, Q) inventory model with two demand classes, poisson demand, backordering, a fixed lead time and a critical level policy. The result of Nahamias and Demmy[5] were generalised by Moon and Kang[6]. They considered an (S, Q) model with compound poisson demand, and derived (approximate) expression for the fill rates of the two demand classes. Cohen et at.[7] consider a periodic review

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(s, S) inventory system where all demand in each period are collected and by the end of each period the inventory is used to satisfy high-priority demand first and the remaining inventory is then made available for low priority demand.

In formulating inventory models, two facets of the problem have been of growing interest, one being the deterioration of items, the other being the variation in the demand rate. Among researchers considering inventory models for deteriorating items, Shah and Jaiswal [8] considred the rate of deterioration to be uniform, Covert and Philip [9] formulated an EOQ model for items with variable rate of deterioration, Misra [10] used a two-parameter Weibull distribution to fit the deterioration rate, and Deb and Chaudhuri [11] suggested a model with variable rate of deterioration allowing shortages to occure. Gupta and Vrat [12] considered a model of stock-dependent consumption rate.

In the proposed model , an inventory model is suggested for deteriorating items with a function of time, quadratic deterioration rate. In the model the rate of replenishment is uniform, the demand rate is varying with stock-level, setup cost is considered, limitation of storage capacity is considered.

2. Fundamental Assumptions and Notations

Assumptions:

We adopt the following assumptions and notations for the models to be discussed.

1. Replenishment rate is infinite but replenishment size is finite,
2. Lead time is zero,
3. No shortages are permitted,
4. The time-horizon is infinite.

Notations:

$q(t)$ - On-hand inventory at time "t"(≥ 0) ;

S_1 - Starting and ending inventory level;

S_2 - Pick of the inventory level;

P - Finite replenishment rate ;

C_s - Set up cost per cycle;

C_h - Inventory holding cost per unit per unit time;

C_p - Procurement cost per unit item ;

T - Duration of the cycle;

3. Formulation of the Model

Here we consider the demand rate of perishable items depends upon on hand inventory. The items undergoes decay at $\theta(t)I(t)$ at time t. Generally, deterioration increases to increase of time t. Here

$$\begin{aligned}\theta(t) &= a + bt + ct^2 \\ \theta\,'(t) &= b + 2ct \\ \theta\,''(t) &= 2c\end{aligned}$$

where a = initial deterioration, b = intital rate of change of deterioration , c = acceleration of deterioration.
In this model, uniform replenishment rate starts with inventory (S_1) and continues upto time $t = t_1$. The inventory piles up during $[0,\ t_1]$, after meeting demands in the market. The inventory level at time $t = t_1$ is S_2. The storage space is limited here. It can store maximum (S_{max}) units. Again, the inventory level reaches at S_1 at time $t = T$. Therefore, the governing equations of this model are:

$$\begin{aligned}\frac{dq(t)}{dt} &= P - \alpha q^{\beta} - \theta(t)q\ ,\ 0 \le t \le t_1 \qquad (1) \\ with &\qquad q(0) = S_1 \ \ and \ \ q(t_1) = S_2\end{aligned}$$

and

$$\begin{aligned}\frac{dq(t)}{dt} &= -\alpha q^{\beta} - \theta(t)q \ \ ,\ \ t_1 \le t \le T \qquad (2) \\ with &\qquad q(T) = S_1.\end{aligned}$$

Since the equ.(1) and equ.(2) can not be solved by classical method. So we can get an approximate solution by Taylor's Series expansion . This approximation is valid for short term review period. Now from equ.(1) , we have

$$\begin{aligned}\frac{dq}{dt} &= P - \alpha q^{\beta} - \theta(t)q \\ \frac{d^2q}{dt^2} &= -\alpha\beta q^{\beta-1}\frac{dq}{dt} - \theta\,'(t)q \\ &\quad -\theta(t)\frac{dq}{dt} \\ \frac{d^3q}{dt^3} &= -\alpha\beta(\beta-1)q^{\beta-2}(\frac{dq}{dt})^2\end{aligned}$$

$$-\alpha\beta q^{\beta-1}\frac{d^2q}{dt^2} - \theta\,''(t)q$$
$$-2\theta\,'(t)\frac{dq}{dt} - \theta(t)\frac{d^2q}{dt^2}$$

