ON A VOLUME FLEXIBLE PRODUCTION POLICY FOR A DETERIORATING ITEM WITH TIME-DEPENDENT DEMAND AND SHORTAGES

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Scope and Purpose: The scope of the model presented here lies in its application in **FMS** (*Flexible Manufacturing system*) which is considered to be one of the best ways to improve production efficiency in modern manufacturing concerns. The purpose of the paper is to incorporate the concept of flexibility in the machine production rate of an item into a quantitative production-inventory model having the potential for application in manufacturing industries. The unit production cost is linked to the variable production rate. Physical decay of the stocked item over time is taken into account and shortages in inventory are allowed. All these practical aspects are incorporated into the model with the purpose of making it more realistic.

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b Corresponding author: Department of Mathematics, Jadavpur University, Kolkata-700 032. INDIA E-mail: k_s_chaudhuri@yahoo.com Tel: +91-033-2414 6717 Fax: +91-033-2414 6484 Abstract: The paper extends the classical *Economic Lot Scheduling Problem* (**ELSP**) to the case of a volume flexible manufacturing system in which the production rate is flexible and the unit production cost is a function of the production rate. The model is developed over a finite planning horizon taking the production rate as a decision variable. It is also assumed that (i) the demand rate is time-dependent, (ii) the stock undergoes decay or deterioration over time at a constant rate and (iii) shortages in inventory are allowed and are completely backlogged. The associated constrained Maximization Problem is solved numerically by using the **Interior Penalty Function Method** for a given set of parameter values to obtain the nearoptimal solutions in the successive production cycles. The production policy is discussed in the light of the critical design production rate of the manufacturing machine.

Keywords: Volume flexible production, time-dependent demand, deteriorating item, shortages in inventory.

1. Introduction

The traditional approach in the Classical Economic Lot Scheduling Problem (**ELSP**) is to take the production rate of a machine to be pre-determined and inflexible[1]. Adler and Nanda[2] extended the **ELSP** model to situations where learning effects would induce an increase in the production rate. The models of Sule([3], [4]), Axsater and Elmaghraby[5], and Muth and Spearmann[6] were concerned with learning effects on the optimal lot size. Proteus[7] and Rosenblat and Lee[8] extended the **EOQ** (*economic order quantity*) and the **ELSP** models to the imperfect production processes. Cheng[9] extended the **ELSP** model to an imperfect production process in which demand would exceed supply.

Schweitzer and Seidmann[10] adopted for the first time, the concept of flexibility in the machine production rate and discussed optimization of processing rates for a **FMS** (flexible manufacturing system). Obviously, the machine production rate is a decision variable in the case of a **FMS** and then the unit production cost becomes a function of the production rate. Khouja and Mehrez[11] and Khouja[12] extended the **ELSP** model to an imperfect production process with a flexible production rate. Silver[13] discussed the effects of slowing down production in the context of a manufacturing equipment dedicated to the production of a family of items, assuming a common cycle for all items. Controllable production rates in a family production context were also considered by Moon, Gallego and Simchi-Levi[14]. Gallego[15] extended the model of Silver[13] by removing the stipulation of a common cycle for all the items.

Nowadays the managers of manufacturing companies have, in their view, mainly four systems to improve production efficiency. These are MRP(materials requirement planning), OPT(optimized production technology), JIT(just-intime) and FMS(flexible manufacturing systems). FMS offers the hope ofeliminating many of the weaknesses of the other three approaches[16]. Volume flexibility is a major component in a FMS. The manufacturing flexibilitywhich is capable of adjusting the production rate with the variability in themarket demand is known as volume flexibility[17]. In the present paper, we consider a volume flexible manufacturing system for a deteriorating item with a time-dependent demand rate, allowing shortages in inventory. None of the authors referred to above took into account the factors of demnd variability, shortages and deterioration of goods. We solve this general model for an imperfect production process taking the production rate to be a decision variable and the unit production cost to be a function of the production rate.

