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A Class of Methods for Projection on a Convex Set

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Abstract

The paper is concerning about the basic optimization problem of projecting a point onto a convex set. We present a class of methods where the problem is reduced to a sequence of projections onto the intersection of several balls. The subproblems are much simpler and more tractable, but the main advantage is that, in so doing, we can avoid solving linear systems completely and thus the methods are very suitable for large scale problems. The methods have been shown to have nice convergence properties under Slater's constraint qualification.

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1 Introduction

In this paper we present a class of methods for solving the optimization problem of projecting a point onto a special kind of convex set:

$$\begin{array}{ll} \text{Minimize} & \|x-a\|^2 \\ \text{subject to} & x \in E_i := \{x | g_i(x) \leq 0\} \\ & \qquad i = 1, 2, \ldots, m, \end{array} \tag{1}$$

where each g_i is a continuously differentiable strictly convex function and $\|.\|$ is the 2-norm. The class of algorithms presented here is a generalization of the algorithms in [Lin, Han, 2003] which solve the special case of Problem (1) where each E_i is an ellipsoid. The main idea is to replace a convex set by balls. More specifically, we solve the problem by iteratively doing projection on the intersection of several balls. There are several advantages for this approach. First, the subproblem is much simpler and can be solved very efficiently. Second, these algorithms can avoid solving linear systems; therefore, they are very suitable for large scale problems.

In Section 2 we describe the class of methods and in Section 3 we study their convergence properties. Then we discuss how to solve the subproblem in Section 4. The major focus of this paper is to present this novel idea. There are still some implementation issues to be solved which will be addressed in future work. However, we did implement the algorithm for solving the special case where each E_i is an ellipsoid and the numerical results presented in [Lin, Han, 2003] shows that this idea has great potential. All discussion is limited to the R^n space. For a set S in R^n we use ∂S and $int(S)$ to denote its boundary and interior, respectively. We also use R_{++} to denote the set of positive real numbers and R^m_+ to denote all m-vectors with nonnegative components. Throughout the paper we assume that $E = \cap_{i=1}^m E_i$ satisfies the Slater condition: E has nonempty interior, or equivalently in this case, E has more than one point. We also use $\mathcal{A}(x)$ to denote the active index set at x for Problem (1).

2 The Class of Algorithms

The class of algorithms we present here are iterative and in each of them we generate a sequence of feasible points $\{x^k\}$ which will be shown to converge to the unique optimal solution. Having a point x^k at the k-th iteration, we first solve a subproblem of the form:

$$\begin{array}{ll} \text{Minimize} & \|x - a\|^2 \\ \text{subject to} & x \in B_i(x^k) \quad i = 1, 2, \ldots, m \end{array} \tag{2}$$

where each $B_i(x^k)$ is a ball approximation to the convex set E_i at x^k. Let y^k be the solution of this subproblem. Generally y^k may not be feasible to the original problem and therefore, we apply a feasibility-restoration procedure to y^k to produce the next feasible point x^{k+1}. To give a clear presentation of our methods, we explain how to generate the balls $B_i(x^k)$ and how to carry out the feasibility-restoration procedure in the following subsections.

2.1 Ball Approximation

Given a general convex set $M = \{x| \quad g(x) \le 0\}$ with $g(x)$ being continuously differentiable and strictly convex, for any point $z \in M$ we consider a ball $B_M(z)$ which can be viewed as a local approximation to M at z. Some properties are needed for this set-mapping $B_M(z)$. For simplicity we study those properties in terms of its center $c(z)$ and radius $\gamma(z)$, thus the ball $B_M(z)$ is given by $B_M(z) := \{x : \|x - c(z)\| \leq \gamma(z)\}$. The center mapping, $c : M \longrightarrow R^n$, and the radius mapping, $\gamma : M \longrightarrow R_{++}$, are required to satisfy the following conditions:

R1. Both c and γ are continuous mappings defined on M.

R2. If $z \in \text{int}(M)$, then $z \in \text{int}(B_M(z))$.

