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Arbitrage and Equilibrium in Asset Exchange Economies: A Survey[1]

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Abstract: This article surveys some recent progress on arbitrage and equilibrium in asset exchange economies. Using the basic geometry of arbitrage, the relationships between various no-arbitrage conditions appeared in the literature are presented. The relationships between some of the basic no-arbitrage conditions together with the existence result of an equilibrium in Dana et al. [16] provide an overview of sufficient conditions for equilibrium.Under certain conditions the various no-arbitrage conditions in the literature are equivalent and necessary and sufficient for the existence of an equilibrium.

Keywords: Arbitrage; Equilibrium; Compactness; Cones; Uniformity

1 Introduction

Since the pioneering contributions of Grandmont [21, 22, 23], Green [25], and Hart [27], the relationship between arbitrage and equilibrium in asset exchange economies allowing short sales has been one of the hot subject in economic and financial studies. An arbitrage opportunity is a mutually compatible set of net trades which are utility nondecreasing and, at most, costless to make. Conditions that limit utility arbitrage are central to establishing existence in general equilibrium models of exchange economies with unbounded short sales (see, for example, Page [40] and Werner [55]). When unbounded short sales are allowed, as is natural in asset market models, agents' choicer sets are bounded from below, and as a consequence, unbounded and mutually compatible arbitrage opportunities can be exhausted may fail to exist, and thus, equilibrium may fail to exist. By assuming that markets admit "no arbitrage", the economy can be bounded endogenously − but this is not

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enough for existence. Since the seminal contributions of Werner [55], much of the research on asset market models have focused upon conditions limiting arbitrage (i.e., no-arbitrage conditions) and upon the relationship between such conditions and the existence of an equilibrium.

No-arbitrage conditions appeared in literature generally fall into three broad categories:

(i) Conditions on **net trades**, for example, Hart [27], Page [41], Nielsen [37], Allouch [2], Page et al. [48] and Allouch [4].

(ii) Conditions on **prices**, for example, Green [25], Grandmont [23, 24], Hammond [26] and Werner [55].

(iii) Conditions on **the set of utility possibilities (namely, compactness)**, for example, Brown and Werner [8] and Dana et al. [16].

In a temporary equilibrium model, Grandmont [23] shows that the overlapping expectations conditions is necessary and sufficient for the existence of an equilibrium. Grandmont's result is the first to give necessary and sufficient conditions for existence of equilibrium in an economics model with asset trading and unbounded short sales. Grandmont's result continues to hold in an asset market setting with unrestricted short selling, provided each investor's asymptotic risk tolerance is zero. In particular, Hart [27], Milne [34], Hammond [26] and Page [38, 41, 42] show that overlapping expectations is sufficient for the existence of an equilibrium in an asset market model in which each investor's asymptotic risk tolerance is zero. They also show that the equilibrium price vector must be contained in the overlap of investor expectations (see Page [39] and Hammond [26]). Thus, it follows as a corollary of the results of Hart [27], Hammond [26], and page [39]) that if each investor's preference are not dependent on prices and each investor's asymptotic risk tolerance is zero, then overlapping expectations is necessary and sufficient for the existence of an equilibrium (see also Milne, [34]). Page [44] generalizes the overlapping expectations condition and shows this generalized condition is necessary and sufficient for the existence of an equilibrium in an asset market model in which preferences are not dependent on prices and investors's are allowed to have asymptotic risk tolerances greater than zero.

At the opposite end of the spectrum from the models of temporary equilibrium and incomplete markets are the general equilibrium models of exchange economies with unbounded short sales (see, for example, page [40]; Werner [55]; Niesen [37]; Page and Wooders [43, 45] and Chichilnisky [11]). The role played by conditions limiting arbitrage in general equilibrium models with short sales is to bound the economy endogenously.

For example, Hart [27] introduces the weak no-market-arbitrage condition on net trades which requires that all mutually compatible arbitrage opportunities be useless. Hart's [27] condition of weak-no-market-arbitrage holds if and only if the projection of set of rational allocations upon the Cartesian product of the agents' subspaces of useful net trades is compact. If in addition, weakly uniform is satisfied, then Hart's condition also implies the compactness of the set of rational utility possibilities.

Werner [55] introduces the no-arbitrage price system condition on prices which requires that there be a nonempty set of prices such that each price contained in this non-empty subset assigns a positive value to any vector of useful net trades belonging to any agent. Werner then assumes that for each agent the set of useful net trades at endowments is non-empty. Werner's [55] condition of no-arbitrage price system implies directly the compactness of the set of utility possibilities but allow the set of rational allocations to be unbounded. An especially intriguing aspect of Werner's existence result is that it does not require local or global nonsatiation (see Werner [55], Theorems 1). This contrasts sharply with classical existence results for bounded exchange economies which require, at minimum, that agents' preferences be globally nonsatiated at rational allocations (e.g., see Debreu [17], Gale and Mas-Colell [19], and Bergstrom [7]).

Allouch et al. [3] extend Werner's price no-arbitrage condition to allow for weak nonsatiation - and in particular, to allow for the possibility that some agents have empty sets of useful net trades at some rational allocations. Allouch et al. [3] show that this extended price no-arbitrage condition is equivalent to Hart's [27] weak no-market-arbitrage condition.

