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New Interior Point Algorithms in Linear Programming[†]

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Abstract

In this paper the abstract of the thesis "New Interior Point Algorithms in Linear Programming" is presented. The purpose of the thesis is to elaborate new interior point algorithms for solving linear optimization problems. The theoretical complexity of the new algorithms are calculated. We also prove that these algorithms are polynomial. The thesis is composed of seven chapters. In the first chapter a short history of interior point methods is discussed. In the following three chapters some variants of the affine scaling, the projective and the path-following algorithms are presented. In the last three chapters new path-following interior point algorithms are defined. In the fifth chapter a new method for constructing search directions for interior point algorithms is introduced, and a new primal-dual path-following algorithm is defined. Polynomial complexity of this algorithm is proved. We mention that this complexity is identical with the best known complexity in the present. In the sixth chapter, using a similar approach with the one defined in the previous chapter, a new class of search directions for the self-dual problem is introduced. A new primal-dual algorithm is defined for solving the self-dual linear optimization problem, and polynomial complexity is proved. In the last chapter the method proposed in the fifth chapter is generalized for target-following methods. A conceptual target-following algorithm is defined, and this algorithm is particularized in order to obtain a new primal-dual weighted-path-following method. The complexity of this algorithm is computed.

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[†]This is the abstract of the author's doctoral thesis, with the same title. The supervisor is **Prof. dr. Iosif Kolumbán.**

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Introduction

In this thesis we discuss interior point methods (IPMs) for solving linear optimization (LO) problems. Linear optimization is an area of mathematical programming dealing with the minimization or maximization of a linear function, subject to linear constrains. These constrains can be expressed by equalities or inequalities. There are many applications of linear optimization. For the important applications of LO in economics Kantorovich [75] and Koopmans [82] received the Nobel Price in Economics in 1976. Dantzig proposed in 1947 the well-known simplex method for solving LO problems. The simplex algorithm has been continuously improved in the past fifty years, and one has been convinced of the practical efficiency of the algorithm. Although the simplex algorithm is efficient in practice, no one could prove polynomial complexity of the algorithm. This property of polynomial complexity is important from the theoretical point of view. For different variants of the simplex algorithm were constructed examples illustrating that in the worst case the number of iterations required by the algorithm can be exponential. Khachiyan developed in 1979 the first polynomial algorithm for solving the LO problem. The ellisoid method of Khachiyan is an important theoretical result, but the practical implementation was not a competitive alternative of the simplex method. Karmarkar proposed his polynomial algorithm in 1984. Karmarkar's algorithm uses interior points of the polytope to approximate the optimal solution. The complexity of this algorithm is smaller than Khachiyan's and the implementation of Karmarkar's algorithm proved to be efficient in practice too, especially when the size of the problem is large. As a consequence the research in the area of LO became very active, and the field of IPMs remained an important research topic in the present too.

The purpose of the thesis is to elaborate new interior point algorithms for solving LO problems, to calculate the theoretical complexity of the new algorithms and to prove that these algorithms are polynomial.

The thesis is composed of seven chapters. In the first chapter a short history of IPMs is presented. Although we can not separate these methods into classes, because of the strong connection between different methods, we can delimit three main directions: affine scaling methods, projective methods with a potential function and path-following methods. This chapter contains many references to articles and books written in the area of IPMs. There is a very extensive bibliography in this theme, therefore only a part of the available articles were cited. In spite of this fact a special effort was made to include the most part of the important articles dealing with interior point algorithms for solving LO problems.

In the following three chapters we present the affine scaling, the projective and the path-following methods. In Chapter 2 we consider two variants of the primal affine scaling algorithm and a dual affine scaling algorithm. The primal algorithm is generalized for the case when the objective function is continuously differentiable. In this chapter we study also two methods of finding the starting interior point.

In Chapter 3 we consider the LO problem in Karmarkar's form. We prove that using a projective transformation the LO problem can be transformed to this form. The potential function is defined and two variants of Karmarkar's algorithm are discussed. Using the potential function we prove the polynomiality of Karmarkar's algorithm.