R3. If $z \in \partial M$, then $z \in \partial B_M(z)$, and $c(z) = z - \lambda \bigtriangledown g(x)$ for some fixed $\lambda > 0$.

It is not particularly difficult to construct mappings that can satisfy the above requirements and the class of methods differ in the choices of them. We give an example below:

For any two positive numbers λ and β,

$$\begin{aligned} c(z) &= z - \lambda \bigtriangledown g(z) \\ \gamma(z) &= \lambda \| \bigtriangledown g(z) \| + \beta \sqrt{-g(z)}. \end{aligned}$$

2.2 Feasibility-Restoration Procedure

For any closed convex set M and two points u and v with $u \in M$, we define a step $\tau(u, v, M)$ as:

$$\tau(u, v, M) = \min\{\tau : v + \tau(u - v) \in M \text{ and } \tau \in [0, 1]\}.$$

Therefore, the point $v + \tau(u, v, M)(u - v)$ is the closest point of M to v on the line-segment $[u, v]$. For the set E in our problem,

$$\tau(u, v, E) = \max\{\tau(u, v, E_i) | i = 1, 2, \ldots, m\}.$$

We have the following lemma regarding the continuity of $\tau(\cdot, \cdot, S)$.

Lemma 2.1 *If M is defined as $\{x | g(x) \leq 0\}$ for some continuously differentiable strictly convex function g, then $\tau(\cdot, \cdot, M)$ is continuous at any (u, v) where $u \in M$, $v \in R^n$ and $u \neq v$.*

Proof: If $g(v) < 0$, then for any $\bar{v}$ sufficiently close to v we have $g(\bar{v}) < 0$, thus $\tau(\bar{u}, \bar{v}, M) = 0 = \tau(u, v, M)$ for any $\bar{u} \in M$. Hence $\tau(\cdot, \cdot, M)$ is continuous at (u, v).

Now suppose $g(v) \geq 0$. We must have $g(v + \tau(u, v, M)(u - v)) = 0$. If $\tau(u, v, M)$ is not continuous at (u, v), then there exist $\{v_k\} \subset R^n$ and $\{u_k\} \subset M$ such that $v_k \to v$, $u_k \to u$ and $\tau(u_k, v_k, M)$ does not converge to $\tau(u, v, M)$. Since $\{\tau(u_k, v_k, M)\}$ is bounded, by passing to a subsequence if necessary, we can assume that $\tau(u_k, v_k, M) \to \bar{\tau} \neq \tau(u, v, M)$. If $v \in M^c$, then when k is sufficiently large, we have $v_k \in M^c$. If $v \in M$, then $\tau(u, v, M) = 0$, hence there are infinitely many k such that $v_k \notin M$. By further passing to a subsequence if necessary, we can also assume $\{v_k\} \subset M^c$. Since $g(v_k) > 0$ and $g(u_k) \leq 0$, we have $g(v_k + \tau(u_k, v_k, M)(u_k - v_k)) = 0$. Hence by letting k go to ∞, we get $g(v + \bar{\tau}(u - v)) = 0$. Obviously $\bar{\tau} \in [0, 1]$. According to the definition of $\tau(\cdot, \cdot, M)$, we have $0 \leq \tau(u, v, M) < \bar{\tau} \leq 1$. Since g is strictly convex, $g(v + \frac{\tau(u,v,M)+\bar{\tau}}{2}(u - v)) < 0$. Thus when k is sufficiently large, $g(v_k + \frac{\tau(u,v,M)+\bar{\tau}}{2}(u_k - v_k)) < 0$, which leads to $\tau(u_k, v_k, M) \leq \frac{\tau(u,v,M)+\bar{\tau}}{2}$. Letting k go to ∞, we get $\bar{\tau} \leq \frac{\tau(u,v,M)+\bar{\tau}}{2}$ which contradicts $\tau(u, v, M) < \bar{\tau}$. So $\tau(\cdot, \cdot, M)$ must be continuous at (u, v). Q.E.D.