Page [41] introduces the condition of no-unbounded-arbitrage condition on net trades stronger than Hart' condition, which requires that all mutually compatible arbitrage opportunities be trivial. Page's [41] condition is equivalent to the compactness of the set of rational allocations, and therefore implies the compactness of the set of rational utility possibilities. Moreover, we show that if agents utility-constant subspaces (at endowments) are linearly independent, then Hart [27] weak no-market-arbitrage condition, Werner [55] no-arbitrage price condition and Page [41] no-unbounded-arbitrage condition are equivalent, and in turn, all are equivalent to the compactness of the set of rational allocations, and therefore implies the compactness of the set of rational utility possibilities. Because the no-arbitrage condition of Hammond [26] −overlapping expectations − is stated in terms of properties of the subjective probability distributions of asset returns, it is difficult to make comparisons in an abstract general equilibrium setting between Hammond's condition and other no-arbitrage conditions. Page [41] shows that under very mild conditions on utility functions and asset return distributions, Hammond's condition of overlapping expectations is equivalent to no-unbounded-arbitrage.

Page et al. [48] introduce the concept of inconsequential arbitrage and, in the context of a model allowing short-sales and half-lines in indifference surfaces, prove that inconsequential arbitrage is sufficient for the existence of an equilibrium. Moreover, with a slightly stronger condition of nonsatiation than that required for the existence of an equilibrium and with a mild uniformity condition on arbitrage opportunities, inconsequential arbitrage, the existence of a Pareto optimal allocation, and the compactness of the set of utility possibilities are equivalent. Thus, when all equilibria are Pareto optimal − for example, when local nonsatiation holds − inconsequential arbitrage is necessary and sufficient for the existence of an equilibrium. By further strengthening this nonsatiation condition, Page et al. [48] obtain a second welfare theorem for exchange economies allowing short sales. In addition, under weak uniformity only that the conditions of Hart and Werner conditions imply inconsequential arbitrage. Under the assumption of no half-lines in indifference surfaces, the conditions of Hart and Werner conditions and inconsequential arbitrage are equivalent.

Dana et al. [16] introduce the concept of strong unbounded arbitrage and show that the absence of strong unbounded arbitrage directly implies the compactness of the individually rational utility set. This result seems to be the first which infers the compactness of U from a no-arbitrage condition. Under the assumption of local nonsatiation at rational allocations, Dana et al. [16] show that compactness of utility possibilities is sufficient for the existence of an equilibrium.

Allouch [4] also introduces the compactness with partial preorder condition (a new condition, bounded arbitrage introduced in Allouch [2]), which eliminates the problem of unboundedness by requiring every sequence of attainable and individually rational allocations to be dominated by an increasing preference subsequence converging to an attainable allocation, and therefore, implies the existence of a competitive equilibrium. Allouch [2] also shows that if local satiation is ruled out, then his condition of bounded arbitrage is equivalent to the compactness of utility possibilities. Allouch's result is implied by Hart [27] and Page [41], but is equivalent to Dana et al. [16] in the case of utility-representable preferences. The compactness with partial preorder condition is weaker than the classical compactness of A the set of individually rational and attainable allocations.

Under a different set of assumptions on the economic model, Chichilnisky defined arbitrage as an opportunity for an agent to increase his utility costlessly beyond the level associated with any given vector in his consumption set. Chichilnisky introduces a new condition, called limited arbitrage which rules out such arbitrage, and asserts that within the context of her model, limited arbitrage is necessary and sufficient for the existence of an equilibrium. Chichilnisky also claims that her condition is necessary and sufficient for boundedness of gains from trade. Because of some ambiguousnesses, the definition given by Chichilnisky may be flawed (see Monterio et al. [35]). This ambiguousnesses disappear in Chichilnisky [12]. Chichilnisky and Heal [14] present limited arbitrage is necessary and sufficient for the existence of an equilibrium and the core in finite or infinite economies.

The stronger conditions of Hammond [26] and Page [41] imply the existence of an equilibrium, without uniformity conditions, by guaranteeing the compactness of the set of rational allocation, while the weaker conditions of Hart [27] and Werner [55] require weak uniformity of preferences to guarantee the compactness of utility possibilities, and therefore to guarantee the existence via the Dana et al. [16] result.

Inconsequential arbitrage and bounded arbitrage work differently. They imply the compactness of the set of utility possibilities without any type of uniformity, and therefore, is sufficient for the existence without uniformity−again via the Dana et al. [16] result.

Using the geometry of arbitrage, Allouch et al. [5] sharpen and extend the result of Page et al. [48] showing the equivalence of the conditions of Hart [27] and Werner [55]. Allouch et al. [5] establish this equivalence without any assumptions concerning uniformity or nonsatiation. In Page et al. [48], the equivalence of Hart and Werner is obtained assuming a very weak form of nonsatiation (due to Werner [55]) and a strong form of uniformity (i.e. uniformity of arbitrage opportunities). In addition, Allouch et al. [3] show under weak uniformity only that the conditions of Hart and Werner imply the condition of inconsequential arbitrage, introduced in Page et al. [48]. Page et al. [48] show this as well, but require Werner nonsatiation and strong uniformity. If the economy satisfies uniformity of arbitrage opportunities, local nonsatiation at rational allocation and weak no-half-lines, then the Hart-Werner no-arbitrage conditions and inconsequential arbitrage are equivalent, and are necessary and sufficient for the compactness of the set of utility possibilities and the existence of an equilibrium. If we strengthen the weak no-half-lines condition to Werner's condition of no-half-lines, then the Hart-Werner no-arbitrage conditions and inconsequential arbitrage are equivalent to no-unbounded-arbitrage, and all are necessary and sufficient for the compactness of the set of rational allocations, the compactness of the set of utility possibilities, and the existence of an equilibrium.

The paper is organized as follows. Basic model of an unbounded exchange economy and some definitions are presented in Section 2. Section 3 is dedicated to present the relationship between the various no-arbitrage conditions found in the literature and the strength of the boundedness implied by these conditions. In Section 4, Sufficient conditions for the existence of an equilibrium is addressed. Finally, Section 5 shows that under certain conditions, the various no-arbitrage conditions appeared in the literature are equivalent and necessary and sufficient for the existence of an equilibrium.