In Chapter 4 we deal with path-following methods. The central path, and the optimal partition is defined and Newton's method is presented. In the final part of this chapter a path-following primal-dual algorithm is studied.

In Chapter 5, 6 and 7 new path-following interior point algorithms are defined. In Chapter 5 we present a new method for constructing search directions for interior point algorithms. Using these results we define a new primal-dual path-following algorithm. We prove that this algorithm is polynomial, and it's complexity is the same as the best known complexity.

In Chapter 6 we consider the self-dual embedding technique. Using a similar method with the one defined in Chapter 5 we introduce a new class of search directions for the self-dual problem. A new primal-dual algorithm is defined for solving the self-dual LO problem, and the polynomiality of this algorithm is proved. This method provides an elegant technique for finding the starting interior point of the algorithm.

In Chapter 7 we generalize the method proposed in Chapter 5 for target-following methods. We define a conceptual target-following algorithm, and we particularize this algorithm in order to obtain a new primal-dual weighted-path-following method. The complexity of this algorithm is computed.

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1 Preliminaries

In the first chapter a history of interior point methods is presented. We discuss the relation between the simplex method and IPMs [25, 79, 53, 20, 92, 74, 49, 3, 111, 78, 67], the ellipsoid method [77, 108, 130, 131], Karmarkar's method, and the impact of his algorithm on the area of optimization [76, 83, 84, 102, 38, 39, 40, 44, 66, 42].
IPMs are classified as affine scaling algorithms [38, 39, 40, 1, 21, 119, 17, 116, 26], projective algorithms [76, 11, 13, 14, 15, 45, 46, 43, 47, 48, 50, 51, 52, 58, 114, 124, 125, 126, 127, 12, 16, 54, 62, 61, 64, 113] and path-following algorithms [109, 99, 57, 105, 91, 81, 35, 36, 71, 88, 89, 59, 60, 104, 90, 65, 70]. The case when there is no strictly feasible starting point led to *infeasible start IPMs* [97, 22, 120, 121, 122, 123]. An alternative technique is the *self-dual embedding method* [129, 90, 118, 69]. The results in the area of IPMs for solving LO problems have been published in recent books on the subject [7, 8, 103, 123, 128, 118, 17, 27, 29]. We also deal with the following topics: convex optimization and *semidefinite programming* [93, 72, 63, 9, 10, 23, 98, 5, 6], multiobjective optimization [18, 19, 115, 2, 30, 33], implementation of IPMs [87, 85, 86, 4, 24, 56, 28, 31, 32, 34] and subspace methods [110, 73].

2 Affine-Scaling Algorithms

2.1 Geometric Approach

We consider the LO problem in the following standard form:

$$\begin{aligned} &min\ c^T x, \\ &Ax = b, \\ &x \geq 0, \end{aligned} \tag{P}$$

where $A \in \Re^{m\times n}$, $rank(A) = m$, $b \in \Re^m$ and $c \in \Re^n$. The dual of this problem can be written in the following form:

$$\begin{aligned} &max\ b^T y, \\ &A^T y + s = c, \\ &s \geq 0. \end{aligned} \tag{D}$$

In this section we point out that the solving procedure of the LO problem with an interior point algorithm can be split up in three different subproblems: finding a starting interior point, generating the next iterate, and determining the stopping procedure. We discuss geometric aspects of the second subproblem.

2.2 Affine-Scaling Primal Algorithm

We consider two variants of the affine-scaling primal algorithm for solving LO problems. We obtain also a generalized form of the primal algorithm. In this case the objective function can be any continuously differentiable function. We discuss the technique of scaling, the step size, the question of how to start the algorithm, and the stopping criterion. We also deal with the minimization of a linear objective function on the intersection of an affine space with an ellipsoid.

2.3 Affine-Scaling Dual Algorithm

In this section the affine-scaling dual algorithm is studied. The dual algorithm is in fact the affine-scaling algorithm applied for the dual problem. We deduce this algorithm in a similar way as the primal algorithm. We discuss the same topics: scaling, step size, stopping criteria. At each step of the algorithm an estimate of the primal problem is computed.