2. Fundamental Assumptions and Notations

Assumptions:

- 1. The inventory system involves only one item and is a self-production system.
- 2. The demand rate for the product is deterministic and is a continuous linear function of time 't'.
- 3. Shortages in inventory are allowed and are completely backlogged.
- 4. The time horizon is infinite.
- 5. The production cost per unit item is a function of the production rate.
- 6. The production rate P_i in the i-th cycle is considered as a decision variable.
- 7. The inventory deteriorates over time at a constant rate.

Notations:

f(t) - % f(t) - f(t) Demand rate function varying with time ${}^{\prime}t^{\prime}\geq 0$.

- h Holding cost per unit per unit time.
- θ Constant deterioration rate of the on-hand inventory, $0 < \theta < 1$.
- $\eta(P_i)$ The production cost per unit item in the i th cycle.
- $S\,$ $\,$ Set up cost per production run.
- C_2 Shortage cost per unit per unit time.
- S_p Selling price per unit.
- $R_{i-1}\,$ $\,$ The total time elapsed up to and including the (i-1)-th cycle.
- T_i The duration of the i th cycle.
- $t_{ij}(j=1,2)$ The times at which the stock-period starts and ends

respectively in the i - th cycle.

 $S_{ij}(j=1,2)$ - The shortage at the beginnig and the end of the i-th cycle.

3. Formulation of the Model

We consider a self-manufacturing system in which the items are produced by a machine and the demand in the market is met by these produced items. The cost for setting up the machine and the whole system is recokned only in the first cycle. The production cost per unit in the *i*-th (i = 1, 2, 3, ...) cycle is

$$\eta(P_i) = r + \frac{g}{P_i} + \alpha P_i + \beta (P_i - P_c) H(P_i - P_c)$$

where

$$H(P_i - P_c) = 1 , \quad P_i > P_c$$
$$= 0 , \quad P_i \le P_c$$

This cost is based on the following factors :

- 1. The material cost r per unit item is fixed.
- 2. As the production rate increases, some costs like labour and energy costs are equally distributed over a large number of units. Hence the per-unit production cost $\left(\frac{g}{P_i}\right)$ decreases as the production rate (P_i) increases.
- 3. The third term(αP_i), associated with tool/die costs, is proportional to the production rate.
- 4. The fourth term is linked to a critical design production $rate(P_c)$ for the machine. The produced items are quite likely to be defective for a high production rate($P_i > P_c$). Then excess labour and energy costs alongwith rework costs will be needed to get perfect items.

Let $I_{i1}(t)$ be the shortage level at any time t in $0 \le t \le t_{i1}$, $I_{i2}(t)$ the inventory level in $t_{i1} \le t \le t_{i2}$ and $I_{i3}(t)$ the shortage level in $t_{i2} \le t \le T_i$.

Then the governing differential equations for the instantaneous inventory level during *i*-th (i = 1, 2, 3, ...) cycle are as follows:

$$\frac{dI_{i1}(t)}{dt} = P_i - f(R_{i-1} + t) \quad , \quad 0 \le t \le t_{i1}$$
(1)

with $I_{i1}(0) = -S_{i1}$ and $I_{i1}(t_{i1}) = 0$;

$$\frac{dI_{i2}(t)}{dt} + \theta I_{i2}(t) = P_i - f(R_{i-1} + t) \quad , \quad t_{i1} \le t \le t_{i2} \tag{2}$$

with
$$I_{i2}(t_{i1}) = 0$$
 and $I_{i2}(t_{i2}) = 0$;

$$\frac{dI_{i3}(t)}{dt} = P_i - f(R_{i-1} + t) \quad , \quad t_{i2} \le t \le T_i \tag{3}$$

with
$$I_{i3}(t_{i2}) = 0$$
 and $I_{i3}(T_i) = -S_{i2}$.