2 The model

We consider an economy $\varepsilon = (X_i, u_i, e_i)_{i=1}^m$ with m agents and l goods. Agent i has consumption set $X_i \subset R^l$, utility function $u_i(.)$, and endowment e_i, Agent i's preferred set at $x_i \in X_i$ is

$$P_i(x_i) = \{x \in X_i \mid u_i(x) > u_i(x_i)\},$$

while the weak preferred set at $x_i \in X_i$ is

$$\hat{P}_i(x_i) = \{x \in X_i \mid u_i(x) \geq u_i(x_i)\}.$$

The set of individually rational utility possibilities is given by

$$A = \{(x_i) \in \prod_{i=1}^{m} X_i \mid \exists \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \quad \text{and} \quad x_i \in \hat{P}_i(e_i), \forall i\}$$

We shall denote ny A_i the projection of A onto X_i.

The set of individually rational utility possibilities is given by

$$U(\varepsilon) = \{(v_i) \in R^m \mid \exists x \in A, \quad \text{such that} \quad u_i(e_i) \leq v_i \leq u_i(x_i), \forall i\}$$

The Pareto frontier $P(\varepsilon)$ is the set of undominated vectors in U:

$$P(\varepsilon) = \{U \in U(\varepsilon) :\sim \exists V \in U(\varepsilon) \text{ with } V > U\}.$$

Definition 2.1 *(a) A rational allocation $x^* \in A$ together with a nonzero vector of prices $p^* \in R^l$ is an equilibrium for the economy ε*

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x > p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x = p^* \cdot e_i$.

(b) A rational allocation $x^* \in A$ together with a nonzero vector of prices $p^* \in R^l$ is a quasi-equilibrium for the economy ε

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x \geq p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x = p^* \cdot e_i$.

Given (x^*, p^*) a quasi-equilibrium, it is well-known that if for each agent i, (i) $p^* \cdot x < p^* \cdot e_i$ for some $x \in X_i$ and (ii) $P_i(x_i^*)$ is relatively open in X_i, then (x^*, p^*) is an equilibrium. Conditions (i) and (ii) will be satisfied if , for example, for each agent $i, e_i \in \text{int}X_i$, and u_i is continuous on X_i. Using irreducibility assumptions, one can also show that a quasi-equilibrium is an equilibrium.

We now introduce our first two assumptions for agents $i = 1, 2, \cdots, m$,

[**A.1**] X_i is closed and convex with $e_i \in X_i$,

[**A.2**] u_i is upper semicontinuous and quasi-concave.

Under these two assumptions, the weak preferred set $\hat{P}_i(x_i)$ is convex and closed for $x_i \in X_i$.

2.1 Arbitrage, Uniformity, and Nonsatiation

2.1.1 Arbitrage

Definition 2.2 *The i^{th} agent's* **arbitrage cone** *at $x_i \in X_i$ as the closed convex cone containing the origin given by*

$$O^+\hat{P}_i(x_i) = \{y_i \in R^l \mid \forall x_i' \in \hat{P}_i(x_i) \text{ and } \lambda \geq 0, x_i' + \lambda y_i \in \hat{P}_i(x_i)\}.$$

Definition 2.3 (Chichilnisky [10]) *The* **global cone** *corresponding to the* i*th agent's utility function* $u_i(\cdot)$ *at consumption vector* $x_i \in X_i$ *is given by,*

$$G_i(x_i) = \{y \in R^l : \forall x \in R^l \ \exists \lambda_x > 0 \text{ such that } u_i(x_i + \lambda_x y) > u_i(x)\}.$$

Definition 2.4 (Page [39]) *The* **increasing cone** *corresponding to the* i*th agent's utility function* $u_i(\cdot)$ *at consumption vector* $x_i \in X_i$ *is given by,*

$$I_i(x_i) = \{y \in R^l : \ u_i(x_i + \lambda y) > u_i(x + \mu y) \text{ if } \lambda > \mu \geq 0\}.$$

In Page and Wooders [45, 46] the definition of the increasing cone is extended to accommodate thick indifference curve:

$$\hat{I}_i(x_i) = \{y \in R^l : \forall \mu \geq 0, \ \exists \lambda > \mu \text{ such that } u_i(x_i + \lambda y) > u_i(x + \mu y)\}.$$

Chichinisky [12] modifies her arbitrage condition by using the increasing cone $\hat{I}_i(x_i)$, but alternatively stated in her paper as:

$$G'_i(x_i) = \{y \in R^l : \neg \exists \max_{\lambda \geq 0} u_i(x_i + \lambda y)\}.$$

The **market cone** of consumer i is

$$D_i(x_i) = \{z \in X : \forall y \in G'_i(x_i), \langle z, y \rangle > 0\}$$

D_i is the convex cone of prices assigning strictly positive value to all directions in $G_i(x_i)$.