3 Projective Algorithms with Potential Function

3.1 Karmarkar's Form

Karmarkar's paper [76] had an important effect on research in the area of optimization. His method is the first polynomial projective method for solving the LO problem. The algorithm has many variants. A common feature of these algorithms is that the LO problem is considered in the following special form:

$$\begin{aligned} min\ c^T x&, \\ Ax = 0&, \\ e^T x = n&, \\ x \geq 0&, \end{aligned} \tag{K}$$

where $A \in \Re^{m \times n}$ and $e = [1, \ldots, 1]^T$ is the n-dimensional all-one vector. Let us consider the LO problem in standard form. We prove that if the set of optimal solutions of the primal problem is not empty, and this set is bounded, then the primal problem can be transformed in the equivalent form (K). We present two different methods of constructing the problem (K).

3.2 Optimal Value

We prove that if the set of optimal values of both the primal, and the dual problems is not empty, then the primal-dual pair is equivalent to the following problem:

$$\begin{aligned} min\ x_1&, \\ A_1x = b^1&, \\ x \geq 0&, \end{aligned} \tag{P_1}$$

and the optimal value is zero. Moreover, in this case there is a strictly feasible starting solution, and we observe that the objective function is reduced to the first component of x.

3.3 Projective Transformation

We apply a projective transformation to problem (P_1). We prove that if the set of optimal solutions of both the primal and the dual problems is non-empty and bounded, then the primal-dual pair is echivalent to

$$\begin{aligned} min\ x_1&, \\ Ax = 0&, \\ e^Tx = n&, \\ x \geq 0&, \end{aligned} \tag{K_1}$$

and the optimal value of problem (K_1) is zero.

3.4 Potential Function

Consider the problem (K) and suppose that the optimal value is zero. The potential function is defined in two different situations: first in the case when x is feasible but not optimal solution, and secondly in the case when x is not feasible. Some properties of the potential function are discussed.

3.5 Variants of Karmarkar's Algorithm

We discuss two variants of Karmarkar's algorithm. The first one is obtained by applying the generalized form of the affine-scaling algorithm to the problem:

$$\begin{aligned} min\ \varphi(x)&, \\ Ax = 0&, \\ x \geq 0&, \end{aligned}$$

where φ is the potential function. This variant is based on the papers of Karmarkar [76], Todd-Burell [114] and Gonzaga [62]. To obtain the second variant we transform the original problem (K) using a function τ. The strictly feasible solution x^0 is transformed in the vector $e = [1, \ldots, 1]^T$, and the condition $e^T x = n$ is satisfied in the scaled space too. We apply a scaled variant of Dikin's algorithm to this problem, with the following slight modification: returning to the original space will be done by using the inverse function τ^{-1}. Thus we obtain the second variant of Karmarkar's algorithm. This method was studied by Karmarkar [76], Roos [100], Terlaky [112] and Schrijver [107]. In the next section we shall prove that this algorithm is polynomial.

3.6 Polynomiality of Karmarkar's Algorithm

In this section two technical lemmas are presented. These are due to Schrijver [107]. We use these lemmas to prove that Karmarkar's algorithm solves the LO problem in polynomial time. We obtain the following final result. Let $\rho = \frac{1}{2}$ and $\sigma = 1 - \ln 2 > 0$. Moreover, let $\varepsilon > 0$. If the optimal value of problem (K) is zero, and we apply the second variant of Karmarkar's algorithm using the initial point $x^0 = e$, then after no more than

$$k \geq \frac{n}{\sigma} \ln \frac{c^T e}{\varepsilon},$$

iterations the algorithm stops, and the value of the objective function is not greater than ε.

