*AMO-Advanced Modeling and Optimization, Volume 5, Number 3, 2003*

# A STOCHASTIC EOQ POLICY IN A FAMILY OF COLD-DRINKS -FOR A RETAILER

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ABSTRACT. This paper presents a stochastic **EOQ** ( *economic order quantity* ) model both for discrete and continuous distribution of demands of multi-item products. A general characterization of the optimal inventory policy is developed analytically.

## 1. Introduction

A well-known stochastic extension of the classical **EOQ** ( *economic order quantity* ) model bases the re-order decision or the stock level ( see Hadley and Whitin[1], Wagner[2] ). Models of storage systems with stochastic supply and demand have been widely analysed in the models of Faddy[3], Harrison and Resnick[4], Miller[5], Moran[6], Pliska[7], Meyer, Rothkopf and Smith[8], Teisberg[9], Chao and Manne[10], Hogan[11] and Devarangan and Weiner[12].

In this paper, we present a general characterization of the optimal inventory policy and interpret it in economic terms. An optimal inventory policy is characterized by conditions: (a) demand rate are partly stochastic and partly deterministic of multi-items with different inventory costs and shortage costs, (b) supply rate is instanteneously infinite and order is placed in the begining of the cycle.

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## 2. Fundamental Assumptions and Notations

1. Model is developed on multi-items products.
2. Lead time is negligible.
3. Demand is uniformly over the period and a function of temperature that follows a probability distributions.
4. production rate is instanteneously infinite.
5. Reorder-time is fixed and known. Thus the set-up cost is not included in the total cost.

Let the holding cost per $i$-th item per unit time be $Ch_i$, the shortage cost per $i$-th item per unit time be $Cs_i$ at any time $t$ , the inventory level be $Q_i$ of $i$-th item, $r_i$ is the demand over the pariod, $P_i$ is the selling price per unit of $i$-th item, $T$ is the cycle length.

## 3. The Model

In this model, we consider $n$ - numbered cold drinks those demands are $r_i$ ( $i = 1,\ 2,\ ..........n$) that depends upon temperature and selling price of $i$-th item. Temperature follows probability distribution over period. Here ,

$$r_i \quad = \quad a_i\tau + \frac{C_i \sum_{j=1,j\neq i}^{n} P_j}{(n-1)P_i}$$

where,

$a_i = \frac{\partial r_i}{\partial \tau}(\geq 0)$ = marginal response of $i$-th cold-drink consumption to a change in $\tau$(temperature) [ $\frac{\sum_{j=1,j\neq i}^{n} P_j}{(n-1)P_i}$ is constant ]

$C_i = \frac{\partial r_i}{\partial(\frac{\sum_{j=1,j\neq i}^{n} P_j}{(n-1)P_i})}(\geq 0)$ =marginal response of $i$-th cold-drink consumption to a change in $\frac{\sum_{j=1,i\neq i}^{n} P_j}{(n-1)P_i}$ ( the ratio of the average selling price of ($j = 1, 2, ......i-1,\ i+1, ...n$) items to the selling price of $i$-th item) [$\tau$ is constant] that depends upon the choice of the consumers. Now, the governing equations are as follows :

Case 1: When Shortage does not occur

$$\frac{dQ_i}{dt} = -\frac{r_i}{T} \quad , \quad 0 \le t \le T \tag{1}$$

(2)

with $Q_i(0) = Q_{i0}$ , for $i = 1, 2, ......n$.

From equ.(1) , we have

$$Q_i(t) = Q_{i0} - \frac{r_i}{T}t \quad , \quad 0 \le t \le T$$

Here $Q_i(T) \ge 0 \Rightarrow Q_{i0} - \frac{r_i}{T}T \ge 0 \Rightarrow Q_{i0} \ge r_i \quad , \quad i = 1, 2, .....n$. Therefore, the inventory of $i$-th item is

$$\int_0^T (Q_{i0} - \frac{r_i}{T}t)dt = (Q_{i0} - \frac{r_i}{2})T,$$

for $r_i \le Q_{i0}$ where $i = 1, 2, .....n$.

When Shortage occurs :

$$\frac{dQ_i}{dt} = -\frac{r_i}{T} \quad , \quad 0 \le t \le t_1 \tag{3}$$

(4)

with $Q_i(0) = Q_{i0}$ , and $Q_i(t_1) = 0$ , for $i = 1, 2, ......n$.

and

$$\frac{dQ_i}{dt} = -\frac{r_i}{T} \quad , \quad t_1 \le t \le T \tag{5}$$

(6)

with $Q_i(T) < 0$ , for $i = 1, 2, ......n$.