Let

$$G_i(e_i) = G_i; I_i(e_i) = I_i; \hat{I}(e_i) = \hat{I}_i; G'_i(e_i) = G'_i; D_i(e_i) = D_i.$$

Note that if the agent, starting at x_i, trades in the $y_i \in O^+\hat{P}_i(x_i)$ direction on any scale $\lambda \geq 0$, then his utility will be nondecreasing. In particular, a set of net trades $y = (y_1, \cdots, y_m)$ is an arbitrage opportunity at $x = (x_1, \cdots, x_m)$ if

$$\sum_{i=1}^{m} y_i = 0 \text{ (i.e., trades are mutually compatible),}$$

and

$$y_i \in O^+\hat{P}_i(x_i) \text{ for all } i \text{ (i.e., trades starting at } x_i \text{ are utility nondecreasing).}$$

2.1.2 Uniformity

A set closely related to the i^{th} agent's arbitrage cone is the linearity space $L_i(x_i)$ of $O^+\hat{P}_i(x_i)$ given by

$$L_i(x_i) = \{y_i \in R^l \mid \forall x_i^{'} \in \hat{P}_i(x_i) \text{ and } \lambda \in R, x_i^{'} + \lambda y_i \in \hat{P}_i(x_i)\}.$$

The set $L_i(x_i)$ consists of the zero vector and all the nonzero vectors y_i such that for each $x_i^{'}$ weakly preferred to x_i (i.e. $x_i^{'} \in \hat{P}_i(x_i)$), any vector z_i on the line through $x_i^{'}$ in the direction $y_i, z_i = x_i^{'} + \lambda y_i$, is also weakly preferred to x_i. The set $L_i(x_i)$ is a subspace of R^l, and is the largest subspace contained in the arbitrage cone $O^+\hat{P}_i(x_i)$. Moreover, since R^l is finite-dimensional, $L_i(x_i)$ is a closed subspace of R^l.

A set of net trades $y = (y_1, \cdots, y_l)$ is useless for consumer i if $u_i(x+y) = u_i(x) = u_i(x-y)$ for all $x \in X_i$; A set of net trades $y = (y_1, \cdots, y_l)$ is useful for consumer i if $u_i(x+y) \geq u_i(x)$ for all $x \in X_i$, and y is not useless [Werner (1987)].

[A.3] [Weak Uniformity] $L_i(x_i) = L_i := L(e_i), \forall x_i \in \hat{P}_i(e_i), \forall i.$

Under weak uniformity, for all $x_i \in \hat{P}_i(e_i), \forall i$ and $y_i \in L_i(x_i)$,

$$u_i(x_i + y_i) = u_i(x_i).$$

Following the terminology of Werner (1987), we refer to arbitrage opportunities $y_i \in O^+\hat{P}_i(x_i)$ such that

$$u_i(x_i + \lambda y_i) = u_i(x_i) \text{ for all } \lambda \in (-\infty, \infty)$$

as useless at x_i. Thus, under [A.3], the useless set is the linearity space $L_i(x_i)$ of $O^+\hat{P}_i(x_i)$; and the useful set is $O^+\hat{P}_i(x_i) \setminus L_i(x_i)$.

Werner [55] makes a uniformity assumption stronger than uniformity of useless net trades (i.e., stronger than weak uniformity, [A,3]). Werner [55] assumes that each agent's arbitrage cone is invariant with respect to the starting point of the trading (i.e., x_i), as long as the starting point is weakly preferred to the agent's endowment (i.e., as long as, $x_i \in \hat{P}_i(e_i)$). That is, Werner assumes: In particular, Werner assumes that all arbitrage opportunities are uniform. Stated formally,

[A'.3] [Weak Uniformity] $O^+\hat{P}_i(x_i) = O^+\hat{P}_i(e_i) := R_i, \forall x_i \in \hat{P}_i(e_i), \forall i.$

Note that if uniformity [A'.3] holds, then weak uniformity [A.3] holds automatically. That is [A'.3] implies that $L_i(x_i) = L_i, \forall x_i \in \hat{P}_i(e_i), \forall i.$

2.1.3 Nonsatiation

Classical existence results for bounded exchange economies which require, at minimum, global nonsatiation at rational allocations.

[A.4] [local Nonsatiation] $\forall x_i \in A_i, \exists \{y_i^n\}_n \subset X_i$ with $\lim_{n\to+\infty} y_i^n = x_i$ and $U_i(y_i^n) > u_i(x_i), \forall n$, that is, for all agents i, $P_i(x_i) \neq \emptyset$ and $clP_i(x_i) = \hat{P}_i(x_i)$ for all $x_i \in A_i$.

[A'.4] [global Nonsatiation] that the economy ε satisfies nonsatiation at rational allocations if for any rational allocation $(x_1, \cdots, x_n) \in A$, $P_i(x_i) \neq \emptyset$, for all agents i.

Werner then assumes that for each agent the set of useful net trades at endowments is nonempty, rather than assume global or local nosatiation. In particular, Werner assumes that

[WNS] [Werner nonsatiation] $R_i \backslash L_i \neq \phi, \forall i$.

This assumption is weaker than the classical assumptions. Allouch et al. [3] weaken Werner's nonsatiation assumption as follows:

[**Weak nonsatiation**] for all agents i, $\forall x_i \in A_i$, if $P_i(x_i) = \emptyset$, then

$$O^+\hat{P}_i(x_i) \backslash L_i(x_i) \neq \emptyset.$$

2.1.4 The geometry of arbitrage

Let $L_i^{\perp}(x_i)$ denote the space orthogonal agent i's subspace $L_i(x_i)$ of useless net trades at x_i. The vector space R^l can be decomposed into the direct sum of the linearity space $L_i(x_i)$ and its orthogonal complement, $L_i^{\perp}(x_i)$. Thus, given $x_i \in X_i$, we have

$$R^l = L_i^{\perp}(x_i) \otimes L_i(x_i),$$

and thus, each vector $x \in R^l$ has a unique representation at the sum of two vectors, one from $L_i(x_i)$ and one from $L_i^{\perp}(x_i)$. in particular, for each $x \in R^l$, there exist uniquely two vectors, $y \in L_i^{\perp}(x_i)$ and $z \in L_i(x_i)$, such that $x = y + z$.