Therefore the mapping $\tau(\cdot, \cdot, E)$ is continuous on the set $\{(x, y) : x \in E, y \in R^n$ and $x \neq y\}$. We also need the following mapping $T : E \times R^n \to E$:

$$T(x, y) = y + \tau(x, y, E)(x - y).$$

It follows from the continuity of $\tau(\cdot, \cdot)$ that $T(\cdot, \cdot)$ is also continuous on the set $\{(x, y) : x \in E, y \in R^n$ and $x \neq y\}$.

In the class of algorithms we are going to present, having x^k, y^k and a, we use the mapping $T(\cdot, \cdot)$ to generate the next feasible estimate x^{k+1}. Furthermore, if $x \neq y$ and $y - x$ is a feasible direction of E at x, then $\tau(x, y, E) < 1$. This follows directly from the definition of a feasible direction: a vector d is a feasible direction of a set S at x if $x + \lambda d \in S$ for all $\lambda \in [0, \epsilon]$ with some $\epsilon > 0$. The above continuity and less-than-one properties of the mapping τ will be needed in our convergence analysis.

2.3 Description of the Algorithms

We now describe the class of algorithms. At an iteration of the algorithm, having a feasible point $x \in E$ as an estimate of the solution, we compute a better estimate $\bar{x}$ by first generating a ball $B_i(z) := \{z : \|z - c_i(x)\| \leq \gamma_i(x)\}$ for each convex set E_i as described in Subsection 2.1. Then we solve the following projection problem:

$$\begin{array}{ll} \text{Minimize} & \|z - a\|^2 \\ \text{subject to} & z \in B_i(x), \quad i = 1, 2, \ldots, m. \end{array} \tag{3}$$

When the solution y of the above subproblem is infeasible to the original problem, we compute a feasible point v on the line segment $[T(x,y), \beta T(x,y) + (1-\beta)x]$ where β is some prespecified constant from $(0,1]$. This can be achieved by some variants of the bisection method. When y is feasible, we just simply set $v = y$. Since we are interested in points closer to a, we can improve v further by choosing our next estimate $\bar{x}$ from $[v, T(a,v)]$ and idealy, we want $\bar{x}$ to be as close to $T(a,v)$ as possible.

The algorithm can now be summarized as follows:

1. Choose a constant $\beta \in (0,1]$. Start from a feasible point x^0.
2. At the k-th iteration, having a feasible point x^k, we do the following:
   (i) For each convex set E_i, generate a ball $B_i(x^k)$ which satisfies the three conditions in Subsection 2.1.
   (ii) Find the projection y^k of the given point a onto $\cap_{i=1}^m B_i(x^k)$.
   (iii) If $y^k \in E$, then set $v^k = y^k$; otherwise choose v^k to be any point in $[T(y^k, x^k), \beta T(y^k, x^k) + (1-\beta)x^k]$.
   (iv) Compute a new estimate $x^{k+1} \in [v^k, T(a, v^k)]$.

3 Convergence Analysis

We first establish that our subproblems are well defined and satisfy the Slater condition when the original problem does. For simplicity, let $B(x)$ denote the intersection $\cap_{i=1}^m B_i(x)$. We note here an interesting fact which will be used very often in the following discussion: E and $B(x)$ have the same feasible direction cone at the point x. This is because that for $i \in \mathcal{A}(x)$, E_i and $B_i(x)$ have the same normal vector $g_i(x)$ and for $i \notin \mathcal{A}(x)$, x is an interior point for both E_i and $B_i(x)$.

Lemma 3.1 *If there exists $y \in int(E) = \{x | g_i(x) < 0 \quad i = 1, 2, \ldots, m\}$, then for any $x \in E$, $int(B(x)) \neq \emptyset$.*

Proof. If $\mathcal{A}(x) = \emptyset$, then $g_i(x) < 0$ for $i = 1, 2, \ldots, m$. Thus we have $x \in \mathrm{int}(B_i(x))$ for all i, hence $x \in \mathrm{int}(B(x))$.

