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Synthesis of Letter Strings in Script Style Based on Minimum Principle

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Abstract: A method for synthesizing a string of letters in script style is developed based on the minimum principle for handwriting, whereby the pen motion is organized so as to minimize the integral of squared jerk subject to visiting a series of spatial points. Let the series of spatial points characterizing a string be a concatenation of those taken from individual letters. The pen motion is best expressed in the form of a linear combination of quintic B-splines. Then, synthesis of the pen motion to write the string is reduced to optimization of the times at which the pen visits the characteristic points. The main contribution of this paper is a number of recurrence formulae that make it possible to evaluate arithmetically the gradients of the integral of squared jerk with respect to visiting times. An optimization algorithm is compiled incorporating the recurrence formulae in a gradient method. Several examples of synthesized letter strings are presented.

Keywords: Simulated handwriting, minimum jerk, splines, gradient method.

1. Introduction

Computer printing in cursive script style may enhance the impression of personal invitation letters or greeting cards even though they are not actually written by hand. There have been static and dynamic approaches to the synthesis of cursive letter strings from individual letters.

In the static approach, a string of letters connected by ligatures is regarded as a patchwork of the graphical images of individual letters and connecter elements. A traditional method employs a specially designed font in which letters begin and end at a common height so that printed letters look connected when aligned along the base line. An improved method by [Wasylyk, 1981] outfits a variety of letter shapes for each category and various types of connector elements to construct a better looking string. An advanced method by [Fenwick, 1995] shapes ligatures by cubic spline interpolation that bridges the facing boundaries of adjacent letters.

The dynamic approach is based on a model of pen motion in human handwriting. One successful attempt is 'Heliscript' by [Dooijes 1989], in which the pen trajectory is modeled as three-dimensional helical curves projected onto a flat surface. Since the trajectory is maintained continuous, a ligature is shaped naturally as the dynamic transition from one letter to another. In addition, [Flash and Hogan, 1985] provided a sound mathematical model of handwriting on the basis of a minimum principle in mechanical dynamics, although their model has not yet been applied to the synthesis of cursive letter strings.

According to the model of [Flash and Hogan, 1985], the pen motion is planned so as to minimize the square integral of its jerk (third-order derivative of the position vector with respect to time) under the constraints of passing through specified via points on a plane and certain boundary conditions. This model has been experimentally confirmed by the same authors. Another experiment by [Abend et al., 1982] suggests that the via points distribute around the points at which the pen has a locally minimal speed. The pen motion is best expressed by a quintic spline function of time having knots located at the via points, according to the general theorem by [de Boor, 1963] and [Schoenberg, 1964] on smoothest interpolation problems. The model was extended by [Kamada, 2003] to include an additional constraint of pausing at some of the via points (referred to herein as pause points). It has been demonstrated that the pause constraint is crucial for the reproduction of steeples, such as that at the left bottom corner of the letter 'h'. The pen motion following the extended model is best expressed by a mixture of quintic and quartic splines. Its practical and mathematically equivalent expression is a linear combination of quintic B-splines having knots at the via points and having double knots at the pause points. Splines having multiple knots are referred to as extended splines by [Curry and Schoenberg, 1966] without explicit reference to their optimal property. Based on the above theory, a letter can be reproduced well as an extended spline interpolation of the data consisting of the via and pause points identified as the points at which the pen speed is locally minimal and zero, respectively, and the times (referred to herein as via times) at which the pen visits these points.

Since the minimum principle for planning how to write a letter should also be true of writing a string of letters in one continuous stroke, the pen motion to write a string in script style may be synthesized as an extended quintic spline interpolation of the concatenated via and pause points taken from individual letters. Good concatenated points can be obtained by lining up via points of individual letters along the baseline. A good initial guess for the optimal via times corresponding to the concatenated points might be the accumulation of the time intervals between adjacent points. However, these via times are not really optimized for the concatenated points as a whole, since each subset of the via times taken from a real letter is optimized for the letter alone by the human brain.

The problem of optimizing the via times for the interpolation of given spatial points is called the data parameterization problem in the general theory of splines. Flash and Hogan, in their experimental confirmation process of their model, solved a special case of the problem, in which a single intermediate via point is allowed, by reducing the problem to a higher order algebraic equation [Flash and Hogan, 1985]. The case of cubic splines having simple knots was solved by [Marin, 1984]. He provided an analytic solution for the one-dimensional data and an iterative optimization procedure for the multi-dimensional data. Unfortunately, those results are not applicable to the present case of extended quintic splines represented in terms of B-splines having simple and double knots.

The present paper provides an iterative optimization procedure to optimize the via times so that the extended quintic spline satisfying the via and pause constraints makes the square integral of the jerk locally minimal. In order to cope with the double knots, the spline is represented in terms of the B-splines throughout this paper.

Following Section 2, which summarizes the handwriting model, the optimization procedure is derived in Section 3 in the following steps: (i) First, the pen motion is substituted by an extended quintic spline. This substitution converts the variational problem of minimizing the square integral of the jerk over the set of continuous functions into a problem of minimizing an objective function of the via times over a subset of Euclidean space. (ii) Integration included in the objective function, which causes trouble in numerical evaluation, is then analytically resolved to represent the objective function by an arithmetic function of the B-splines and their derivatives. (iii) Next, partial differentials of the B-splines (and their higher order derivatives) with respect to the via times, which construct the partial differentials of the objective function in an arithmetic form, are derived. (iv) Finally, an optimization procedure is compiled by incorporating the partial differentials into a standard gradient method. In Section 4, the effectiveness of this procedure in the synthesis of letter strings in script style from the data of individual letters is presented.

2. Preliminaries

A brief summary of the minimum principle, the expression for the pen motion in writing letters, and several well-known formulae concerning the splines are prepared in this section.

The motion of the pen is modeled as a vector-valued continuous function $\boldsymbol{r}(t) = (x(t), y(t))'$ of time $t \in [0, T]$ on the plane having $x - y$ orthogonal axes, as illustrated in Fig. 1, where the vector denoted by a prime indicates its transpose. It is assumed that the pen motion $\boldsymbol{r}(t)$ is planned so as to minimize the square integral of jerk,

$$P[\boldsymbol{r}] := \int_{0<t<T} \left\| \dddot{\boldsymbol{r}}\ (t) \right\|^2 dt \longrightarrow \min. \tag{1}$$

under the constraints of visiting the via points $\{\boldsymbol{r}_k\}_{k=1}^{K}$ on the plane consecutively at the via times $\{t_k\}_{k=1}^{K}$, $(0 < t_1 < t_2 < \cdots < t_K < T)$,

$$\boldsymbol{r}(t_k) = \boldsymbol{r}_k, k = 1, 2, 3, \cdots, K, \tag{2}$$

and pausing at pause points $\{t_{\ell_i}\}_{i=1}^{I}$, $(1 \leq \ell_1 < \ell_2 < \cdots < \ell_I \leq K)$,

$$\dot{\boldsymbol{r}}(t_{\ell_i}) = \boldsymbol{0}, i = 1, 2, 3, \cdots, I, \tag{3}$$

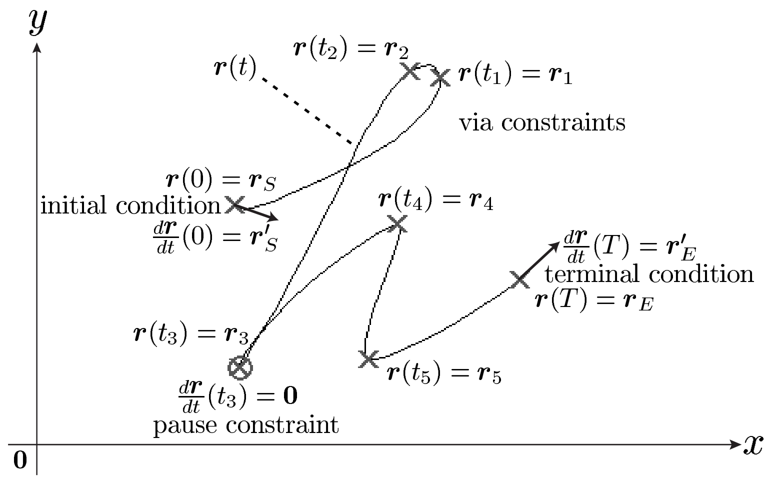

Figure 1: Constraints on handwriting motion ($\times$ denotes a via point and $\otimes$ denotes a pause point).

and the boundary conditions

$$\left\{ \begin{array}{llll} \boldsymbol{r}(0) = & \boldsymbol{r}_S, & \boldsymbol{r}(T) = & \boldsymbol{r}_E \\ \dot{\boldsymbol{r}}(0) = & \dot{\boldsymbol{r}}_S, & \dot{\boldsymbol{r}}(T) = & \dot{\boldsymbol{r}}_E, \end{array} \right. \tag{4}$$

where $||\cdot||$ denotes the Euclidean norm in the plane and a dot over a function indicates differentiation with respect to the variable t.