From equation (2), we have

$$Q_i(t) = Q_{i0} - \frac{r_i}{T}t \quad , \quad 0 \le t \le t_1$$

Now $Q_i(t_1) = 0 \Rightarrow t_1 = \frac{Q_{i0}T}{r_i}$. The equation(3) gives us

$$Q_i(t) = -\frac{r_i}{T}(t - t_i) \quad , \quad t_1 \le t \le T.$$

So $Q_i(T) < 0 \Rightarrow -\frac{r_i}{T}(T-t_1) < 0 \Rightarrow T > t_1 \Rightarrow T > \frac{Q_{i0}T}{r_i} \Rightarrow Q_{i0} < r_i$ . Therefore, the inventory during $(0, t_1)$ is

$$\int_0^{t_1} (Q_{i0} - \frac{r_i}{T}t)\ dt = Q_{i0}t_1 - \frac{r_i}{2T}t_1^2$$
$$= \frac{1}{2}\frac{Q_{i0}^2}{r_i}T\ ,\ r_i > Q_{i0},\quad for\ \ i = 1, 2, ...., n$$

The shortage during $(t_1\ ,\ T)$ is

$$\int_{t_1}^{T} -Q_i(t)\ dt = \frac{r_i}{2T}(T-t_1)^2$$
$$= \frac{1}{2}r_iT(1-\frac{Q_{i0}}{r_i})^2,\ r_i < Q_{i0},\quad for\ \ i = 1, 2, ...., n$$

Since , $Q_{i0} \geq r_i$
$\Rightarrow Q_{i0} \geq a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}$
$\Rightarrow \tau \leq \frac{1}{a_i}(Q_{i0} - \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}) = \tau^*$ ( say). i.e., $Q_{i0} = a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}$.
Also, $Q_{i0} < r_i \Rightarrow \tau > \tau^*$ and $Q_{i0} \geq r_i \Rightarrow \tau \leq \tau^*$
Case 1 : Uniform demand and discrete units.
$\tau$ is random variable with probability $p(\tau)$ such that $\sum_{\tau=\tau_0}^{\infty} p(\tau) = 1$ and $p(\tau) \geq 0$.
Therefore the expected average cost is
$Eac(\tau^*) = \frac{1}{T}\sum_{i=1}^n\{Ch_i\sum_{\tau=\tau_0}^{\tau^*}(Q_{i0} - \frac{r_i}{2})Tp(\tau) + \frac{1}{2}Cs_i\sum_{\tau=\tau^*+1}^{\infty}\frac{Q_{i0}}{r_i}p(\tau)T$
$+\frac{1}{2}Cs_i\sum_{\tau=\tau^*+1}^{\infty} r_iT(1-\frac{Q_{i0}}{r_i})^2p(\tau)\}$

$= \sum_{i=1}^n Ch_i\{\sum_{\tau=\tau_0}^{\tau^*}(a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i} - \frac{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}}{2})p(\tau)\}$
$+\frac{1}{2}\sum_{i=1}^n Ch_i\{\sum_{\tau=\tau^*+1}^{\infty}(a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i})^2\frac{p(\tau)}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}}\}$

$+\frac{1}{2}\sum_{i=1}^n Cs_i\{\sum_{\tau=\tau^*+1}^{\infty}(a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i})(1 - \frac{a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}})^2p(\tau)\}$

Now,

$Eac(\tau^*+1) = Eac(\tau^*) + \sum_{i=1}^n(Ch_i + Cs_i)a_i(\sum_{\tau=\tau_0}^{\tau^*} p(\tau))$
$+\sum_{i=1}^n\sum_{\tau=\tau^*+1}^{\infty}\{(Ch_i + Cs_i)a_i(a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i})\frac{p(\tau)}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^n p_j}{p_i}}$

$+\frac{1}{2}(Ch_i+Cs_i)a_i^2\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}\}-\sum_{i=1}^{n}Cs_ia_i$

In order to find the optimum value of $Q_{i0}^*$ i.e., $\tau^*$ so as to minimize $Eac(\tau^*)$, the following conditions must hold: $Eac(\tau^*+1)\ >Eac(\tau^*)$ and $Eac(\tau^*-1)\ >Eac(\tau^*)$ i.e., $Eac(\tau^*+1)-Eac(\tau^*)\ >\ 0$ and $Eac(\tau^*-1)-Eac(\tau^*)\ >\ 0$. Now, $Eac(\tau^*+1)-Eac(\tau^*)\ >\ 0$ implies

$$\sum_{\tau=\tau_0}^{\tau^*}p(\tau)+\sum_{i=1}^{n}\sum_{\tau=\tau^*+1}^{\infty}\{(Ch_i+Cs_i)a_i(a_i\tau^*+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i})\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}$$
$$+\frac{a_i^2}{2}(Ch_i+Cs_i)\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}\}\frac{1}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}>\frac{\sum_{i=1}^{n}Cs_ia_i}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}$$

Similarly $Eac(\tau^*-1)-Eac(\tau^*)\ >\ 0$ implies

$$\sum_{\tau=\tau_0}^{\tau^*-1}p(\tau)+\sum_{i=1}^{n}\sum_{\tau=\tau^*}^{\infty}\{(Ch_i+Cs_i)a_i(a_i\tau^*-a_i+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i})\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}$$
$$+\frac{a_i^2}{2}(Ch_i+Cs_i)\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}\}\frac{1}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}<\frac{\sum_{i=1}^{n}Cs_ia_i}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}$$

Therefore for minimum value of $Eac(\tau^*)$ , the following condition must be satisfied:

$$F(\tau^*-1)<\frac{\sum_{i=1}^{n}Cs_ia_i}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}<F(\tau^*) \quad (7)$$