If $\mathcal{A}(x) \neq \emptyset$, then $x \neq y$. Letting $s = y - x$, we will show that for sufficiently small $\epsilon > 0$, point $x + \epsilon s$ is in $\mathrm{int}(B_i(x))$ for all i, so that it lies in $\mathrm{int}(B)$. This holds for each $i \notin \mathcal{A}(x)$, since by requirement R2, x lies in $\mathrm{int}(B_i(x))$. Now consider any $i \in \mathcal{A}(x)$. By the definition of s and the convexity of $g_i(\cdot)$, we have $s^T \bigtriangledown g_i(x) \leq g_i(y) - g_i(x) < 0$. Since by requirement R3, $x - z_i(x)$ has the same direction as $\bigtriangledown g_i(x)$, it follows that $s^T(x - z_i(x)) < 0$. From this together with the consequence $x \in \partial B_i(x)$ of R3, it follows as desired that $x + \epsilon s$ is in $\mathrm{int}(B_i(x))$ for sufficiently small $\epsilon > 0$. Q.E.D.

We now establish a key lemma that shows the point y obtained from the subproblem (3) is, indeed, a substantial improvement over the current point x.

Lemma 3.2 *For any $x \in E$ and the solution y of the subproblem (3),*

$$\|x - a\|^2 \geq \|y - a\|^2 + \|x - y\|^2.$$

Proof: If any two of x, y and a are the same, the conclusion is obviously true.

If they are three distinct points, then they form a triangle. If $a - y$ and $x - y$ make an accute angle, then y can not be the optimal solution of the subproblem (3) since $y + t(x - y)$ is better for small $t > 0$. Therefore the angle between $a - y$ and $x - y$ is greater than or equal to $\frac{\pi}{2}$, hence the conclusion holds. Q.E.D.

Using the same notation as in Subsection 2.3, we know that the feasible point v lies on the line-segment $[x, y]$, therefore, by the convexity of the objective function we have

$$\|x - a\| \geq \|v - a\| \geq \|y - a\|.$$

Moreover, the new point $\bar{x}$ is a further improvement over v and hence we also have

$$\|x - a\| \geq \|v - a\| \geq \|\bar{x} - a\|.$$

Consequently, both $\{\|x^k - a\|\}$ and $\{\|v^k - a\|\}$ are nonincreasing and have the same limit. We summarize the above results in the following lemma.

Lemma 3.3 *The following statements are true for the three sequences $\{x^k\}, \{v^k\}$ and $\{y^k\}$:*
(1) $\|x^k - a\| \geq \|v^k - a\| \geq \|y^k - a\|$.
(2) $\|x^k - a\| \geq \|v^k - a\| \geq \|x^{k+1} - a\|$.
(3) $\lim \|x^k - a\| = \lim \|v^k - a\|$.

We also need the following lemma about the relationship between the sequences $\{y^k\}$ and $\{x^k\}$.

Lemma 3.4 $\lim(x^k - y^k) = 0$.