Lemma 2.1 *Let $\varepsilon = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying [A.1]-[A.2]. The following statements are true:*

1. $\forall i, \forall x_i \in X_i$,

(a) $\hat{P}_i(x_i) = (\hat{P}_i(x_i) \cap L_i^{\perp}(x_i)) \oplus L_i(x_i)$,

(b) $O^+\hat{P}_i(x_i) = (O^+\hat{P}_i(x_i) \cap L_i^{\perp}(x_i)) \oplus L_i(x_i)$.

2. If in addition [A.3] holds (i.e., if weak uniformity holds), then

$$u_i(x_i + y_i) = u_i(x_i), \forall x_i \in \hat{P}_i(x_i) \text{ and } \forall y_i \in L_i.$$

3. Let $A^{\perp}$ be the projection of A onto $\prod_{i=1}^m L_i^{\perp}$. Then $A^{\perp}$ is closed and convex.

4. Let $O^+(A), O^+(A^{\perp})$ denote the recession cones of A and $A^{\perp}$ respectively. Then

$$O^+(A^{\perp}) = \{(y_i) \in \prod_{i=1}^m (R_i \cap L_i^{\perp}) \mid \sum_{i=1}^m y_i \in \sum_{i=1}^m L_i\}.$$

5. Let $B = O^+(A^\perp) + \prod_{i=1}^m L_i$*. Then*

$$O^+(A) = \{(y_i) \in B \mid \sum_{i=1}^m y_i = 0\}.$$

3 No-arbitrage conditions and compactness

3.1 Weak no market arbitrage

Hart [27] introduced the weak no-market-arbitrage condition (WNMA). Hart's condition, a condition on net trades, requires that all mutually compatible arbitrage opportunities be useless.

Definition 3.1 *The economy* ε *satisfies the WNMA condition if,* $\sum_{i=1}^m y_i = 0$ *and* $y_i \in R_i$ *for all* i*, then* $y_i \in L_i$ *for all* i*.*

Next result tells us that Hart's condition is equivalent to the condition that $A^\perp$ be compact. More importantly, it tell us that if the economy satisfies weak uniformity, then Hart's condition implies that the set of rational utility possibilities is compact.

Theorem 3.1 *Let* ε *be an economy satisfying [A.1]-[A.2]. The following statements are true:*

1. WNMA holds if and only if $A^\perp$ *is compact. In this case,*

$$O^+(A) = \{(y_i) \in \prod_{i=1}^m L_i \mid \sum_{i=1}^m y_i = 0\}.$$

2. If in addition [A.3] holds, then if ε *satisfies WNMA, then the set of rational utility possibilities,* U *is compact.*

3.2 No unbounded arbitrage

Page [41] introduced the no-unbounded-arbitrage condition (NUBA). Page's condition, a condition on net trades stronger than Hart's, requires that all mutually compatible arbitrage opportunities be trivial.

Definition 3.2 *The economy* ε *satisfies the NUBA condition if,* $\sum_{i=1}^m y_i = 0$ *and* $y_i \in R_i$ *for all* i*, then* $y_i = 0$ *for all* i*.* y *is an unbounded arbitrage if* $y_i \in R_i$ *for all* i*,* $y_i \neq 0$ *for some* i*, and* $\sum_{i=1}^m y_i = 0$*.*

Next result tells us that Page's condition is equivalent to the condition that A be compact. More importantly, it tell us that if agents' linearity spaces are linearly independent, then Hart's condition and Page's condition are equivalent.

Theorem 3.2 *Let ε be an economy satisfying [A.1]-[A.2]. The following statements are equivalent:*

1. *ε satisfies NUBA.*
2. *A is compact.*
3. *$A^{\perp}$ is compact and the linearity spaces, L_i, are linearly independent.*
4. *ε satisfies WNMA and the linearity spaces, L_i, are linearly independent.*

3.3 No arbitrage price system

Werner [55] introduced the no-arbitrage price system condition (NAPS). Werner's condition, a condition on prices, requires that there be a nonempty set of prices such that each price contained in this nonempty subset assigns a positive value to any vector of useful net trades belonging to any agent. Werner then assumed that for each agent the set of useful net trades at endowments is nonempty. In particular, Werner assumes that

[WNS] [Werner nonsatiation] $R_i \backslash L_i \neq \phi, \forall i.$

Definition 3.3 *The economy ε satisfies the [WNS], the NAPS condition is satisfied if,*

$$\bigcap_{i=1}^{m} S_i^W \neq \phi,$$

where

$$S_i^W = \{p \in R^l \mid p \cdot y > 0, \forall y \in R_i \setminus L_i\}$$

is Werner's cone of no-arbitrage prices.

Allouch et al. [5] extended Werner's condition to allow for the possibility that for some agent the set of useful net trades is empty, that is, to allow for the possibility that for some agent, $R_i \setminus L_i = \phi$. More importantly, Allouch et al. [5] proved, under very mild conditions, that their extended version of Werner's condition is equivalent to Hart's condition. This result extends an earlier result by Page et al. [48] on the equivalence of Hart and Werner conditions.