The solution $\boldsymbol{r}(t)$ of the minimization problem is analytically derived by [Kamada, 2003] as an extended quintic spline [Curry and Schoenberg, 1966] in the form of a linear combination

$$\boldsymbol{r}(t) = \sum_{j=-5}^{K+I} \boldsymbol{d}_j M_j^6(t) \tag{5}$$

of the quintic B-splines $M_j^6(t)$, $(j = -5, -4, \cdots, K+I)$ on the basis of the knots $\{\tau_j\}_{j=-5}^{K+I+6}$ set by

$$\tau_j = \left\{ \begin{array}{lll} t_j, & 1 \le j \le \ell_1 & \\ t_{j-i}, & \ell_i + i \le j \le \ell_{i+1} + i, & (i = 1, 2, \ldots, I-1) \\ t_{j-I}, & \ell_I + I \le j \le K + I & \end{array} \right. \tag{6}$$

so that knots are placed at $\{t_k\}_{k=1}^{K}$ and, in particular, double knots (or two knots at the same time) are placed at $\{t_{\ell_i}\}_{i=1}^{I}$. The other knots are arbitrary as long as they satisfy $\tau_{-5} < \tau_{-4} < \cdots < \tau_{-1} < \tau_0 = 0$ and $T = \tau_{K+I+1} < \tau_{K+I+2} < \cdots < \tau_{K+I+6}$.

The array $D = (\boldsymbol{d}_{-5} \quad \boldsymbol{d}_{-4} \quad \cdots \quad \boldsymbol{d}_{K+I})$ of the coefficient vectors is uniquely determined by the constraints (2), (3), the boundary conditions (4), and the natural boundary condition

$$\dddot{\boldsymbol{r}}(0) = \dddot{\boldsymbol{r}}(T) = \boldsymbol{0}, \tag{7}$$

namely,

$$D = XB^{-1}, \tag{8}$$

where

$$B = \left(\begin{array}{cccc} M_{-5}^{6}(0) & M_{-4}^{6}(0) & \cdots & M_{K+I}^{6}(0) \\ \dot{M}_{-5}^{6}(0) & \dot{M}_{-4}^{6}(0) & \cdots & \dot{M}_{K+I}^{6}(0) \\ \dddot{M}_{-5}^{6}(0) & \dddot{M}_{-4}^{6}(0) & \cdots & \dddot{M}_{K+I}^{6}(0) \\ \hline M_{-5}^{6}(t_1) & M_{-4}^{6}(t_1) & \cdots & M_{K+I}^{6}(t_1) \\ M_{-5}^{6}(t_2) & M_{-4}^{6}(t_2) & \cdots & M_{K+I}^{6}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ M_{-5}^{6}(t_K) & M_{-4}^{6}(t_K) & \cdots & M_{K+I}^{6}(t_K) \\ \hline \dot{M}_{-5}^{6}(t_{\ell_1}) & \dot{M}_{-4}^{6}(t_{\ell_1}) & \cdots & \dot{M}_{K+I}^{6}(t_{\ell_1}) \\ \dot{M}_{-5}^{6}(t_{\ell_2}) & \dot{M}_{-4}^{6}(t_{\ell_2}) & \cdots & \dot{M}_{K+I}^{6}(t_{\ell_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{M}_{-5}^{6}(t_{\ell_I}) & \dot{M}_{-4}^{6}(t_{\ell_I}) & \cdots & \dot{M}_{K+I}^{6}(t_{\ell_I}) \\ \hline M_{-5}^{6}(T) & M_{-4}^{6}(T) & \cdots & M_{K+I}^{6}(T) \\ \dot{M}_{-5}^{6}(T) & \dot{M}_{-4}^{6}(T) & \cdots & \dot{M}_{K+I}^{6}(T) \\ \dddot{M}_{-5}^{6}(T) & \dddot{M}_{-4}^{6}(T) & \cdots & \dddot{M}_{K+I}^{6}(T) \end{array} \right) \tag{9}$$

and

$$X = \left(\boldsymbol{r}_S \quad \dot{\boldsymbol{r}}_S \quad \boldsymbol{0} \;\middle|\; \boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \cdots \quad \boldsymbol{r}_K \;\middle|\; \underbrace{\boldsymbol{0} \quad \cdots \quad \boldsymbol{0}}_{I} \;\middle|\; \boldsymbol{r}_E \quad \dot{\boldsymbol{r}}_E \quad \boldsymbol{0} \right). \tag{10}$$

The quintic B-spline $M_j^m(t)$ of order m having the knots $\tau_j \leq \tau_{j+1} \leq \cdots \leq \tau_{j+m}$ is a piecewise polynomial of degree $m-1$ defined by [Curry and Schoenberg, 1966] as

$$M_j^m(t) := [\tau_j, \tau_{j+1}, \ldots, \tau_{j+m}](\cdot - t)_+^{m-1}, \tag{11}$$

where $[\tau_j, \tau_{j+1}, \ldots, \tau_{j+m}]$ denotes the operator of divided difference such as

$$[\tau_j, \tau_{j+1}, \ldots, \tau_{j+m}]g(\cdot) := \begin{cases} \dfrac{[\tau_{j+1}, \ldots, \tau_{j+m}]g(\cdot) - [\tau_j, \ldots, \tau_{j+m-1}]g(\cdot)}{\tau_{j+m} - \tau_j} & \text{if } \tau_{j+m} > \tau_j \\ \left. \dfrac{d^i g(\tau)}{d\tau^i} \right|_{\tau = \tau_j} & \text{if } \tau_{j+m} = \tau_j. \end{cases}$$
$$i = 1, 2, \ldots, m. \tag{12}$$

The value of the B-splines and their (higher order) derivatives with respect to the variable t can be evaluated recursively by the recurrence formulae of de Boor and Cox [de Boor, 1978]

$$M_j^m(t) \quad = \quad \begin{cases} \dfrac{t - \tau_j}{\tau_{j+m} - \tau_j} M_j^{m-1}(t) + \dfrac{\tau_{j+m} - t}{\tau_{j+m} - \tau_j} M_{j+1}^{m-1}(t) & \text{if } \tau_j < t < \tau_{j+m}, \\ 0 & \text{otherwise,} \end{cases}$$
$$(m \geq 2) \tag{13}$$

$$M_j^1(t) = \begin{cases} \dfrac{1}{\tau_{j+1} - \tau_j} & \text{if } \tau_j \le t < \tau_{j+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

and another well-known recurrence formula

$$\left(\frac{d}{dt}\right)^n M_j^m(t) = \frac{m-1}{\tau_{j+m} - \tau_j} \left\{ \left(\frac{d}{dt}\right)^{n-1} M_j^{m-1}(t) - \left(\frac{d}{dt}\right)^{n-1} M_{j+1}^{m-1}(t) \right\}, \quad n = 1, 2, \cdots, m-1; m \ge 2. \tag{15}$$

3. Numerical method to optimize via times

3.1 Objective function

The analytically derived minimizer $\boldsymbol{r}(t)$ of the functional $P[\boldsymbol{r}]$ is an extended quintic spline represented in the form of (5) and the coefficients are uniquely determined by (8). Since the matrix B in (8) is dependent on the via times $\{t_k\}_{k=0}^K$, so is the minimizer $\boldsymbol{r}(t)$. Consequently, the functional $P[\boldsymbol{r}]$ can be reduced to an ordinary objective function of the via times, as shown in the following.