4 Path-Following Algorithms

4.1 Introduction

Consider the standard primal-dual pair. Let

$$\mathcal{P} = \{x \in \Re^n \mid Ax = b,\ x \geq 0\},$$

$$\mathcal{D} = \{(y, s) \in \Re^m \times \Re^n \mid A^T y + s = c,\ s \geq 0\},$$

be the set of strictly feasible solution of the primal, and the dual problem respectively. Suppose that both problems have at least one feasible solution, which is also interior point. Thus

$$\exists\, x > 0, \qquad x \in \mathcal{P},$$

$$\exists\ (y,s) \in \mathcal{D}, \qquad s > 0.$$

This condition is called the *interior point condition* (IPC). We mention that instead of $(y,s) \in \mathcal{D}$ we shall often write simply $s \in \mathcal{D}$. We have the following lemma.

Lemma 4.1 *Let $\tilde{x} \in \mathcal{P}$ and $\tilde{s} \in \mathcal{D}$. Then we have $x^T s = \tilde{s}^T x + \tilde{x}^T s - \tilde{x}^T \tilde{s}$ for each $x \in \mathcal{P}$ and $s \in \mathcal{D}$.* ■

From this lemma we obtain the following consequence.

Consequence 4.2. *For every $K > 0$, the set $\{(x,s) \in \mathcal{P} \times \mathcal{D} \mid x^T s \leq K\}$ is bounded.* ■

Let us consider the function:

$$\psi : R^n_{++} \times R^n_{++} \to R^n_{++}, \qquad \psi(x,s) = xs,$$

where $R^n_{++} = \{x \in \Re^n \mid x > 0\}$ and $xs = [x_1 s_1, \ldots, x_n s_n]^T$. It is well-known the following theorem. An elegant proof was done by Roos and Vial [106].

Theorem 4.3 *For every $w \in \Re^n_{++}$ there is exactly one pair $(x,s) \in \mathcal{P} \times \mathcal{D}$, $x > 0$, $s > 0$ such that $\psi(x,s) = w$.* ■

4.2 Central Path

The *central path* is discussed in this section. We point out that if the IPC holds, then the primal-dual central path is formed by the unique solutions of the following system:

$$\begin{aligned} Ax &= b, & x &\geq 0, \\ A^T y + s &= c, & s &\geq 0, \\ xs &= \mu e, \end{aligned}$$

where xs is the coordinatewise product of the vectors x and s, the n-dimensional all-one vector is denoted by e, and $\mu > 0$. Let $(x(\mu), s(\mu))$ be the solution of the above system. Then we have the following lemma.

Lemma 4.4 *The following assertions hold.*
a) We have $x(\mu)^T s(\mu) = n\mu$.
b) The set formed by the points $(x(\mu), s(\mu))$ has at least one accumulation point for $\mu \to 0$ and this point is optimal solution of the pair (P)-(D). ■

In the following section we use this lemma to prove the Goldman-Tucker [55] theorem.

4.3 Optimal Partition

Let $\mathcal{P}^*$ and $\mathcal{D}^*$ be the set of optimal solutions of the primal and the dual problem respectively. We introduce the notations:

$$B = \{i \mid \exists\, x \in \mathcal{P}^*, \quad x_i > 0, \quad 1 \leq i \leq n\},$$

$$N = \{i \mid \exists\, s \in \mathcal{D}^*, \quad s_i > 0, \quad 1 \leq i \leq n\}.$$

We have the following theorem.

Theorem 4.5 (Goldman, Tucker) *There exists a pair of optimal solutions (x^*, s^*) of the primal and dual problems, such that $x^* + s^* > 0$.* ■

From this theorem results that the sets B and N form a partition of the index set.

4.4 Newton's Method

Let $f : \Re^n \to \Re^n$ be a continuously differentiable function, and let $J(x)$ be the Jacobi matrix attached to f. Consider the system:

$$f(x) = 0.$$

Suppose we are given the vector x^0. Then we obtain a sequence of points using the formula:

$$x^{k+1} = x^k - J(x^k)^{-1} f(x^k).$$

If we introduce a step direction vector Δx^k, thus

$$x^{k+1} = x^k + \Delta x^k,$$

and we have

$$J(x^k)\Delta x^k = -f(x^k).$$

If x^0 is sufficiently close to a solution of f, then this sequence is convergent. The analysis of Newton's method is very important from the point of view of IPMs. We shall use these results later in the thesis to develop new IPMs.