Definition 3.4 *For each agent i, define*

$$S_i = \begin{cases} S_i^W & if R_i \setminus L_i \neq \phi, \\ L_i^{\perp} & if R_i \setminus L_i = \phi. \end{cases}$$

Definition 3.5 *The economy ε satisfies the NAPS condition if*

$$\bigcap_{i=1}^{m} S_i \neq \phi.$$

Remark 3.1 *Note that if the economy ε satisfies Werner's nonsatiation condition, i.e., $R_i \setminus L_i \neq \phi, \forall i$, then the NAPS condition given in Definition 3.5 above reduces to werner's original condition given in Definition 3.3.*

Lemma 3.1 *Let ε be an economy satisfying [A.1]-[A.2]. The following statements are true:*

1. *For any i, such that $R_i \setminus L_i \neq \phi$, we have:*

$$S_i = \{p \in L_i^{\perp} \mid p \cdot y > 0, \forall y \in (R_i \cap L_i^{\perp}) \setminus \{0\}\}.$$

2. *$\forall i = 1, \cdots, m,\ S_i = -ri(R_i^0)$ where (R_i^0) is the polar cone of R_i.*

Page et al. [48] show that under [A.1]-[A.2], [A'.3] and WNS], WNMA holds if and only if $\bigcap_{i=1}^{m} S_i^W \neq \phi$ (i.e., Hart's condition holds if and only if Werner's condition holds). Allouch [5] extend this result by proving, under [A.1]-[A.2] only, that WNMA holds if and only if $\bigcap_{i=1}^{m} S_i \neq \phi$.

Theorem 3.3 *Let ε be an economy satisfying [A.1]-[A.2]. The following statements are equivalent:*

1. *ε satisfies WNMA.*
2. *ε satisfies NAPS.*

Page and Wooders [45] state that if $L_i = \{0\}, \forall i$, then NUBA holds if and only if $\bigcap_{i=1}^{m} S_i^W \neq \phi$. In fact, this result is a consequence of a sharper result:

Corollary 3.1 *Let ε be an economy satisfying [A.1]-[A.2]. The following statements are equivalent:*

1. *ε satisfies NUBA.*
2. *$\bigcap_{i=1}^{m} S_i \neq \phi$. and the linearity spaces are linearly independent.*

Remark 3.2 *By the Corollary 3.1 there is an absence of arbitrage opportunities if and only if there exists a price system limiting arbitrage opportunities contained in the $L_i^{\perp}$ spaces and there are no arbitrage opportunities in the linearity spaces. Thus, when the linearity spaces are equal to zero, nonemptiness of the set of no-arbitrage prices (i.e., $\bigcap_{i=1}^{m} S_i \neq \phi$.) is necessary and sufficient to rule out arbitrage opportunities in the economy.*

3.4 Inconsequential arbitrage

Page et al. [48] extended the Hart [27] model to an abstract general equilibrium setting without uniformity conditions and introduce a condition limiting arbitrage, called inconsequential arbitrage(IC). Their condition is weaker that the weak no-market-arbitrage condition and implies compactness of the utility set U.

A set of trades $y = (y_1, \cdots, y_m) \in R^{lm}$ is an arbitrage in the economy ε if y is the limit of some sequence $\{\lambda^n x^n\}_n$ where $\lambda^n \downarrow 0$ and $\{x^n\}_n \subseteq A$ is a sequence of rational allocations. They denote the set os all arbitrages by

$$\mathrm{arb}(\varepsilon) = \{y \in R^{lm} \mid \exists \{x^n\}_n \subseteq A \text{ and } \lambda^n \downarrow 0 \text{ such that } y = \lim_{n\to+\infty} \lambda^n x^n\}$$

and they denote by

$$\mathrm{arbseq}(y) = \{\{x^n\}_n \subseteq A \mid \exists \lambda^n \downarrow 0 \text{ such that } y = \lim_{n\to+\infty} \lambda^n x^n\}$$

the set of all arbitrage sequences corresponding to $y \in \mathrm{arb}(\varepsilon)$.

$y \in \mathrm{arb}(\varepsilon)$ is in the back-up set, denoted by $\mathrm{bus}(\varepsilon)$, if for all $y \in \mathrm{arb}(\varepsilon)$ and $\{x^n\}_n \in \mathrm{arbseq}(y)$, there exists an $\epsilon > 0$ such that for all n sufficiently large

$$x_i^n - \epsilon y_i \in X_i \text{ and } u_i(x_i^n - \epsilon y_i) \geq u_i(x_i^n), \forall i.$$

Definition 3.6 *The economy ε satisfies the (IC) condition if*

$$arb(\varepsilon) \subset bus(\varepsilon).$$

In words, an arbitrage $y \in \mathrm{arb}(\varepsilon)$ is inconsequential (i.e. is contained in the backup set at endowments $\mathrm{bus}(\varepsilon)$)) if for sufficiently large allocations $x \in A$ in the $y = (y_1, \cdots, y_n)$ 'directions' from the endowment ω, each agent j can reduce his consumption by a small amount in the $-y_j$ direction without reducing his utility.

Theorem 3.4 *Let ε be an economy satisfying [A.1]-[A.2]. The following statements are true :*

1. *NUBA holds $\Rightarrow$ IC holds.*
2. *If, in addition, [A.3] holds, then WNMA holds $\Rightarrow$ IC holds.*
3. *IC holds $\Rightarrow$ U is compact.*

The above theorem shows that under weak uniform condition, the Hart/Werner conditions imply inconsequential arbitrage. In general, no unbounded arbitrage implies inconsequential arbitrage. While the condition of no unbounded arbitrage focuses on expanding utility nondecreasing or increasing trades, inconsequential arbitrage focuses on contracting net trades without decreasing utility. Meanwhile, inconsequential arbitrage inconsequential arbitrage directly implies compactness of the set of utility possibilities.

3.5 Strong unbounded arbitrage

Dana et al. [16] refined "no unbounded arbitrage" condition of Page [41] and provided a new concept of no-arbitrage, called "no strong unbounded arbitrage" (NSUBA).