Substituting (5) for $\boldsymbol{r}$ in (1), we have

$$\begin{aligned} P[\boldsymbol{r}] &= \int_0^T \left\| \sum_{j=-5}^{K+I} \boldsymbol{d}_j \dddot{M}_j^6(t) \right\|^2 dt \\ &= \operatorname{tr}\left(\sum_{i=-5}^{K+I} \sum_{j=-5}^{K+I} \boldsymbol{d}_i \int_0^T \dddot{M}_i^6(t) \dddot{M}_j^6(t) dt \ \boldsymbol{d}_j' \right) \\ &= \operatorname{tr}(DGD'), \end{aligned} \tag{16}$$

where G is set as

$$G = \begin{bmatrix} \int_0^T \dddot{M}_{-5}^6(t) \dddot{M}_{-5}^6(t)dt & \int_0^T \dddot{M}_{-5}^6(t) \dddot{M}_{-4}^6(t)dt & \cdots & \int_0^T \dddot{M}_{-5}^6(t) \dddot{M}_{K+I}^6(t)dt \\ \int_0^T \dddot{M}_{-4}^6(t) \dddot{M}_{-5}^6(t)dt & \int_0^T \dddot{M}_{-4}^6(t) \dddot{M}_{-4}^6(t)dt & \cdots & \int_0^T \dddot{M}_{-4}^6(t) \dddot{M}_{K+I}^6(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^T \dddot{M}_{K+I}^6(t) \dddot{M}_{-5}^6(t)dt & \int_0^T \dddot{M}_{K+I}^6(t) \dddot{M}_{-4}^6(t)dt & \cdots & \int_0^T \dddot{M}_{K+I}^6(t) \dddot{M}_{K+I}^6(t)dt \end{bmatrix}. \tag{17}$$

Since the via times determine the knots by (6) and because the knots determine the B-splines, all the matrices B, G and D are functions of the via times. Consequently, $P[\boldsymbol{r}]$ given by (16) is also a function of the via times, which shall be denoted as

$$P(\boldsymbol{t}) = \operatorname{tr}(DGD'), \qquad \boldsymbol{t} = (t_1, t_2, \cdots, t_K). \tag{18}$$

Then the problem is to minimize the objective function $P(\boldsymbol{t})$ over the feasible set of via times

$$F := \{\boldsymbol{t} = (t_1, t_2, \cdots, t_K)\,|0 < t_1 < t_2 < \cdots < t_K < T\} \tag{19}$$

in the K-dimensional Euclidean space.

3.2 Resolution of integrals

Elements of matrix G, which form a major part of the objective function $P(\boldsymbol{t})$ by (18), are defined by (17) as integrated products of two functions. Naive replacement of the integrals by sums, which is typical of numerical integration, would result in slow computation and degraded accuracy. In the following, the integrals are resolved to be expressed as arithmetic functions of the B-splines and their derivatives by repeating partial integration and exploiting the fact that a spline differentiated repeatedly becomes a series of Dirac delta functions eventually.

Three-fold partial integration of an element G_{ij} of G is

$$\begin{aligned}
G_{ij} &= \int_0^T \dddot{M}_i^6(t)\, \dddot{M}_j^6(t) dt \\
&= \Big[\dddot{M}_i^6(t)\, \ddot{M}_j^6(t)\Big]_0^T - \int_0^T \ddddot{M}_i^6(t)\, \ddot{M}_j^6(t) dt \\
&= \Big[\dddot{M}_i^6(t)\, \ddot{M}_j^6(t)\Big]_0^T - \Big[\ddddot{M}_i^6(t)\, \dot{M}_j^6(t)\Big]_0^T + \int_0^T \overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(t)\, \dot{M}_j^6(t) dt \\
&= \Big[\dddot{M}_i^6(t)\, \ddot{M}_j^6(t)\Big]_0^T - \Big[\ddddot{M}_i^6(t)\, \dot{M}_j^6(t)\Big]_0^T + \Big[\overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(t) M_j^6(t)\Big]_0^T - \int_0^T \overset{\cdot\cdot\cdot\cdot\cdot\cdot}{M}_i^6(t) M_j^6(t) dt.
\end{aligned} \tag{20}$$

The integrand of the last term of (20) includes the sixth-order derivative of the quintic B-spline $M_i^6(t)$. Since the fourth-order derivative of the quintic B-spline is a piecewise linear function which is discontinuous at double knots, the fifth-order derivative is a sum of a piecewise constant function, which is discontinuous at simple knots, and a delta function δ at the double knots. Consequently, the sixth-order derivative is a sum of delta functions at simple knots and derivatives of delta function at double knots, as follows:

$$\begin{aligned}
\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{M}_i^6(t) = &\sum_{\substack{0 \le p \le 6 \\ \text{and} \\ \tau_{i+p} < \tau_{i+p+1}}} \Big[\overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(\tau_{i+p}+0) - \overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(\tau_{i+p}-0)\Big]\delta(t-\tau_{i+p}) \\
&+ \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ \tau_{i+p} = \tau_{i+p+1}}} \Big[\ddddot{M}_i^6(\tau_{i+p+1}+0) - \ddddot{M}_i^6(\tau_{i+p}-0)\Big]\dot{\delta}(t-\tau_{i+p}).
\end{aligned} \tag{21}$$

The coefficients for δ and $\dot{\delta}$ in (21) are the difference of the function values at the corresponding discontinuities. Then, we can resolve the integral as follows:

$$
\begin{aligned}
\int_0^T \overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(t) M_j^6(t)dt \;=\; & \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} < \tau_{i+p+1} < T}} \left[\overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}+0) - \overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}-0) \right] M_j^6(\tau_{i+p}) \\
& - \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} = \tau_{i+p+1} < T}} \left[\overset{\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p+1}+0) - \overset{\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}-0) \right] \dot{M}_j^6(\tau_{i+p})
\end{aligned}
\tag{22}
$$

by the property

$$
\int_0^T f(t)\delta(t-\tau)dt = f(\tau) \quad \text{and} \quad \int_0^T f(t)\dot{\delta}(t-\tau)dt = -\dot{f}(\tau)
$$

of the delta function for $0 < \tau < T$ and twice differentiable f. Equations (20) and (22) give an arithmetic expression of G_{ij} as follows:

$$
\begin{aligned}
G_{ij} \;=\; & \left[\overset{\cdot\cdot\cdot}{M_i^6}(t) \, \overset{\cdot\cdot}{M_j^6}(t) \right]_0^T - \left[\overset{\cdot\cdot\cdot\cdot}{M_i^6}(t) \dot{M}_j^6(t) \right]_0^T + \left[\overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(t) M_j^6(t) \right]_0^T \\
& - \Bigg\{ \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} < \tau_{i+p+1} < T}} \left[\overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}+0) - \overset{\cdot\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}-0) \right] M_j^6(\tau_{i+p}) \\
& \qquad - \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} = \tau_{i+p+1} < T}} \left[\overset{\cdot\cdot\cdot\cdot}{M_i^6}(\tau_{i+p}+0) - \overset{\cdot\cdot\cdot\cdot}{M_j^6}(\tau_{i+p}-0) \right] \dot{M}_j^6(\tau_{i+p}) \Bigg\}.
\end{aligned}
\tag{23}
$$

The B-splines and their higher order derivatives in (23) can be computed arithmetically by (13), (14) and (15).

3.3 Resolution of partial differentials

Optimization of the objective function $P(t_1, t_2, \cdots, t_K)$ by a gradient method requires the gradient

$$
\mathrm{grad}P = \left(\frac{\partial P}{\partial t_1}, \frac{\partial P}{\partial t_2}, \cdots, \frac{\partial P}{\partial t_K} \right)
\tag{24}
$$

composed of the partial differentials $\frac{\partial P}{\partial t_k}$, $(k = 1, 2, \cdots, K)$. Arithmetic expressions for the partial differentials shall be derived analytically in the following.