Proof: By Lemma 3.2 we only need to show $\lim(\|x^k - a\|^2 - \|y^k - a\|^2) = 0$. We prove it by contradiction. Assume that this is not true, then there exists a subsequence $\{n_k\}$ such that $\lim \|x^{n_k} - a\| > \lim \|y^{n_k} - a\|$. Because both $\{x^k\}$ and $\{y^k\}$ are bounded, we may assume, by extracting a further subsequence if necessary, $\lim x^{n_k} = x^*$ and $\lim y^{n_k} = y^*$ for some points $x^* \in E$ and $y^* \in R^n$. Of course, $x^* \neq y^*$. We first show that the step $\tau(x^*, y^*, E)$ is strictly less than 1. It follows from the continuity of the ball mapping $B_i(x)$ and $y^{n_k} \in B(x^{n_k})$ that $y^* \in B(x^*)$. Therefore the nonzero vector $y^* - x^*$ is a feasible direction of $B(x^*)$ at x^*, hence it is also a feasible direction of E at x^*. Thus the step $\tau(x^*, y^*, E) < 1$ and therefore $T(x^*, y^*) \neq x^*$. It follows from $\|x^* - a\| > \|y^* - a\|$, $\tau(x^*, y^*, E) < 1$ and the convexity of the objective function that $\|T(x^*, y^*) - a\| < \|x^* - a\|$. By the continuity of the mapping $T(\cdot, \cdot)$ at (x^*, y^*), we have $T(x^{n_k}, y^{n_k}) \to T(x^*, y^*)$. Note that $\|T(x^{n_k}, y^{n_k}) - a\| \leq \|x^{n_k} - a\|$, so from Lemma 3.3 we have $\|x^{n_k+1} - a\| \leq \|v^{n_k} - a\| \leq \beta\|T(y^{n_k}, x^{n_k}) - a\| + (1 - \beta)\|x^{n_k} - a\|$. Letting n_k go to ∞ we get $\|x^* - a\| \leq \beta\|T(y^*, x^*) - a\| + (1 - \beta)\|x^* - a\|$. This is a contradiction since we just showed that $\|T(y^*, x^*) - a\| < \|x^* - a\|$. Q.E.D.

The following lemma shows the close connection between Problem (1) and the subproblem.

Lemma 3.5 *A vector u^* is the unique optimal solution of Problem (1) if and only if $u^* \in E$ and u^* is the optimal solution of the problem:* $\min\{\|x - a\| : x \in B(u^*)\}$.

Proof: This follows directly from comparing the Karish-Kuhn-Tucker conditions of the two problems. Q.E.D.

The following lemma is a special case of the standard parametric optimization. The results for much more general cases are contained in Proposition 23 on page 120 in [Aubin, Ekeland, 1984] and the Maximum theorem on page 116 in [Berge, 1997].

Lemma 3.6 *Let $\{u^k\}$ be a sequence in E convergent to a point u^*. If for each k, w^k solves* $\min\{\|x - a\| : x \in B(u^k)\}$ *and* $\lim w^k = w^*$, *then the point w^* solves the problem* $\min\{\|x - a\| : x \in B(u^*)\}$.

We now give our main convergence theorem below.

Theorem 3.7 *The three sequences $\{x^k\}$, $\{v^k\}$ and $\{y^k\}$ all converge to the solution of Problem (1).*

Proof: It follows from Lemma 3.4 that all three sequences have the same set of accumulation points because $\lim(x^k - y^k) = 0$ and v^k lies on the line-segment $[x^k, y^k]$. Let u^* be an accumulation point, we want to show that u^* solves the problem. Let $\{n_k\}$ be a subsequence such that $\lim z^{n_k} = \lim y^{n_k} = u^*$. It follows from Lemma 3.6 that u^* solves $\min\{\|x-a\| : x \in B(u^*)\}$ and by Lemma 3.5, the vector u^* is optimal to the original problem. By the strict convexity of the objective function, u^* is the unique optimal solution. It then follows from the boundedness of the three sequences that they all converge to u^*. Q.E.D.

4 Projecting a point onto the intersection of several balls

This section is basically a copy of the Section 4 of [Lin, Han, 2003]. The only purpose of putting it here is to make this paper self-contained.

In this section we briefly discuss how to project a point onto the intersection of several balls, an operation needed for Step 2(ii) of our projection algorithm. The problem can be expressed as

$$\begin{array}{ll} \text{Minimize} & \|x-a\|^2 \\ \text{subject to} & x \in B_i := \{x \| \|x-a_i\|^2 \le r_i^2\} \quad i = 1, 2, \ldots, m. \end{array} \tag{4}$$

Letting $B = \cap_{i=1}^m B_i$, we assume $\mathrm{int}(B) \neq \emptyset$ and $a \notin B$.