Now using the initial condition, we have

$$[\frac{dq}{dt}]_{t=t_0} = P - \alpha S_1^{\beta} - aS_1 = M_1(S_1)\ (>0)\ (say),$$

for feasibility of the model,

$$\begin{aligned}[\frac{d^2q}{dt^2}]_{t=t_0} &= -\alpha\beta P S_1^{\beta-1} + \alpha^2\beta S_1^{2\beta-1} + \alpha\beta a S_1^{\beta} \\ &\quad -aP + a\alpha S_1^{\beta} + a^2 S_1 - bS_1 = N_1(S_1)\ \ (say)\end{aligned}$$

$$\begin{aligned}[\frac{d^3q}{dt^3}]_{t=t_0} &= -\alpha^3\beta(\beta-1)S_1^{3\beta-1} - \alpha^3\beta^2 S_1^{3\beta-2} \\ &\quad -(3a\alpha^2\beta^2 - 2P\alpha^2\beta(\beta-1))S_1^{2\beta-1} - 2a\alpha^2\beta(\beta-1)S_1^{2\beta} \\ &\quad +\alpha^2\beta^2 P S_1^{2\beta-2} - a^2\alpha\beta(\beta-1)S_1^{\beta+1} \\ &\quad -\{a^2\alpha + 2a^2\alpha\beta - \alpha\beta b - 2aP\alpha\beta(\beta-1) - 2\alpha b\}S_1^{\beta} \\ &\quad +(2aP\alpha\beta - \alpha\beta(\beta-1)P^2)S_1^{\beta-1} \\ &\quad -(a^3 - 3ab + 2c)S_1 + (a^2 - 2b)P = R_1(S_1)\ \ (say).\end{aligned}$$

Neglecting higher order of the expansion of $q(t)$, we have

$$\begin{aligned}q(t) &= q(0) + [\frac{dq}{dt}]_{t=t_0}.t \\ &\quad +[\frac{d^2q}{dt^2}]_{t=t_0}\frac{t^2}{2} + [\frac{d^3q}{dt^3}]_{t=t_0}\frac{t^3}{6} \\ &= S_1 + M_1(S_1).t + \frac{N_1(S_1)}{2}.t^2 + \frac{R_1(S_1)}{6}.t^3\ ,\ \ 0 \le t \le t_1 \qquad (3)\end{aligned}$$

Similarly, from equ.(2), we have

$$\begin{aligned}q(t) &= S_2 + M_2(S_2)(t-t_1) + \frac{N_2(S_2)}{2}(t-t_1)^2 \\ &\quad +\frac{R_2(S_2)}{6}(t-t_1)^3\ ,\ \ t_1 \le t \le T \qquad (4)\end{aligned}$$

where

$$\begin{aligned}
M_2(S_2) &= -\alpha S_2^\beta - (a + bt_1 + ct_1^2)S_2 \\
N_2(S_2) &= \alpha^2\beta S_2^{2\beta-1} + \alpha(\beta+1)(a + bt_1 + ct_1^2)S_2^\beta \\
&\quad +\{(a + bt_1 + ct_1^2)^2 - (b + 2ct_1)\}S_2 \\
R_2(S_2) &= -\alpha^3\beta(\beta-1)S_2^{3\beta-1} - \alpha^3\beta^2 S_2^{3\beta-2} \\
&\quad -3a\alpha^2\beta^2 S_2^{2\beta-1} - 2a\alpha^2\beta(\beta-1)S_2^{2\beta} \\
&\quad -a^2\alpha\beta(\beta-1)S_2^{\beta+1} - \{a^2\alpha + 2a^2\alpha\beta \\
&\quad -\alpha\beta b - 2\alpha b\}S_2^\beta - (a^3 - 3ab + 2c)S_2
\end{aligned}$$