From (16), we have

$$
\frac{\partial P}{\partial t_k} \;=\; \frac{\partial}{\partial t_k} \mathrm{tr}(D'GD)
$$

$$= \ \mathrm{tr}\left(\frac{\partial D}{\partial t_k}GD' + D\frac{\partial G}{\partial t_k}D' + DG\left(\frac{\partial D}{\partial t_k}\right)'\right). \tag{25}$$

Differentiating both sides of $DB = X$ that follows from (8), we have

$$\frac{\partial D}{\partial t_k}B + D\frac{\partial B}{\partial t_k} = O,$$

which implies

$$\frac{\partial D}{\partial t_k} = -D\frac{\partial B}{\partial t_k}B^{-1} = -XB^{-1}\frac{\partial B}{\partial t_k}B^{-1}. \tag{26}$$

Substituting (8) and (26) for D and $\frac{\partial D}{\partial t_k}$ in (25), respectively, we have an expression of $\frac{\partial P}{\partial t_k}$ as

$$\begin{aligned}\frac{\partial P}{\partial t_k} \ = \ \mathrm{tr}\Bigg(& -(XB^{-1}\frac{\partial B}{\partial t_k}B^{-1})G(XB^{-1})' + (XB^{-1})\frac{\partial G}{\partial t_k}(XB^{-1})' \\ & -(XB^{-1})G\left(XB^{-1}\frac{\partial B}{\partial t_k}B^{-1}\right)'\Bigg).\end{aligned} \tag{27}$$

Matrix B is composed of $M_j^6(t)$, $\dot{M}_j^6(t)$ and $\dddot{M}_j^6(t)$ evaluated at $t = 0, t_1, t_2, \cdots, t_K, T$. The arithmetic expression (23) for matrix G is composed of $M_j^6(t)$, $\dot{M}_j^6(t)$, $\ddot{M}_j^6(t)$ and $\dddot{M}_j^6(t)$ evaluated at $t = \tau_1, \tau_2, \cdots, \tau_{K+I}$, and also of $\ddddot{M}_j^6(t)$ and $\overset{\cdot\cdot\cdot\cdot\cdot}{M}_j^6(t)$ evaluated at $t = \tau_q \pm 0$. The derivatives $\dot{M}_j^6(t)$, $\ddot{M}_j^6(t)$, $\dddot{M}_j^6(t)$, $\ddddot{M}_j^6(t)$ and $\overset{\cdot\cdot\cdot\cdot\cdot}{M}_j^6(t)$ are reduced by (15) to lower order B-splines $M_j^5(t)$, $M_j^4(t)$, $M_j^3(t)$, $M_j^2(t)$ and $M_j^1(t)$, respectively. These B-splines can be evaluated by the recurrence formulae (13) and (14).

On the other hand, arithmetic expressions for $\frac{\partial B}{\partial t_k}$ and $\frac{\partial G}{\partial t_k}$ have yet to be derived. The partial differential $\frac{\partial B}{\partial t_k}$ of B in (9) includes $\frac{\partial}{\partial t_k}M_j^6(0)$, $\frac{\partial}{\partial t_k}M_j^6(T)$, $\frac{\partial}{\partial t_k}\dot{M}_j^6(0)$, $\frac{\partial}{\partial t_k}\dot{M}_j^6(T)$, $\frac{\partial}{\partial t_k}\dddot{M}_j^6(0)$, $\frac{\partial}{\partial t_k}\dddot{M}_j^6(T)$, and $\frac{\partial}{\partial t_k}M_j^6(t_p)$, $(p = 0, 1, 2, \cdots, K)$. The partial differential $\frac{\partial G}{\partial t_k}$ of G, the elements of which are expressed by (23), includes

$$\begin{aligned}\frac{\partial G_{ij}}{\partial t_k} \ = \ & \Bigg[\left(\frac{\partial}{\partial t_k}\dddot{M}_i^6(T)\right)\ddot{M}_j^6(T) + \dddot{M}_i^6(T)\left(\frac{\partial}{\partial t_k}\ddot{M}_j^6(T)\right) \\ & \quad -\left(\frac{\partial}{\partial t_k}\dddot{M}_i^6(0)\right)\ddot{M}_j^6(0) - \dddot{M}_i^6(0)\left(\frac{\partial}{\partial t_k}\ddot{M}_j^6(0)\right)\Bigg] \\ & -\Bigg[\left(\frac{\partial}{\partial t_k}\ddddot{M}_i^6(T)\right)\dot{M}_j^6(T) + \ddddot{M}_i^6(T)\left(\frac{\partial}{\partial t_k}\dot{M}_j^6(T)\right) \\ & \quad -\left(\frac{\partial}{\partial t_k}\ddddot{M}_i^6(0)\right)\dot{M}_j^6(0) - \ddddot{M}_i^6(0)\left(\frac{\partial}{\partial t_k}\dot{M}_j^6(0)\right)\Bigg] \\ & +\Bigg[\left(\frac{\partial}{\partial t_k}\overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(T)\right)M_j^6(T) + \overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(T)\left(\frac{\partial}{\partial t_k}M_j^6(T)\right) \\ & \quad -\left(\frac{\partial}{\partial t_k}\overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(0)\right)M_j^6(0) - \overset{\cdot\cdot\cdot\cdot\cdot}{M}_i^6(0)\left(\frac{\partial}{\partial t_k}M_j^6(0)\right)\Bigg]\end{aligned}$$

$$
\begin{aligned}
- \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} < \tau_{i+p+1} < T}} \Bigg\{ & \left[\left(\frac{\partial}{\partial t_k} \overset{.....}{M}{}_i^6 \, (\tau_{i+p}+0) \right) - \left(\frac{\partial}{\partial t_k} \overset{.....}{M}{}_i^6 \, (\tau_{i+p}-0) \right) \right] M_j^6(\tau_{i+p}) \\
& + \left[\overset{.....}{M}{}_i^6 \, (\tau_{i+p}+0) - \overset{.....}{M}{}_i^6 \, (\tau_{i+p}-0) \right] \left(\frac{\partial}{\partial t_k} M_j^6(\tau_{i+p}) \right) \Bigg\} \\
+ \sum_{\substack{0 \le p \le 6 \\ \text{and} \\ 0 < \tau_{i+p} = \tau_{i+p+1} < T}} \Bigg\{ & \left[\left(\frac{\partial}{\partial t_k} \overset{....}{M}{}_i^6 \, (\tau_{i+p}+0) \right) - \left(\frac{\partial}{\partial t_k} \overset{....}{M}{}_j^6 \, (\tau_{i+p}-0) \right) \right] \dot{M}_j^6(\tau_{i+p}) \\
& + \left[\overset{....}{M}{}_i^6 \, (\tau_{i+p}+0) - \overset{....}{M}{}_j^6 \, (\tau_{i+p}-0) \right] \left(\frac{\partial}{\partial t_k} \dot{M}_j^6(\tau_{i+p}) \right) \Bigg\}. \qquad (28)
\end{aligned}
$$

It is possible to reduce the partial differentials (with respect to t_k) of higher order derivatives (with respect to t) of the B-splines to those of the lower order B-splines by

$$
\frac{\partial}{\partial t_k} \left\{ \left(\frac{d}{dt} \right)^n M_j^m(t) \right\} = \begin{cases}
\dfrac{m-1}{(\tau_{j+m}-\tau_j)^2} \left\{ \left(\dfrac{d}{dt} \right)^{n-1} M_j^{m-1}(t) - \left(\dfrac{d}{dt} \right)^{n-1} M_{j+1}^{m-1}(t) \right\} \\
\quad + \dfrac{m-1}{\tau_{j+m}-\tau_j} \left\{ \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_j^{m-1}(t) - \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_{j+1}^{m-1}(t) \right\} \\
\qquad \qquad \qquad \qquad \qquad \qquad \text{if } t_k = \tau_j, \\
-\dfrac{m-1}{(\tau_{j+m}-\tau_j)^2} \left\{ \left(\dfrac{d}{dt} \right)^{n-1} M_j^{m-1}(t) - \left(\dfrac{d}{dt} \right)^{n-1} M_{j+1}^{m-1}(t) \right\} \\
\quad + \dfrac{m-1}{\tau_{j+m}-\tau_j} \left\{ \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_j^{m-1}(t) - \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_{j+1}^{m-1}(t) \right\} \\
\qquad \qquad \qquad \qquad \qquad \qquad \text{if } t_k = \tau_{j+m}, \\
\dfrac{m-1}{\tau_{j+m}-\tau_j} \left\{ \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_j^{m-1}(t) - \dfrac{\partial}{\partial t_k} \left(\dfrac{d}{dt} \right)^{n-1} M_{j+1}^{m-1}(t) \right\} \\
\qquad \qquad \qquad \qquad \qquad \qquad \text{otherwise,}
\end{cases}
$$

$$
n = m-1, m-2, \cdots, 2, 1; \quad m \ge 2, \qquad (29)
$$

which is derived by differentiating both sides of (15) with respect to t_k. Then, the evaluation of $\frac{\partial B}{\partial t_k}$ and $\frac{\partial G}{\partial t_k}$ is arithmetically reduced to the evaluation of $\frac{\partial}{\partial t_k} M_j^m(t_p)$, ($j = -5, -4, \cdots, K+I$; $m \ge 1$; $p = 0, 1, 2, \cdots, K+1$), by taking $\{\tau_0, \tau_1, \cdots, \tau_{K+I}\} \subset \{0, t_1, t_2, \cdots, t_K, T\}$ into account and setting $t_0 = 0$ and $t_{K+1} = T$ for the convenience of notation, $\frac{\partial}{\partial t_k} M_j^2(\tau_q \pm 0)$ in the case $\tau_q = \tau_{q+1}$, and $\frac{\partial}{\partial t_k} M_j^1(\tau_q \pm 0)$ in the case $\tau_q < \tau_{q+1}$.