Definition 3.7 *A "strong unbounded arbitrage" is an unbounded arbitrage y with the property (P): There exist sequences $\lambda_n \in R+$ and $y^n \in (R^l)^m$ such that:*

(i) $\lambda_n \to +\infty$ and $y^n \to y$;

(ii) $e + \lambda_n y^n \in A, \forall n$;

(iii) $\forall x \in A, \exists i$ such that $\underline{\lim} u_i(e_i + \lambda_n y_i^n) > u_i(x_i)$.

The following theorem shows that no strong unbounded arbitrage directly implies the compactness of U. This result seems to be the first which infers the compactness of U from a no-arbitrage condition.

Theorem 3.5 *Let ε be an economy satisfying [A.1]-[A.2]. If there is no strong unbounded arbitrage (NSUBA), then U is compact.*

3.6 Bounded arbitrage

Allouch [2] introduced a new condition, bounded arbitrage (The compactness with partial preorder (CPP) condition called in Allouch [4]), defined as follows:

The economy satisfies bounded arbitrage (CPP) if for all sequences of rational allocations $\{x^n\}_n \in A$ there exists

a subsequence $\{x^{n_k)\}_{k \in A}}$;

a rational allocation $z \in A$ and

a sequence $z^k \subset \prod_{i=1}^{n} X_i$ converging to z such that

$$z_j^k \in P_j^{'}(x_j^{n_k}).$$

Here, following Gale and Mas-Colell (1975), the augmented preference correspondence $x_j \to P_j^{'}(x_j)$ is given by

$$P_j^{'}(x_j) := \{\hat{x}_j \in X_j : \hat{x}_j = (1-\lambda)x_j + \lambda x_j^{'} \text{ for } 0 \ \leq \lambda \leq 1, x_j^{'} \in P_j(x_j)\}.$$

Thus, by bounded arbitrage, for every sequence of rational allocations there is a subsequence that is augmented preference-dominated by a sequence converging to a rational allocation. Note that bounded arbitrage implies global nonsatiation at rational allocations.

Theorem 3.6 *Let ε be an economy satisfying [A.1]-[A.2]. Then bounded arbitrage directly implies compactness of the set of utility possibilities. In addition, if ε satisfies local nonsatiation, then bounded arbitrage is equivalent to compactness of the set of utility possibilities.*

3.7 Limited arbitrage

Definition 3.8 *ε satisfies limited arbitrage if,*

$$(LA) \qquad \bigcap_{i=1}^{l} D_i \neq \phi$$

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. When global cones are independent of the initial endowments, this condition (LA) is satisfied simultaneously at every set of endowments. The concept of limited arbitrage can be interpreted in terms of gains from trade:

$$\text{gains from trade} = G(\varepsilon) = \sup(\sum_{i=1}^{m} u_i(x_i) - u_i(e_i)),$$

where $(x_1, \cdots, x_m) \in A$.

Condition(C) Let $z = (x_1, \cdots, x_m) \in A$. If a sequence $(x^n) \in A$ satisfies $\| x^n \| \to \infty$ and $x_i^n \in \hat{P}_i(x_i)$, then $\exists N$ s.t. $\sum_{i=1}^{m} e_i - x_i^n \notin \sum_{j \neq i} \hat{P}_j(x_j)$ for $n > N$.

It is useful to show the connection between limited arbitrage and the notion of 'no-arbitrage' used in finance. The concepts are generally different, but in certain cases they coincide. In financial markets, an arbitrage opportunity exists when gains can be made at no cost or, equivalently, by taking no risks. The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy ε where the traders' initial endowments are zero, $e_i = 0$, and the normalized gradient of a closed set of indifference vectors define a closed set. Here no-arbitrage at the initial endowments means that there are no trades which could increase the trader's utilities at zero cost: gains from trade must be zero. By contrast, limited arbitrage means that no trader can increase utility beyond a given bound at zero cost: gains from trade must be bounded.

When the traders' utilities are linear functions, the two concepts coincide.

Lemma 3.2 *The economy ε satisfies limited arbitrage if and only if it has bounded gains from trade which are attainable, i.e., $\exists x^* \in A$ such that:*

$$G(\varepsilon) = (\sum_{i=1}^{m} u_i(x_i^*) - u_i(e_i)) < \infty.$$

4 Sufficient conditions for existence of equilibrium

Under the assumption of local nonsatiation at rational allocations, Dana et al. [16] show that compactness of utility possibilities is sufficient for the existence of an equilibrium. In the above section, in economic models of exchange economies allowing short sales, these above no-arbitrage conditions guarantee compactness of the set of rational utility possibilities.

[A.5] $\forall i, e_i \in \text{int}X_i$ and $\forall x_i \in A_i, P_i(x_i)$ is relatively open in X_i.

Assumption [A.5] allows us to conclude that a quasi-equilibrium for the economy is in fact an equilibrium for the economy. The following theorem states compactness of rational utility possibilities is sufficient for the existence of a quasi-equilibrium due to Dana et al. [16].

Theorem 4.1 *(compactness of the utility set is sufficient for existence)*

Let ε be an economy satisfying assumption [A.1], [A,2], and [A,4]. If the set of rational utility possibilities U is compact, then ε has a quasi-equilibrium. Moreover, if [A.5] holds, then ε has an equilibrium.

Putting together Theorem 4.1 and Theorem 3.1, 3.2, 3.2, 3.4, 3.5 and 3.6, we can summarize the relationship between the no-arbitrage conditions we have discussed and existence of equilibrium as follows:

Theorem 4.2 *(No-arbitrage conditions implying existence)*

Let ε be an economy satisfying [A.1], [A.2] and [A.4]. The following statements are true :

1. If IC holds, then ε has a quasi-equilibrium.

2. If in addition the economy satisfies [A.3],(weak uniformity), then

(a) if WNMA holds, then ε has a quasi-equilibrium,

(b) if NAPS holds, then ε has a quasi-equilibrium.