4.5 Primal-Dual Path-Following Algorithm

Consider the LO problem in standard form, and suppose that the IPC holds for a starting strictly feasible pair. In this section we develop the standard primal-dual path-following algorithm. We apply Newton's method to the system which defines the central path. Thus, we obtain step direction vectors, by solving a system of linear equations. To guard against hitting the boundary, by violating the nonnegativity constraints, we determine the maximum allowable step size. We perform a step by taking a fraction of this step size. Hence we obtain a new interior point. We repeat the above procedure till a stopping condition will be satisfied.

5 A New Class of Search Directions

5.1 Introduction

In this chapter we introduce a new method for finding search directions for interior point methods in linear optimization. For some particular cases we obtain the directions defined recently by Peng, Roos and Terlaky. We develop a new short-update primal-dual interior point algorithm based on one particular member of the new family of search directions. We prove that this algorithm has also the best known iteration bound for interior point methods.

Let us consider the LO problem in standard form, and suppose that the IPC is satisfied. It is well known that using the self-dual embedding technique we can always construct a LO problem in such a way that the IPC holds. Thus, IPC can be assumed without loss of generality. Furthermore, the self-dual embedding model yields $x^0 = s^0 = e$, and hence $\mu^0 = \frac{(x^0)^T s^0}{n} = 1$.

Finding the optimal solution of the primal-dual pair is equivalent to solving the system:

$$\begin{aligned} Ax &= b, \qquad && x \geq 0, \\ A^T y + s &= c, && s \geq 0, \\ xs &= 0, && \end{aligned} \tag{1}$$

where xs is the coordinatewise product of the vectors x and s, i.e.

$$xs = [x_1 s_1, x_2 s_2, \dots, x_n s_n]^T.$$

We shall use also the notation

$$\frac{x}{s} = \left[\frac{x_1}{s_1}, \frac{x_2}{s_2}, \dots, \frac{x_n}{s_n}\right]^T,$$

for each vector x and s such that $s_i \neq 0$, for all $1 \leq i \leq n$. In fact for an arbitrary function f, and an arbitrary vector x we will use the notation

$$f(x) = [f(x_1), f(x_2), \dots, f(x_n)]^T.$$

The first and the second equations of system (1) are called the *feasibility condisions.* They serve for maintaining feasibility. The last equation is named the *complementarity condition.* Primal-dual IPMs generally replace the complementarity condition by a parameterized equation. Thus we obtain:

$$\begin{array}{rcll} Ax &=& b, & x \geq 0, \\ A^T y + s &=& c, & s \geq 0, \\ xs &=& \mu e, & \end{array} \tag{2}$$

where $\mu > 0$, and e is the n-dimensional all-one vector, i.e. $e = [1, 1, \ldots, 1]^T$. If the IPC holds, then for a fixed $\mu > 0$ the system (2) has a unique solution, called the *μ-center* (Sonnevend [109]). The set of μ-centers for $\mu > 0$ formes a well-behaved curve, the *central path.* Polynomial-time IPMs generally follow the central path approximately by using Newton's method to obtain search directions. In the following section we present a new method for constructing search directions for IPMs.

5.2 A New Class of Directions

In this section we define a new method for finding search directions for IPMs. Let $\Re^+ = \{x \in \Re \mid x \geq 0\}$, and let us consider the function

$$\varphi \in C^1, \qquad \varphi : \Re^+ \to \Re^+,$$

and suppose that the inverse function φ^{-1} exists. We observe that the system of equations which defines the central path (2) can be written in the following equivalent form:

$$\begin{array}{rcll} Ax &=& b, & x \geq 0, \\ A^T y + s &=& c, & s \geq 0, \\ \varphi(xs) &=& \varphi(\mu e). & \end{array} \tag{3}$$

Now we can use Newton's method for the system (3) to obtain a new class of directions. An alternative variant is the following. The system (2) is equivalent to

$$\begin{array}{rcll} Ax &=& b, & x \geq 0, \\ A^T y + s &=& c, & s \geq 0, \\ \varphi\left(\dfrac{xs}{\mu}\right) &=& \varphi(e), & \end{array} \tag{4}$$

and we can use Newton's method for the system (4). Thus we can define new search directions. In the remaining part of this section we deal with the system (4). The advantage of this variant is that we can introduce the vector

$$v = \sqrt{\frac{xs}{\mu}},$$

and we can use it for scaling the linear system obtained by applying Newton's method.