Although an explicit formula for the B-splines is available as written in [Greville, 1971], the via times, which are tied to the knots by (6), are scattered elusively in the formula. Instead of struggling with the explicit formula, we make use of the recurrence formulae (13) and (14) for the B-splines to derive another recurrence formula for their partial differentials.

Substituting t_p for t in (13) and (14), we have

$$M_j^m(t_p) = \begin{cases} \dfrac{t_p-\tau_j}{\tau_{j+m}-\tau_j} M_j^{m-1}(t_p) + \dfrac{\tau_{j+m}-t_p}{\tau_{j+m}-\tau_j} M_{j+1}^{m-1}(t_p) & \text{if } \tau_j < t_p < \tau_{j+m}, \\ 0 & \text{otherwise,} \end{cases} \quad (m \geq 3) \tag{30}$$

$$M_j^2(t_p) = \begin{cases} \dfrac{t_p-\tau_j}{\tau_{j+2}-\tau_j} M_j^1(t_p) + \dfrac{\tau_{j+2}-t_p}{\tau_{j+2}-\tau_j} M_{j+1}^1(t_p) & \text{if } \tau_j < \tau_{j+1} = t_p < \tau_{j+2}, \\ \dfrac{1}{t_p-\tau_j} & \text{if } \tau_j \leq t_p = \tau_{j+1} = \tau_{j+2}, \\ \dfrac{1}{\tau_{j+2}-t_p} & \text{if } \tau_j = \tau_{j+1} = t_p < \tau_{j+2}, \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

$$M_j^1(t_p) = \begin{cases} \dfrac{1}{\tau_{j+1}-t_p} & \text{if } \tau_j = t_p < \tau_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Differentiating both sides of (30) with respect to t_k, we have

$$\frac{\partial}{\partial t_k} M_j^m(t_p) = \begin{cases} \frac{1}{\tau_{j+m}-\tau_j}\left(M_j^{m-1}(t_p) - M_{j+1}^{m-1}(t_p)\right) & \\ \quad + \left(\frac{t_k-\tau_j}{\tau_{j+m}-\tau_j}\right) \frac{\partial}{\partial t_k} M_j^{m-1}(t_p) & \\ \quad + \left(\frac{\tau_{j+m}-t_k}{\tau_{j+m}-\tau_j}\right) \frac{\partial}{\partial t_k} M_{j+1}^{m-1}(t_p) & \text{if } \tau_j < t_p = t_k < \tau_{j+m}, \\ \frac{t_p-\tau_{j+m}}{(\tau_{j+m}-t_k)^2}\left(M_j^{m-1}(t_p) - M_{j+1}^{m-1}(t_p)\right) & \\ \quad + \left(\frac{t_p-t_k}{\tau_{j+m}-t_k}\right) \frac{\partial}{\partial t_k} M_j^{m-1}(t_p) & \\ \quad + \left(\frac{\tau_{j+m}-t_p}{\tau_{j+m}-t_k}\right) \frac{\partial}{\partial t_k} M_{j+1}^{m-1}(t_p) & \text{if } \tau_j = t_k < t_p < \tau_{j+m}, \\ \frac{\tau_j-t_p}{(t_k-\tau_j)^2}\left(M_j^{m-1}(t_p) - M_{j+1}^{m-1}(t_p)\right) & \\ \quad + \left(\frac{t_p-\tau_j}{t_k-\tau_j}\right) \frac{\partial}{\partial t_k} M_j^{m-1}(t_p) & \\ \quad + \left(\frac{t_k-t_p}{t_k-\tau_j}\right) \frac{\partial}{\partial t_k} M_{j+1}^{m-1}(t_p) & \text{if } \tau_j < t_p < t_k = \tau_{j+m}, \\ \left(\frac{t_p-\tau_j}{\tau_{j+m}-\tau_j}\right) \frac{\partial}{\partial t_k} M_j^{m-1}(t_p) & \\ \quad + \left(\frac{\tau_{j+m}-t_p}{\tau_{j+m}-\tau_j}\right) \frac{\partial}{\partial t_k} M_{j+1}^{m-1}(t_p) & \text{if } \tau_j < t_p < \tau_{j+m} \\ & \quad \text{and } t_k \notin \{t_p, \tau_j, \tau_{j+m}\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } \ j = -5, -4, \cdots, K+I; \quad m \geq 3; \quad p = 0, 1, 2, \cdots, K+1. \tag{33}$$

Similarly, differentiating both sides of (31) and (32) with respect to t_k, we have

$$\frac{\partial}{\partial t_k} M_j^2(t_p) = \begin{cases} \frac{1}{\tau_{j+2}-\tau_j}\left(M_j^1(t_k) - M_{j+1}^1(t_k)\right) \\ \qquad + \left(\frac{t_p-\tau_j}{\tau_{j+2}-\tau_j}\right)\frac{\partial}{\partial t_k} M_j^1(t_p) \\ \qquad + \left(\frac{\tau_{j+2}-t_k}{\tau_{j+2}-\tau_j}\right)\frac{\partial}{\partial t_k} M_{j+1}^1(t_p) & \text{if } \tau_j < \tau_{j+1} = t_p = t_k < \tau_{j+2}, \\ \frac{t_p-\tau_{j+2}}{(\tau_{j+2}-t_k)^2}\left(M_j^1(t_p) - M_{j+1}^1(t_p)\right) \\ \qquad + \left(\frac{t_p-t_k}{\tau_{j+2}-t_k}\right)\frac{\partial}{\partial t_k} M_j^1(t_p) \\ \qquad + \left(\frac{\tau_{j+2}-t_p}{\tau_{j+2}-t_k}\right)\frac{\partial}{\partial t_k} M_{j+1}^1(t_p) & \text{if } \tau_j = t_k < \tau_{j+1} = t_p < \tau_{j+2}, \\ \frac{\tau_j-t_p}{(t_k-\tau_j)^2}\left(M_j^1(t_p) - M_{j+1}^1(t_p)\right) \\ \qquad + \left(\frac{t_p-\tau_j}{t_k-\tau_j}\right)\frac{\partial}{\partial t_k} M_j^1(t_p) \\ \qquad + \left(\frac{t_k-t_p}{t_k-\tau_j}\right)\frac{\partial}{\partial t_k} M_{j+1}^1(t_p) & \text{if } \tau_j < \tau_{j+1} = t_p < t_k = \tau_{j+2}, \\ \frac{-1}{(t_k-\tau_j)^2} & \text{if } \tau_j < t_k = t_p = \tau_{j+1} = \tau_{j+2}, \\ \frac{1}{(t_p-t_k)^2} & \text{if } \tau_j = t_k < t_p = \tau_{j+1} = \tau_{j+2}, \\ \frac{1}{(\tau_{j+2}-t_k)^2} & \text{if } \tau_j = \tau_{j+1} = t_k = t_p < \tau_{j+2}, \\ \frac{-1}{(t_k-t_p)^2} & \text{if } \tau_j = \tau_{j+1} = t_p < \tau_{j+2} = t_k, \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

and

$$\frac{\partial}{\partial t_k} M_j^1(t_p) = \begin{cases} \frac{1}{(\tau_{j+1}-t_k)^2} & \text{if } \tau_j = t_p = t_k < \tau_{j+1}, \\ \frac{-1}{(t_k-t_p)^2} & \text{if } \tau_j = t_p < t_k = \tau_{j+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

respectively.