4. Solution of the Model

Let f(t) = a + bt, $a \ge 0$, b > 0. Here a denotes the initial demand rate and b stands for the rate at which the demand rate increases per unit of time. The solutions of (1), (2) and (3) are then given by

$$I_{i1}(t) = (P_i - a - bR_{i-1})(t - t_{i1}) - \frac{1}{2}b(t^2 - t_{i1}^2) \quad , \quad 0 \le t \le t_{i1} \quad ; \tag{4}$$

$$I_{i2}(t) = \left(\frac{P_i - a - bR_{i-1}}{\theta}\right) \left(1 - e^{\theta(t_{i1} - t)}\right) - \frac{b}{\theta^2} \{(\theta t - 1) - (\theta t_{i1} - 1)e^{\theta(t_{i1} - t)}\}, \quad t_{i1} \le t \le t_{i2}$$
(5)

and

$$I_{i3}(t) = (P_i - a - bR_{i-1})(t^2 - t_{i2}^2) \quad , \quad t_{i2} \le t \le T_i.$$
(6)

Using the condition $I_{i1}(0) = -S_{i1}$, we have from (4),

$$t_{i1} = \frac{1}{b} [(P_i - a - bR_{i-1}) - \{(P_i - a - bR_{i-1})^2 - 2bS_{i1}\}^{\frac{1}{2}}].$$
 (7)

It is obvious from the above result that $t_{i1} = 0$ when $S_{i1} = 0$, i.e., the first cycle starts with no-shortage.

Since $I_{i2}(t_{i2}) = 0$, we have from (5),

$$\left(\frac{P_{i-a-bR_{i-1}}}{\theta}\right)\left(1-e^{\theta(t_{i1}-t_{i2})}\right) - \frac{b}{\theta^{2}}\left\{\left(\theta t_{i2}-1\right) - \left(\theta t_{i1}-1\right)e^{\theta(t_{i1}-t_{i2})}\right\} = 0$$

or,
$$\{b(\theta t_{i1}-1)-\theta(P_i-a-bR_{i-1})\}e^{\theta(t_{i1}-t_{i2})}+\{\theta(P_i-a-bR_{i-1})+b(1-\theta t_{i2})\}=0$$

or,

$$F(P_i, t_{i2}) = 0,$$
 (8)

where $t_{i1} = \tau(P_i)$.

Both t_{i1} and t_{i2} being real and distinct, the constraint $P_i > a + bR_{i-1} + (2bS_{i1})^{\frac{1}{2}}$ must be satisfied. Therefore the total inventory during *i*-th cycle is

$$H_{i} = \int_{\tau}^{t_{i2}} I_{i2}(t) dt$$

$$= \int_{\tau}^{t_{i2}} [\{\frac{b}{\theta^{2}}(\theta t_{i1} - 1) - \frac{1}{\theta}(P_{i} - a - bR_{i-1})\}e^{\theta(\tau - t)} - \frac{b}{\theta}t$$

$$+ \{\frac{P_{i} - a - bR_{i-1}}{\theta} + \frac{b}{\theta^{2}}\}]dt$$

$$= \phi(P_{i})\int_{\tau}^{t_{i2}} e^{-\theta t} dt - \frac{b}{\theta}\int_{\tau}^{t_{i2}} t dt + \psi(P_{i})\int_{\tau}^{t_{i2}} dt$$

$$= \frac{\phi(P_{i})}{\theta}\{e^{-\theta \tau} - e^{-\theta t_{i2}}\} - \frac{b}{2\theta}(t_{i2}^{2} - \tau^{2})$$

$$+ \psi(P_{i})(t_{i2} - \tau)$$
(9)

where

$$\phi(P_i) = \frac{1}{\theta^2} \{ b(\theta \tau - 1) - \theta(P_i - a - bR_{i-1}) \} e^{\theta \tau}$$

and $\psi(P_i) = \frac{1}{\theta^2} \{ b + \theta(P_i - a - bR_{i-1}) \}.$

The total shortage in inventory in the i-th cycle is given by

$$S_i = S_{i1} + S_{i2}$$

where

$$S_{i1} = \int_{0}^{t_{i1}} \{-I_{i1}(t)\} dt$$

= $-\int_{0}^{t_{i1}} \{(P_{i} - a - bR_{i-1})(t - t_{i1}) - \frac{1}{2}b(t^{2} - t_{i1}^{2})\} dt$
= $\frac{1}{2}(P_{i} - a - bR_{i-1})t_{i1}^{2} - \frac{1}{3}bt_{i1}^{3},$