Using the condition $q(t_1) = S_2$ in equ.(3), we have

$$R_1(S_1)t_1^3 + 3N_1(S_1)t_1^2 + 6M_1(S_1)t_1 + (S_1 - S_2) = 0 \qquad (5)$$

Here $S_1 - S_2 < 0$, as $S_2 > S_1$ and $M_1(S_1) > 0$. Therefore, by Descarte's rule, it may have at least one positive real root. This equation can be solved by Cardon's method. One real root of the above equation is (*see Appendix*)

$$\begin{aligned}
t_1 &= \frac{1}{R_1}[\{-3R_1^2(S_1 - S_2) + 3N_1M_1R_1 - N_1^3 \\
&\quad +\sqrt{(3R_1^2(S_1 - S_2) - 3N_1M_1R_1 + N_1^3)^2 + (2M_1R_1 - N_1^2)^3}\ \}^{\frac{1}{3}} \\
&\quad +\{-3R_1^2(S_1 - S_2) + 3N_1M_1R_1 - N_1^3 \\
&\quad -\sqrt{(3R_1^2(S_1 - S_2) - 3N_1M_1R_1 + N_1^3)^2 + (2M_1R_1 - N_1^2)^3}\ \}^{\frac{1}{3}}] - \frac{N_1}{R_1}
\end{aligned}$$

Similarly, using the condition $q(T) = S_1$ in equ.(4), we have

$$\begin{aligned}
R_2(S_2)(T - t_1)^3 + 3N_2(S_2)(T - t_1)^2 & \\
+6M_2(S_2)(T - t_1) + (S_2 - S_1) &= 0 \qquad (6)
\end{aligned}$$

Here $S_1 - S_2 < 0$, as $S_2 > S_1$ and $M_2(S_2) < 0$. Therefore, by Descarte's rule, it may have at least one positive real root. This equation can be solved by Cardon's method. One real root of the above equation is (see *Appendix*)

$$\begin{aligned}
T &= \frac{1}{R_2}[\{-3R_2^2(S_2 - S_1) + 3N_2M_2R_2 - N_2^3 \\
&\quad +\sqrt{(3R_2^2(S_2 - S_1) - 3N_2M_2R_2 + N_2^3)^2 + (2M_2R_2 - N_2^2)^3}\ \}^{\frac{1}{3}} \\
&\quad +\{-3R_2^2(S_2 - S_1) + 3N_2M_2R_2 - N_2^3 \\
&\quad -\sqrt{(3R_2^2(S_2 - S_1) - 3N_2M_2R_2 + N_2^3)^2 + (2M_2R_2 - N_2^2)^3}\ \}^{\frac{1}{3}}] \\
&\quad -\frac{N_2}{R_2} + t_1
\end{aligned}$$

Therefore, the total average cost is

$$
\begin{aligned}
AVC(S_1, S_2) &= \frac{1}{T}[C_h\{\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt\} + C_s + kPt_1] \\
&= \frac{1}{T}[C_h\{S_1t_1 + M_1(S_1)\frac{t_1^2}{2} + N_1(S_1)\frac{t_1^3}{6} + R_1(S_1)\frac{t_1^4}{24} \\
&\quad + S_2(T - t_1) + M_2(S_2)\frac{(t_2 - t_1)^2}{2} + N_2(S_2)\frac{(T - t_1)^3}{6} \\
&\quad + R_2(S_2)\frac{(T - t_1)^4}{24}\} + C_s + C_pPt_1] \qquad (7)
\end{aligned}
$$

Now we have to

$$
\begin{aligned}
&Minimize && AVC(S_1, S_2) \\
&such\ that && S_1 > 0, \\
& && P > \alpha S_1^{\beta} + aS_1, \\
& && S_2 > S_1, \\
& && S_2 \leq S_{max}.
\end{aligned}
$$

The above constrained minimization problem can be solved by *Interior Penalty Function Method* or any other *Software* .

4.Conclusion

One of the important problems to a supply manager in modern organization is the control and maintenance of inventories of deteriorating items. For some items, such as steel, hardware, toys, and glassware, the rate of deterioration is so low that there is little need for considering deterioration in the determination of the economic lot size. However, there are numerous types of storage so that in time they become partially or unfit for consumption. For example, lysis or the disintegration of red blood cells renders blood unacceptable for transfusion twenty-one days after the blood is drawn. Fresh produce, meats, and other foodstuffs becomes unusable after a certain time has elapsed. Photographic film and drugs are further examples of items that have a limited useful lifetime. It is now evident that in many systems, the impact of deterioration or perishability cannot be neglected.