3. If NUBA holds, then ε has a quasi-equilibrium.

4. If NSUBA holds, then ε has a quasi-equilibrium.

5. If bounded arbitrage holds, then ε has a quasi-equilibrium.

Remark 4.1 *part 2(b) of the above Theorem improves upon the existence result of Werner's in the following sense. Allouch et al. [3] show that an extended version of Werner's no-arbitrage price condition is sufficient for existence under weak uniformity [A.3]. Werner in his proof of existence requires the stronger condition of uniformity [A'.3]. However, Werner makes a different assumption concerning non-satiation. In particular, Werner assumes [WNS] rather than local nonsatiation as Allouch et al. [5] do. In Allouch et al. [3], they investigate the relationship between existence and nonsatiation using our extended version of Werner's no-arbitrage-price system condition. Part 2(a) improves upon the existence result of Hart. In particular, Allouch et al. [5] extend Hart's condition to an abstract general equilibrium model and show that Hart's condition is sufficient for existence under weak uniformity [A.3]. Like Werner, Hart in his proof of existence requires that the stronger condition of uniformity [A'.3] hold.*

5 Necessary and sufficient conditions for existence of equilibrium

In this last section, if the economy satisfies the additional condition of weak no-half-lines, then the conclusions of Theorem 4.2 can be greatly strengthened. In particular, under weak no-half-lines the conditions of Hart and Werner and inconsequential arbitrage, are equivalent, and all are equivalent to the compactness of the set of the set of rational utility possibilities and the existence of equilibrium.

[A.6] **[Weak No-half-lines]** $\forall x_i \in \hat{P}_i(x_i)$, if $y \in R^l$, satisfies $u_i(x_i + \lambda y) = U_i(x_i), \forall \lambda \geq 0$, then $y \in L_i$.

If the economy satisfies uniformity [A'.3] as well as weak no-half-lines, then any potential arbitrage (i.e., any net trade vector contained in any agent's arbitrage cone) is either a direction in which the agent's utility is eventually increasing or a direction in which the agent's utility is eventually increasing or a direction or a direction in which the agent's utility is constant.

An agent's utility is eventually increasing at x_i in direction y_i if given any $\lambda \geq 0$, there exists a $\lambda' > \lambda$ such that $u_i(x_i + \lambda' y_i) > u_i(x_i + \lambda y_i)$.

Lemma 5.1 *Let ε be an economy satisfying assumption [A.1], [A,2], [A'.3], [A,4], and [A.6]. Then any equilibrium price is a no-arbitrage price.*

Theorem 5.1 *Let ε be an economy satisfying assumption [A.1], [A,2], [A'.3], [A,4], [A.5], and [A.6]. Then the following statements are equivalent:*

1. *ε satisfies the no-arbitrage price system condition (Werner [55]).*
2. *ε satisfies the weak-no-arbitrage condition (Hart [27]).*
3. *ε satisfies inconsequential arbitrage (Page et al. [48]).*
4. *ε satisfies bounded arbitrage(CPP) (Allouch [2, 4]).*
5. *The set of rational utility possibilities , U, is compact.*
6. *ε has an equilibrium.*

If we strengthen the weak no-half-lines condition then no-unbounded-arbitrage (NUBA) and compactness of the set rational allocations can be added to our list of equivalence.

[A'.6] **[No-half-lines]** $\forall x_i \in \hat{P}_i(x_i)$, if $y \in R^l$, satisfies $u_i(x_i+\lambda y) = U_i(x_i), \forall \lambda \geq 0$, then $y = 0$.

Corollary 5.1 *Let ε be an economy satisfying assumption [A.1], [A,2], [A'.3], [A,4], [A.5], and [A'.6]. Then the following statements are equivalent:*

1. *ε satisfies the no-arbitrage price system condition (Werner [55]).*
2. *ε satisfies the weak-no-arbitrage condition (Hart [27]).*

3. ε satisfies the no-unbounded-arbitrage condition (Page [41]).

4. ε satisfies bounded arbitrage(CPP) (Allouch [2, 4]).

5. The set of rational allocations, A, is compact.

6. ε satisfies inconsequential arbitrage (Page et al. [48]).

7. The set of rational utility possibilities , U, is compact.

8. ε has an equilibrium.

[**A.7**] u_i is continuous.

[**A.8**] u_i is uniformly non-satiated, and satisfies one of the following two mutually exclusive conditions: (a) the normalized gradient to any closed set of indifferent vectors define a closed set or (b) no indifference surface contains half lines.

Theorem 5.2 *Let ε be an economy satisfying assumption [A.1], [A,2], [A,7] and [A,8]. Then the following statements are equivalent:*

1. The economy ε has limited arbitrage.

2. $U(\varepsilon)$ is compact.

3. $P(\varepsilon)$ is compact.

Corollary 5.2 *Let ε be an economy satisfying assumption [A.1], [A,2], [A'.3], [A,4], [A.5], [A'.6],[A,7] and [A,8]. Then the following statements are equivalent:*

1. ε satisfies the no-arbitrage price system condition (Werner [55]).

2. ε satisfies the weak-no-arbitrage condition (Hart [27]).

3. ε satisfies the no-unbounded-arbitrage condition (Page [41]).

4. The set of rational allocations, A, is compact.

5. ε satisfies inconsequential arbitrage (Page et al. [48]).

6. ε satisfies bounded arbitrage(CPP) (Allouch [2, 4]).

7. The set of rational utility possibilities , U, is compact.

8. ε has an equilibrium.

9. The economy ε has limited arbitrage.

10. $P(\varepsilon)$ is compact.

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