$$S_{i2} = \int_{t_{i2}}^{T_i} \{-I_{i3}(t)\} dt$$

= $-\int_{t_{i2}}^{T_i} \{(P_i - a - bR_{i-1})(t - t_{i2}) - \frac{1}{2}(t^2 - t_{i2}^2)\} dt$
= $-\frac{1}{2}(P_i - a - bR_{i-1})(T_i - t_{i2})^2 + \frac{1}{6}b(T_i^3 + 2t_{i2}^3 - 3T_it_{i2}^2).$

After a little calculation, S_i becomes

$$S_{i} = \frac{2}{3b^{2}} \{ (P_{i} - a - bR_{i-1})^{2} - 2bS_{i1} \}^{\frac{3}{2}} + \frac{T_{i}}{6} \{ bT_{i}^{2} - 3T_{i}(P_{i} - a - bR_{i-1}) + 6S_{i1} \}.$$
 (10)

Again, the number of items deteriorated during the i -th cycle is

$$D_i = \theta \int_{\tau}^{t_{i2}} I_{i2}(t) dt = \theta H_i.$$
(11)

The average profit during the i -th cycle is

$$\Pi(P_i, t_{i2}, T_i) = \frac{S_p}{T_i} \{ (a + bR_{i-1})(t_{i2} - t_{i1}) + \frac{b}{2}(t_{i2}^2 - t_{i1}^2) \} - \frac{1}{T_i} \{ \delta_i S + hH_i + C_2 S_i \} - \{ rP_i + g + \alpha P_i^2 + \beta P_i (P_i - P_c) H(P_i - P_c) \}.$$
(12)

Hence our problem is:

To Maximize $\Pi(P_i, t_{i2}, T_i)$

such that

 $a + bR_{i-1} + (2bS_{i1})^{\frac{1}{2}} - P_i \le 0 ,$ $P_c - P_i \le 0 \quad or \quad -P_c + P_i \le 0,$ $F(P_i , t_{i2}) = 0 ,$ $-T_i + t_{i2} \le 0,$ $-t_{i2} + t_{i1} \le 0,$

where P_i , t_{i2} , T_i are the decision variables.

We use the **Interior Penalty Function Method** (*see Appendix*) for numerical solution of the problem. The primal problem is reformulated below:

Primal Problem : (General Form)

Minimize $\Pi(\bar{X})$ =-Maximize $\Pi(\bar{X})$

such that

 $G_j(\bar{X}) \le 0, \quad j = 1, 2, \dots, m$

 $H_i(\bar{X}) = 0, \quad i = 1, 2, \dots l$

where $\Pi(\bar{X})$, $G_j(\bar{X})$, $H_i(\bar{X})$ are continuous functions of $\bar{X}\epsilon R^n.$

5. Numerical Example

We take the parameter values as a = 200, b = 5.0, $P_c = 220$, r = 90, g = 2500, $\alpha = 0.01$, $\beta = 0.04$, S = 200, h = 3, $C_2 = 5$, $S_p = 130.0$, $\theta = 0.01$ in appropriate units.

We have

$$\eta(P_i) = r + \frac{g}{P_i} + \alpha P_i + \beta (P_i - P_c) , \quad P_i > P_c$$
$$= r + \frac{g}{P_i} + \alpha P_i , \quad P_i \le P_c.$$

Then

$$\frac{d\eta}{dP_i} = -\frac{g}{P_i^2} + \alpha + \beta \quad , \quad P_i > P_c$$

$$= -\frac{g}{P_i^2} \qquad , \qquad P_i \le P_c$$

and

$$\frac{d^2\eta}{dP_i^2} = \frac{2g}{P_i^3} > 0 \quad for \quad all \quad P_i > 0.$$

We thus have :

(i)
$$\eta_{min}$$
 at $P_i = \sqrt{\frac{g}{\alpha + \beta}} = 223.6$ when $P_i > P_c$

and

(*ii*)
$$\eta_{min}$$
 at $P_i = \sqrt{\frac{g}{\alpha}}$ when $P_i \le P_c$.