It is real fact that, in a supermarket, a large piles of goods motivated the customer to by more. So the demand rate should be a function of the stock-level. In the existing literature, deterioration rate is considered as constant, linear function of time, and weibull distribution. But we have considered the deterioration is a quadratic function of time. Because, when deterioration starts then it is accelerated with time. So the purpose of our model is to reduce the inventory cost and deterioration at optimal inventory-level , production-run-time and inventory-review-level.

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Appendix:

Here $R_1(S_1)t_1^3 + 3N_1(S_1)t_1^2 + 6M_1(S_1)t_1 + 6(S_1 - S_2) = 0$,
or,

$$t_1^3 + 3B_1 t_1^2 + 3C_1 t_1 + D_1 = 0 \tag{8}$$

wher $B_1 = \frac{N_1}{R_1}$, $C_1 = \frac{2M_1}{R_1}$, $D_1 = \frac{6(S_1-S_2)}{R_1}$. Let the roots of equ.(8) are diminished by h such that the 2nd degree term is removed. Let α, β, γ be the roots of this equation . To diminish the roots of this equation by h , we put $t_1 = y + h$ and the transformed equation is

$$\begin{aligned} y^3 + 3(h + B_1)y^2 + 3(h^2 + 2B_1h + C_1)y & \\ +(h^3 + 3B_1h^2 + 3C_1h + D_1) &= 0 \end{aligned} \qquad (9)$$

Next we remove the second term by effecting $h+B_1 = 0$, so that $h = -B_1 = -\frac{N_1}{R_1}$ and the equation becomes

$$y^3 + 3(C_1 - B_1^2)y + (D_1 - 3B_1C_1 + 2B_1^3) = 0 \qquad (10)$$

Using the symbols

$$\begin{aligned} H &= C_1 - B_1^2 \\ &= 2\frac{M_1}{R_1} - (\frac{N_1}{R_1})^2 \\ &= \frac{1}{R_1^2}(2M_1R_1 - N_1^2), \\ G &= D_1 - 3B_1C_1 + 2B_1^3 \\ &= \frac{2}{R_1^3}\{3R_1^2(S_1 - S_2) - 3N_1M_1R_1 + N_1^3\}, \end{aligned}$$

the transformed equation is

$$y^3 + 3Hy + G = 0 \qquad (11)$$

The roots of this equarion are thus $\alpha-h, \beta-h, \gamma-h$, i.e., $\alpha+B_1, \beta+B_1, \gamma+B_1$. By cardon's method, we have at least one real root of equ. (11) that is $[\frac{-G+\sqrt{G^2+4H^3}}{2}]^{\frac{1}{3}} + [\frac{-G-\sqrt{G^2+4H^3}}{2}]^{\frac{1}{3}}$. Therefore, the real root of equ.(8) is

$$\begin{aligned} t_1 &= [\frac{-G + \sqrt{G^2 + 4H^3}}{2}]^{\frac{1}{3}} + [\frac{-G - \sqrt{G^2 + 4H^3}}{2}]^{\frac{1}{3}} + B_1 \\ &= \frac{1}{R_1}[\{-3R_1^2(S_1 - S_2) + 3N_1M_1R_1 - N_1^3 \\ &\quad +\sqrt{(3R_1^2(S_1 - S_2) - 3N_1M_1R_1 + N_1^3)^2 + (2M_1R_1 - N_1^2)^3}\ \}^{\frac{1}{3}} \\ &\quad +\{-3R_1^2(S_1 - S_2) + 3N_1M_1R_1 - N_1^3 \\ &\quad -\sqrt{(3R_1^2(S_1 - S_2) - 3N_1M_1R_1 + N_1^3)^2 + (2M_1R_1 - N_1^2)^3}\ \}^{\frac{1}{3}}] \\ &\quad -\frac{N_1}{R_1} \end{aligned} \qquad (12)$$