$M_j^2(\tau_q \pm 0)$ in the case $\tau_q = \tau_{q+1}$ and $M_j^1(\tau_q \pm 0)$ in the case $\tau_q < \tau_{q+1}$ are derived from (13) and (14) as follows:

$$M_j^2(\tau_q + 0) = \begin{cases} \dfrac{\tau_{j+2} - \tau_{j+1}}{(\tau_{j+2} - \tau_j)^2} & \text{if } \tau_j < \tau_{j+1} = \tau_q < \tau_{j+2} \\ \dfrac{1}{\tau_{j+2} - \tau_{j+1}} & \text{if} \tau_j = \tau_{j+1} = \tau_q < \tau_{j+2} \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

$$M_j^2(\tau_q - 0) = \begin{cases} \dfrac{\tau_{j+2} - \tau_{j+1}}{(\tau_{j+2} - \tau_j)^2} & \text{if } \tau_j < \tau_{j+1} = \tau_q < \tau_{j+2} \\ \dfrac{1}{\tau_{j+1} - \tau_j} & \text{if } \tau_j < \tau_q = \tau_{j+1} = \tau_{j+2} \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

$$M_j^1(\tau_q+0) = \begin{cases} \dfrac{1}{\tau_{j+1}-\tau_j} & \text{if } \tau_q=\tau_j<\tau_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

$$M_j^1(\tau_q-0) = \begin{cases} \dfrac{1}{\tau_{j+1}-\tau_j} & \text{if } \tau_j<\tau_q=\tau_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

Differentiation of both sides of (36)–(39) yields $\frac{\partial}{\partial t_k}M_j^2(\tau_q \pm 0)$ for $\tau_q=\tau_{q+1}$ and $\frac{\partial}{\partial t_k}M_j^1(\tau_q \pm 0)$ for $\tau_q<\tau_{q+1}$ as follows:

$$\frac{\partial}{\partial t_k}M_j^2(\tau_q+0) = \begin{cases} \dfrac{2(\tau_{j+2}-\tau_{j+1})}{(\tau_{j+2}-\tau_j)^3} & \text{if } t_k=\tau_j<\tau_{j+1}=\tau_q<\tau_{j+2} \\ \dfrac{-1}{(\tau_{j+2}-\tau_j)^2} & \text{if } \tau_j<t_k=\tau_{j+1}=\tau_q<\tau_{j+2} \\ \dfrac{-\tau_{j+2}+2\tau_{j+1}+\tau_j}{(\tau_{j+2}-\tau_j)^3} & \text{if } \tau_j<\tau_{j+1}=\tau_q<\tau_{j+2}=t_k \\ \dfrac{1}{(\tau_{j+2}-\tau_{j+1})^2} & \text{if} t_k=\tau_j=\tau_{j+1}=\tau_q<\tau_{j+2} \\ \dfrac{-1}{(\tau_{j+2}-\tau_{j+1})^2} & \text{if} \tau_j=\tau_{j+1}=\tau_q<\tau_{j+2}=t_k \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

$$\frac{\partial}{\partial t_k}M_j^2(\tau_q-0) = \begin{cases} \dfrac{2(\tau_{j+2}-\tau_{j+1})}{(\tau_{j+2}-\tau_j)^3} & \text{if } t_k=\tau_j<\tau_{j+1}=\tau_q<\tau_{j+2} \\ \dfrac{-1}{(\tau_{j+2}-\tau_j)^2} & \text{if } \tau_j<t_k=\tau_{j+1}=\tau_q<\tau_{j+2} \\ \dfrac{-\tau_{j+2}+2\tau_{j+1}+\tau_j}{(\tau_{j+2}-\tau_j)^3} & \text{if } \tau_j<\tau_{j+1}=\tau_q<\tau_{j+2}=t_k \\ \dfrac{1}{(\tau_{j+1}-\tau_j)^2} & \text{if } t_k=\tau_j<\tau_q=\tau_{j+1}=\tau_{j+2} \\ \dfrac{-1}{(\tau_{j+1}-\tau_j)^2} & \text{if } \tau_j<\tau_q=\tau_{j+1}=\tau_{j+2}=t_k \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$\frac{\partial}{\partial t_k}M_j^1(\tau_q+0) = \begin{cases} \dfrac{1}{(\tau_{j+1}-\tau_j)^2} & \text{if } t_k=\tau_q=\tau_j<\tau_{j+1} \\ \dfrac{-1}{(\tau_{j+1}-\tau_j)^2} & \text{if } \tau_q=\tau_j<\tau_{j+1}=t_k \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

$$\frac{\partial}{\partial t_k}M_j^1(\tau_q-0) = \begin{cases} \dfrac{1}{(\tau_{j+1}-\tau_j)^2} & \text{if } t_k=\tau_j<\tau_q=\tau_{j+1} \\ \dfrac{1}{(\tau_{j+1}-\tau_j)^2} & \text{if } \tau_j<\tau_q=\tau_{j+1}=t_k \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

Now we have all the necessary formulae to compute B, G, $\boldsymbol{d}$, $\dfrac{\partial B}{\partial t_k}$, $\dfrac{\partial \boldsymbol{d}}{\partial t_k}$, $\dfrac{\partial G}{\partial t_k}$, and eventually $\dfrac{\partial P}{\partial t_k}$ arithmetically. Although these formulae look quite complicated, they are compatible with computer implementation that allows for recursive procedures with many *if* clauses.

3.4 Numerical algorithm

An optimization procedure can be compiled by incorporating the partial differentials into any gradient method over a convex polyhedral feasible set.

The feasible set F of $\boldsymbol{t} = (t_1, t_2, \cdots, t_K)$ specified as an open set by (19) shall be modified to be a closed set

$$D := \{\boldsymbol{t} = (t_1, t_2, ..., t_K) |\; t_k + \epsilon \leq t_{k+1},\; k = 1, 2, \ldots, K\}, \quad (t_0 = 0, t_{K+1} = T) \tag{44}$$

by introducing a small positive parameter $\epsilon > 0$, since any numerical methods work only for closed feasible sets. This parameter ϵ indicates the limit beyond which a time t_k must not be closer to its neighbors t_{k-1} or t_{k+1}. There is virtually no difference between F and D since, in practice, we can choose ϵ to be arbitrarily small.

Taking the conditional gradient method from [Vasiliev, 1996], we have the following algorithm:

Step 0. Choose $\boldsymbol{t}^{(0)} = (t_1^{(0)}, t_2^{(0)}, \ldots, t_K^{(0)}) \in D$ as the initial values of the via times. Set $n := 0$. Choose a threshold $\theta > 0$.

Step 1. Set the knots $\{\tau_j\}_{j=-5}^{K+I+6}$ by (6) according to $\boldsymbol{t} = \boldsymbol{t}^{(n)}$.

Step 2. Compute B for $\boldsymbol{t} = \boldsymbol{t}^{(n)}$ as expressed by (9) using the recurrence formulae (13), (14) and (15).

Step 3. Compute D from B and X by (8).

Step 4. Compute $\left\{ \dfrac{\partial B}{\partial t_k} \right\}_{k=0}^{K}$ for $\boldsymbol{t} = \boldsymbol{t}^{(n)}$ by the recurrence formulae (29) and (33)–(35).

Step 5. Compute G for $\boldsymbol{t} = \boldsymbol{t}^{(n)}$, which is composed of the elements represented by (23), using the recurrence formulae (13), (14) and (15).

Step 6. Compute $\left\{ \dfrac{\partial G}{\partial t_k} \right\}_{k=0}^{K}$ for $\boldsymbol{t} = \boldsymbol{t}^{(n)}$, which is composed of the elements represented by (28), using the recurrence formulae (29), (33)–(35), and (40)–(43).

Step 7. Compute $\mathrm{grad}P(\boldsymbol{t}^{(n)}) = \left(\dfrac{\partial P}{\partial t_1}, \dfrac{\partial P}{\partial t_2}, \cdots, \dfrac{\partial P}{\partial t_K} \right)$ for $\boldsymbol{t} = \boldsymbol{t}^{(n)}$ by (27) from B, G, $\dfrac{\partial B}{\partial t_k}$, $\dfrac{\partial G}{\partial t_k}$, and X.