We may , therefore , take $P_c = 220$. For $P_i > P_c$, Table 1 shows the average maximum profit and the corresponding optimum solution $(t_{i1}^*, t_{i2}^*, T_i^*)$ for different cycles. Similarly for $P_i < P_c$ and and $P_i > P_c$, Table 2 shows the average maximum profit and the corresponding optimum solutions.

Table -1: (For $P_i > P_c$)									
Cycle	t^*_{i1}	t^*_{i2}	T^*	P_i^*	Π^*_{max}				
Number									
01	0.000000	11.55398	11.66928	229.4408	5807.156				
02	0.177713	05.30644	05.33011	259.8940	7013.229				
03	0.002702	05.37641	05.39721	289.6619	8388.826				

	Table -2: (For $P_i < P_c$ & $P_i > P_c$)									
	Cycle	t_{i1}^*	t^*_{i2}	T^*	P_i^*	Π^*_{max}				
	Number									
	01	0.00000000	4.852452	4.900913	212.2290	5298.807				
	02	0.02649300	5.586026	5.609819	229.9313	7019.038				
	03	0.01052856	8.651609	8.737083	267.1449	7530.061				
	04	0.01383033	2.930659	2.946561	300.1275	7692.180				
* Ind	* Indicates the near-optimal solution									

6. Observation and Conclusion :

The maximum average profit (Π_{max}^*) , in Table 1 $(P_i > P_c)$ gradully increases from one cycle to another. Besides this, the optimum production rate (P_i^*) increases gradully. If we start with the condition $P_i < P_c$ with the same parameter values as in Table 1, it is found (Table 2) that there exists no feasible solution after the first cycle. Therefore, we have no alternative but to continue operation of the production system with the production rates $P_i > P_c$ for the rest of the cycles in the planning horizon. Hence the computations in the third and successive cycles are carried out with the condition $P_i > P_c$.

We may now compare the trade-offs between the pure strategy $(P_i > P_c)$ in Table 1 and the mixed strategy $(P_i < P_c \text{ and } P_i > P_c)$ in Table 2 for deciding a better manufacturing policy. The TAP(total average profit) $\frac{\sum \Pi_{max} T_i}{\sum T_i}$ for the pure strategy is 6716.326 while that for the mixed strategy is 6929.719. Thus a better production policy is to follow a mixed strategy. We have confirmed this result for several values of the planning horizon $\sum T_i$. For higher values of $\sum T_i$, the trade-off in the case of the mixed strategy will be higher.

References

- Hax A. C. and Candea D. Production and inventory management. Prentice-Hall, Englewood Cliffs, NJ, 1984.
- [2] Adler G. L. and Nanda R. The effects of learning on optimal lot size determination-single product case. AIIE Trans., 1974, 6, 14-20.
- [3] Sule D. R. A note on production time variation in determinaing EMQ under influence of learning and forgetting. *AIIE Trans.*, 1981, 13, 91-95.
- [4] Sule D. R. The effects of alternate periods learning and forgetting on economic manufacturing quantity. AII Trans., 1978, 10, 338-343.
- [5] Axsater S. and Elmaghraby S. E. A note on EMQ under learning and forgetting. AIIE Trans., 1981, 13, 86-90.
- [6] Muth, E. J. and Spearmann, K. Learning effects in economic lot size. Management Science, 1983, 29, 264-269.
- [7] Proteus E. L. Optimal lot sizing, process quality improvement, and setup cost reduction. Ops. Res., 1986, 34, 137-144.
- [8] Rosenblatt M. J. and Lee H. L. Economic production cycles with imperfect production processes. *IIE Trans.*, 1986, 17, 48-54.