Where,

$$F(\tau^*)=\sum_{\tau=\tau_0}^{\tau^*}p(\tau)+\sum_{i=1}^{n}\sum_{\tau=\tau^*+1}^{\infty}\{(Ch_i+Cs_i)a_i(a_i\tau^*+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i})$$
$$\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}+\frac{a_i^2}{2}(Ch_i+Cs_i)\frac{p(\tau)}{a_i\tau+\frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n}p_j}{p_i}}\}\frac{1}{\sum_{i=1}^{n}(Ch_i+Cs_i)a_i}$$

Case 2: Uniform demand and continuous units.
When uncertain demand is estimated as a continuous random variable , the cost equation of the inventory involves integrals instead of summation signs. The discrete point probabilities $p(\tau)$ are replaced by the probability differential $f(\tau)$ for small interval. In this case $\int_0^\infty f(\tau)\ \ d\tau=1$ and $f(\tau)\geq\ 0$.

Proceeding exactly in the same manner as in *Case 1*, The total expected average cost during period $(0\ ,\ T)$ is

$$\begin{aligned} Eac(\tau^*) &= \frac{1}{2}\sum_{i=1}^{n} Ch_i[\int_{\tau=\tau_0}^{\tau^*}(2a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i} - a_i\tau)f(\tau)\ d\tau \\ &+ (a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i})^2 \int_{\tau=\tau^*}^{\infty}\frac{f(\tau)\ d\tau}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}}] \\ &+ \frac{1}{2}\sum_{i=1}^{n} Cs_i \int_{\tau=\tau^*}^{\infty}\frac{(a_i\tau - a_i\tau^*)^2}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}} f(\tau)d\tau \end{aligned} \tag{8}$$

Now ,

$$\begin{aligned} \frac{dEac(\tau^*)}{d\tau^*} &= \sum_{i=1}^{n} Ch_i a_i \int_{\tau=\tau_0}^{\tau^*} f(\tau)d\tau \\ &+ \sum_{i=1}^{n} Ch_i a_i (a_i\tau^* + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}) \int_{\tau^*}^{\infty}\frac{f(\tau)d\tau}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}} \\ &- \sum_{i=1}^{n} Cs_i a_i^2 \int_{\tau^*}^{\infty}(\tau - \tau^*)\frac{f(\tau)d\tau}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}} \end{aligned}$$

and

$$\frac{d^2Eac(\tau^*)}{d\tau^{*2}} = \sum_{i=1}^{n}(Ch_i + Cs_i)a_i^2 \int_{\tau^*}^{\infty}\frac{f(\tau)\ d\tau}{a_i\tau + \frac{C_i}{n-1}\frac{\sum_{j=1,i\neq j}^{n} p_j}{p_i}} > 0$$

For minimum value of $Eac(\tau^*)$, $\frac{dEac(\tau^*)}{d\tau^*} = 0$ and $\frac{d^2Eac(\tau^*)}{d\tau^{*2}} > 0$ must be satisfied.

## 4.Conclusion

From physical phenomenon, it is true that the demand of cold drinks depends upon the increase of temperature. As , in the market, there is various types of cold drinks and their selling price is different, so their consumption depends upon their selling price. That is why we consider the consumption of $i$-th cold drink is a function of temperature and selling price. Generally the procurement cost of the cold drinks is smaller than their selling price. Consequently, supply of cold drinks to a retailer is sufficiently large. In reality, the discrete case is more realistic than the continuous one. But we discuss both the cases.

## References


1. G. Hadly and T. Whitin, Analysis of Inventory System, Prentice-Hall, Englewood Cliffs, **NJ**, 1963.

2. H. M. Wagner, Statistical Management of Inventory Systems, John Wiley and Sons, 1962.

3. M. J. Faddy , Optimal control of finite dams, Adv. Appl. Prob. **6**(1974) 689-710.

4. J. M. Harrison and S. I. Resnick, The stationary distribution and first exit probabilities of a storage process with general release rules, Math. Opns. Res. **1** (1976) 347-358.

5. R. G.(Jr.) Miller, Continuous time stochastic storage processes with random linear inputs and outputs, J. Math. Mech., **12** (1963) 275-291.

6. P. A. Moran, The theory of storage, Metuen, London, 1959

7. S. R. Pliska, A diffusion process model for the optimal operations of a reservior system, J. Appl. Prob. **12** (1975) 859-863.

8. R. R. Meyer, M. H. Rothkopf and S. A. Smith, Reliability and inventory in a production-storage system, Mgmt. sci., **9** (1963) 259-267.

9. T. J. Teisberg, A dynamic programming model of the U. S. stochastic petroleum reserve, Bell. J. Econ. **12**(1981) 526-546.

10. H. Chao and A. S. Manne, It oil stock-piles and import reductions: A dynamic programming approach, Opns. Res. **31**(1983) 632-651.

11. W. W. Hogan, Oil stockpiling : help thy neighbor , Energy. J., **4**, (1983) 49-71.

12. S. Devarangan and R. Weiner stockpile Behavior as an International Game, Harvard University, 1983.