Step 8. Find a solution $\bar{\boldsymbol{t}}^{(n)}$ of the linear programming problem

$$\mathrm{grad}P(\boldsymbol{t}^{(n)}) \;\; (\bar{\boldsymbol{t}}^{(n)} - \boldsymbol{t}^{(n)})' \;\; \to \; \min, \quad \bar{\boldsymbol{t}}^{(n)} = (\bar{t}_1^{(n)}, \bar{t}_2^{(n)}, \ldots, \bar{t}_K^{(n)}) \in D,$$

by the simplex method.

Step 9. Compute the criterion h_n of minimality as

$$h_n = \left| \mathrm{grad}P(\boldsymbol{t}^{(n)}) \;\; (\bar{\boldsymbol{t}}^{(n)} - \boldsymbol{t}^{(n)})' \right|.$$

Step 10. If $h_n \leq \theta$, then terminate, and $\boldsymbol{t}^{(n)}$ is the solution. Otherwise, take a point $\boldsymbol{t}^{(n+1)} = (t_1^{(n+1)}, t_2^{(n+1)}, \cdots, t_K^{(n+1)}) \in D$ that is constructed as

$$\boldsymbol{t}^{(n+1)} = \boldsymbol{t}^{(n)} + \alpha_n(\bar{\boldsymbol{t}}^{(n)} - \boldsymbol{t}^{(n)}),\ \alpha_n \in (0, 1]$$

such that $P(\boldsymbol{t}^{(n+1)}) < P(\boldsymbol{t}^{(n)})$, set $n := n + 1$, and go to **Step 1**.

Convergence of this algorithm to a local minimum is immediate from Theorem 3.5 of [Vasiliev, 1996] since the feasible set D is closed, bounded and convex.

4. Experiment

The optimization procedure developed in the previous section is applied to the synthesis of letter strings in script style from the data of individual letters.

Individual letters written by a volunteer on an electromagnetic tablet and digitized into 5 dot/mm by the sampling interval of 10 ms are shown in Fig. 2. Characteristic points were extracted as via points and pause points where the pen speed was locally minimal and zero, respectively. The via times at which to visit the via points were taken from the data. In addition, the initial and terminal parameters were the pen positions and velocity taken from the data. The via points, as well as the initial and terminal points, are denoted by $\times$ in Fig. 2. Points denoted by $\otimes$ are pause points.

Based on the idea that the characteristic points of a letter characterize the letter even when it is written as a part of a letter string, we can synthesize a string of letters from the concatenation of characteristic points for the individual letters. The string "file", for example, is synthesized as illustrated in Fig. 3. The points taken from the individual letters in Fig. 3(a) are aligned as shown in Fig. 3(b) so that the terminal point of a previous letter and the initial point of the next letter share the same horizontal position. The boundary conditions, except for the initial point of the first letter and the terminal point of the last letter, are discarded, as shown in Fig. 3(c), because no boundaries exist between the letters in a continuously written string. Then, the string is synthesized as an extended spline interpolation of the concatenated points. Figure 3(d) shows a synthesis using the initial guess of the via times, which are estimated simply by accumulating the time intervals between the adjacent points for the case in which the letters were written individually. Figure 3(e) plots the synthesis using the optimized via times.

Several examples of synthesized strings are shown in Fig. 4. The dark and light curves represent syntheses using initial and optimized via times, respectively. Most of the light curves (with the initial times) deviate considerably from our expectation, and a number of overshoots are observed. In addition, an unnecessary switch-back is observed in "eye". The dark curves (with the optimized times) look more similar to what is usually written. Each letter is adapted to suit the context.

The computational time of the present optimization method is plotted in Fig. 5 along with that of the naive method employing sum and difference rather than integral and differential, respectively, for the purpose of comparison. The sums substituting the integrals are those of the integrand evaluated by the

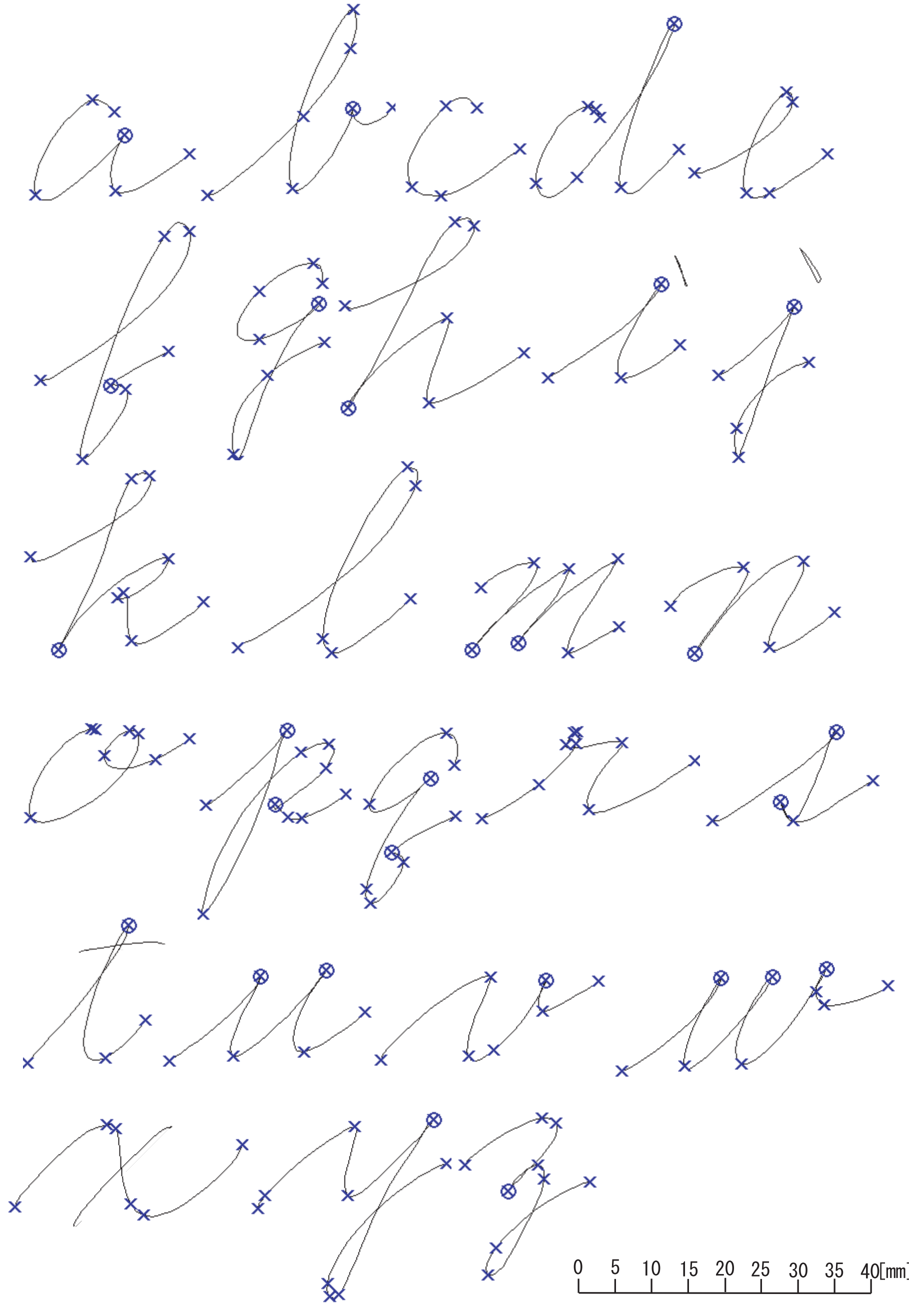

Figure 2: Original letters ($\times$ denotes a via point and $\otimes$ denotes a pause point).