As usual, the Lagrangian function is defined as

$$\begin{aligned} L(x,\lambda) &= \|x-a\|^2 + \sum_{i=1}^m \lambda_i(\|x-a_i\|^2 - r_i^2) \\ &= (1+\sum_{i=1}^m \lambda_i)\|x - \frac{a+\sum_{i=1}^m \lambda_i a_i}{1+\sum_{i=1}^m \lambda_i}\|^2 - \frac{\|a+\sum_{i=1}^m \lambda_i a_i\|^2}{1+\sum_{i=1}^m \lambda_i} \\ &\quad + \sum_{i=1}^m \lambda_i(\|a_i\|^2 - r_i^2) + \|a\|^2, \end{aligned}$$

where $\lambda \in R_+^m$; the dual function is

$$\begin{aligned} g(\lambda) &= \inf_{x \in R^n} L(x,\lambda) \\ &= L(\frac{a+\sum_{i=1}^m \lambda_i a_i}{1+\sum_{i=1}^m \lambda_i}, \lambda) \\ &= -\frac{\|a+\sum_{i=1}^m \lambda_i a_i\|^2}{1+\sum_{i=1}^m \lambda_i} + \sum_{i=1}^m \lambda_i(\|a_i\|^2 - r_i^2) + \|a\|^2, \end{aligned}$$

and thus the dual problem is defined as

$$\begin{array}{ll} \text{Maximize} & g(\lambda) \\ \text{subject to} & \lambda \in R^m_+. \end{array}$$

The primal problem is a convex programming problem satisfying Slater's condition, so strong duality holds. Let x^* denote the primal optimal solution which is unique due to the strict convexity of the primal objective function, λ^* denote a dual optimal solution. Since strong duality holds, we have

$$\begin{aligned} & \|x^* - a\|^2 \\ = \ & g(\lambda^*) \\ = \ & \inf_{x \in R^n} L(x, \lambda^*) \\ \leq \ & L(x^*, \lambda^*) \\ = \ & \|x^* - a\|^2 + \sum_{i=1}^m \lambda_i^* (\|x^* - a_i\|^2 - r_i^2) \\ \leq \ & \|x^* - a\|^2, \end{aligned}$$

where the last inequality is due to the fact that x^* is a feasible point of the primal problem and $\lambda^* \geq 0$. Thus we get

$$L(x^*, \lambda^*) = \inf_{x \in R^n} L(x, \lambda^*) = L\left(\frac{a + \sum_{i=1}^m \lambda_i^* a_i}{1 + \sum_{i=1}^m \lambda_i^*}, \lambda^*\right).$$

Note $L(x, \lambda^*)$ is a strictly convex function of x, so we must have $x^* = \frac{a + \sum_{i=1}^m \lambda_i^* a_i}{1 + \sum_{i=1}^m \lambda_i^*}$.

From the above discussion, we see that in order to find x^* we only need to solve the dual problem. The dual problem is a simple convex programming problem with only nonnegative constraints and readily computable gradients and Hessians. A lot of standard algorithms can solve this problem very efficiently. For example, the nonmonotone spectral projected gradient method proposed in [Birgin, Martínez, Raydan, 2000] is very fast and only involves simple arithmetic computation, i.e., no matrix-factorization is needed. In [Lin, Han, 2003] we use this method to solve the ball-constrained subproblem.

Due to its special structure, the primal problem can also be attacked directly, for example, by interior-point methods. "SeDuMi" [Sturm, 1999] is probably the best noncommercial software for doing this job.

Both the nonmontone spectral projected gradient method and the interior-point methods are excellent algorithms, but we don't think they fully exploit the special structure of this ball-constrained subproblem. It will not be surprising that a special method dedicated to this problem can outperform these general methods.

5 Comments and Future Research

We have presented a new idea for projecting a point onto a special kind of convex set. This idea, based on approximating a convex set by several balls, can avoid solving linear systems. Therefore it can be expected to be very economic in terms of memory usage. The numerical experiments in [Lin, Han, 2003] demonstrates this property for the case when the convex set is the intersection of several ellipsoids. For future research, we are trying to solve the implementation issues and in particular to develop a dedicated method of solving the subproblem. We are also interested in generalizing this idea to other problems.

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