- [9] Cheng T. C. E. An economic order quantity model with demanddependent unit production cost and imperfect production processes. *IIE Trans.*, 1991, 23, 23-28.
- [10] Schweitzer P. J. and Seidman A. Optimizing processing rates for flexible manufacturing systems. Mgmt. Sc., 1991, 37, 454-466.
- [11] Khouja M and Mehrez A. An economic production lot size model with imperfect quality and variable production rate. J. Opl. Res. Soc., 1994 , 45, 1405-1417.
- [12] Khouja M. The economic production lot size model under volume flexibility. *Computers Ops. res.*, 1995, 22, 515-523.
- [13] Silver E. A. Deliberately slowing down output in a family production context. Int. J. Prod. Res., 1990, 28, 17-27.
- [14] Moon I., Gallego G. and Simchi-Levi D. Controllable production rates in a family production context. Int. J. Prod. Res., 1991, 29, 2459-2470.
- [15] Gallego G. Reduced production rates in the economic lot scheduling problem. Int. J. Prod. Res., 1993, 31, 1035-1046.
- [16] Aggarwal S. C. Making sense of production operations system. Harvard Business Review, 1985, September-October, 8-16.
- [17] Sethi A. K. and Sethi P. S. Flexibility in manufacturing: A survey. Int. J. Flexible Manufacturing System, 1990, 2, 289-328.

- [18] Fiacco A. V. and Mc Cormick G. P. Extension of SUMT for non linear programming : equality constraints and extrapolation. *Management Science*, 1966, **12**, 816-828.
- [19] Fiacco A. V. and Mc Cormick G. P. Nonlinear programming: Sequential unconstrained minimization techniques, Wiley, New York, 1968.

Appendix

Interior Penalty Function Method :

This method generally deals with an unconstrained minimization problem. The general form of the problem equivalent to the Primal Problem is ([18], [19]):

Minimize
$$\chi_k(\bar{X}, r_k) = \Pi(\bar{X}) - r_k \sum_{j=1}^m \frac{1}{G_j(\bar{X})} + \frac{1}{\sqrt{r_k}} \sum_{i=1}^l H_i^2(\bar{X}).$$
 (13)

where r_k is a positive penalty parameter.

If χ_k is minimized for a sequence of decreasing values of r_k , the following theorem proves that the unconstrained minima \bar{X}_k^* (k = 1, 2, ..., m) converges to the solution \bar{X}^* of the primal problem stated above.

Theorem :

If the primal problem has a solution, the unconstrained minima \bar{X}_k^* of $\chi_k(\bar{X}, r)$ for a sequence of values $r_1 > r_2 > \dots > r_k$, converges to optimal solution of the primal problem as to the optimal solution of the primal.

The Iterative Procedure :

- Step 1. Start with an initial feasible point \bar{X}_1 , satisfying all the constraints with strict inequality sign, i.e., $G_j(\bar{X}_1) < 0$ for j=1,2,....m. and a suitable initial value of r_1 where $r_1 = -\frac{\Pi(\bar{X}_1)}{\sum_{j=1}^m \frac{1}{G_j(\bar{X}_1)}}$. Set k=1.
- Step 2. Minimize $\chi_k(\bar{X}_K, r_k)$ by using any method of unconstrained minimization(we use here the *Devidon Fletcher -Powell Method*) and obtain the solution \bar{X}_k^* .
- Step 3. Test whether $|\frac{\Pi(\bar{X}_{k}^{*}) \Pi(\bar{X}_{k+1}^{*})}{\Pi(\bar{X}_{k}^{*})}| \leq \epsilon_{1} |\bar{X}_{k}^{*} \bar{X}_{k-1}^{*}| < \epsilon_{2}$ where ϵ_{1} and ϵ_{2} are arbitrary small positive numbers. If it is satisfied, then terminate the process; otherwise, go to the next Step.
- Step 4. Find the value of next penalty parameter r as $r_{k+1} = Cr_k$ where 0;C;1.
- Step 5. Set the new value of k=k+1, take the new starting point as $\bar{X}_1 = \bar{X}_k^*$ and go to Step 2.

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