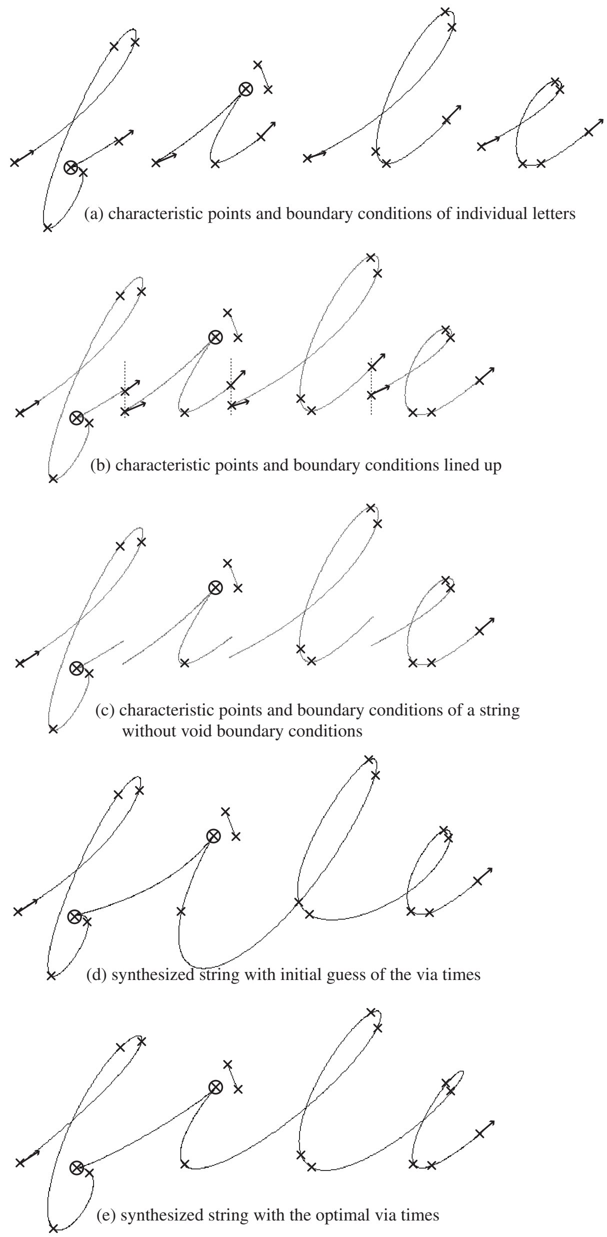

Figure 3: How to synthesize a string of letters in script style.

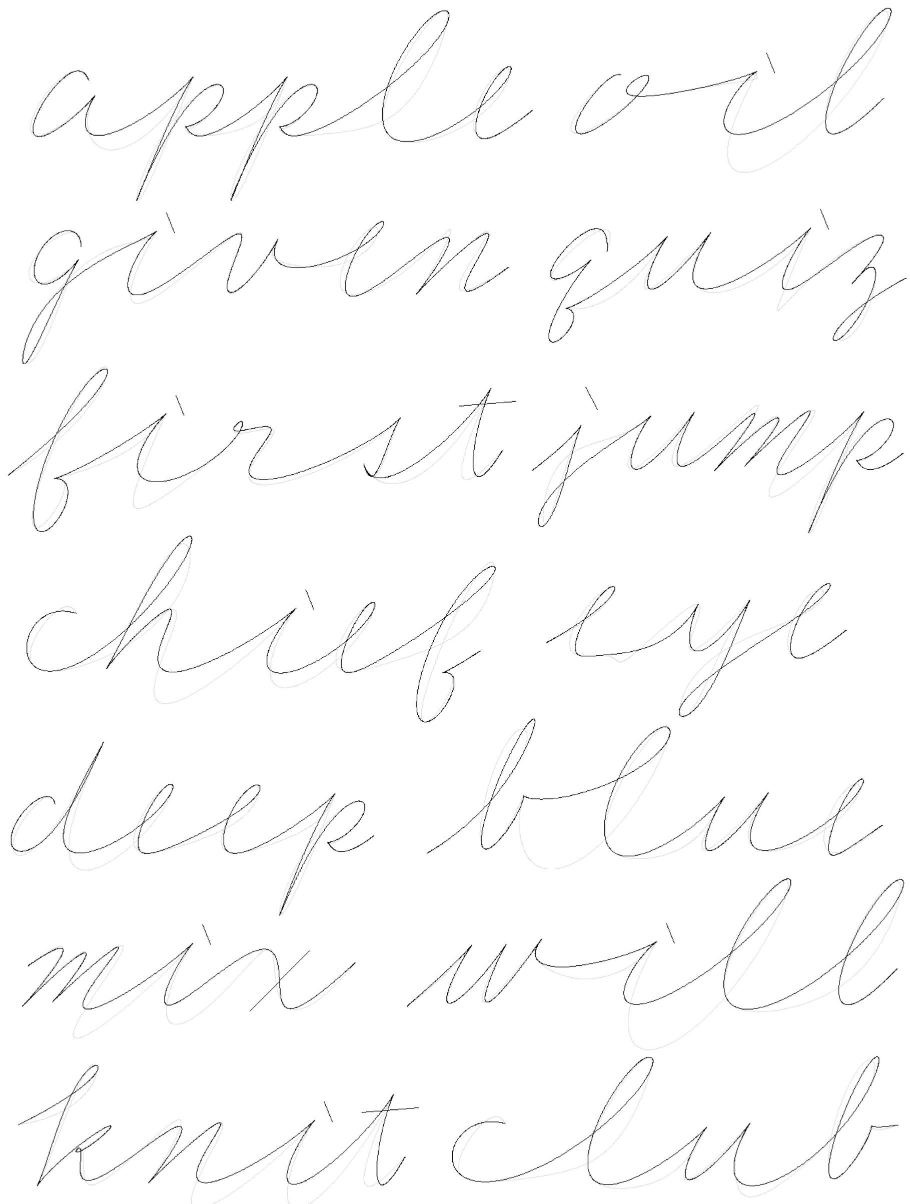

Figure 4: Letter strings synthesized on the basis of individual letters in Fig. 2. (Dark and light curves represent strings synthesized with initial and optimized via times, respectively.)

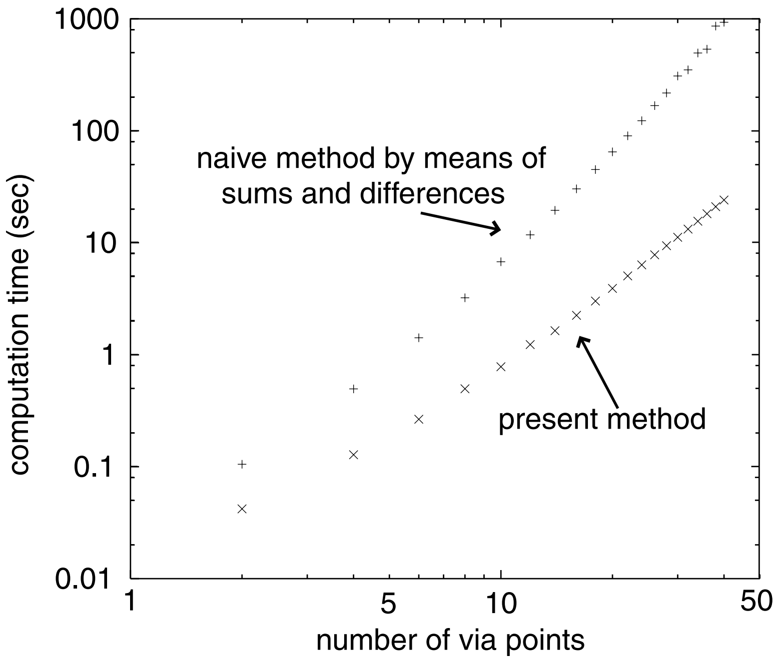

Figure 5: Computational time for optimization of the via times.

sampling interval of 10 ms. The differences substituting the differentials are those of the primitive functions evaluated at the same rate. The program was written in Java and run on the Java virtual machine emulated on an 800 MHz Intel Cerelon. The mathematical effort made in this paper to write down the integrals and differentials in terms of arithmetic operations resulted in an algorithm that is roughly ten times faster than the naive algorithm.

5. Conclusions

For the purpose of synthesizing a string of letters continuously written in script style, an optimization procedure was developed to optimize the via times such that the pen visits the via points characterizing the string so that the integral of squared jerk is minimized. A number of recurrence formulae were derived which allow us to compute the gradients arithmetically even though the gradients are defined in terms of differential and integral operators. Incorporating the gradients in the conditional gradient method, we compiled an algorithm that yielded synthesized strings of good shapes.

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