Now assume that we have $Ax = b$, and $A^T y + s = c$ for a triple (x, y, s) such that $x > 0$ and $s > 0$, i.e. x and (y, s) are strictly feasible. Applying Newton's method for the non-linear system (4) we get

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ \frac{s}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta s &= \varphi(e) - \varphi\left(\frac{xs}{\mu}\right). \end{aligned} \tag{5}$$

We introduce the notations

$$d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}.$$

We have

$$\mu v(d_x + d_s) = s\Delta x + x\Delta s, \tag{6}$$

and

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{7}$$

Consequently the linear system (5) can be written in the following form

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_s &= 0, \\ d_x + d_s &= p_v, \end{aligned} \tag{8}$$

where

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi\prime(v^2)},$$

and $\bar{A} = Adiag(\frac{x}{v})$, where for any arbitrary vector ξ, we denote by $diag(\xi)$ the diagonal matrix having the elements of the vector ξ on the diagonal, and in the same order.

We mention that $\varphi(t) = t$ yields $p_v = v^{-1} - v$, and we obtain the standard primal-dual algorithm. Recently Peng, Ross and Terlaky [96] observed, that a new search direction can be obtained by taking $p_v = v^{-3} - v$. The same authors analysed in [95] the case $p_v = v^{-q} - v$, where $q > 1$. They have introduced also a class of search directions based on self-regular proximities (Peng, Ross and Terlaky [94]). Our general approach can be particularized in such a way as to obtain, the directions defined in [95] and [96]. For $\varphi(t) = t^2$ we get $p_v = \frac{1}{2}(v^{-3} - v)$, and for $\varphi(t) = t^{\frac{q+1}{2}}$, where $q > 1$ we obtain $p_v = \frac{2}{q+1}(v^{-q} - v)$. We conclude that these search directions differ from those defined in [95] and [96] only by a constant multiplier. In the following section we use a different function to develop a new primal-dual algorithm.

5.3 A New Primal-Dual Algorithm

In this section we take $\varphi(t) = \sqrt{t}$, and we present a new primal-dual interior-point algorithm based on the appropriate search directions. We have

$$p_v = 2(e - v), \tag{9}$$

We define a proximity measure to the central path

$$\sigma(xs, \mu) = \frac{\|p_v\|}{2} = \|e - v\| = \left\| e - \sqrt{\frac{xs}{\mu}} \right\|,$$

where $\|\cdot\|$ is the Euclidean norm (l_2 norm). We introduce the notation

$$q_v = d_x - d_s.$$

Note that from (8) we have $d_x^T d_s = 0$, thus the vectors d_x and d_s are orthogonal, and this implies

$$\|p_v\| = \|q_v\|.$$

As a consequence we mention that the proximity measure can be expressed also with the vector q_v, thus

$$\sigma(xs, \mu) = \frac{\|q_v\|}{2}.$$

We have

$$d_x = \frac{p_v + q_v}{2} \quad \text{and} \quad d_s = \frac{p_v - q_v}{2},$$

hence

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{10}$$

Now we are ready to define the algorithm.

Algorithm 5.1 *Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter (default $\theta = \frac{1}{2\sqrt{n}}$), and $0 < \tau < 1$ the proximity parameter (default $\tau = \frac{1}{2}$). Suppose that for the triple (x^0, y^0, s^0) the interior point condition holds, and let $\mu^0 = \frac{(x^0)^T s^0}{n}$. Furthermore, suppose $\sigma(x^0 s^0, \mu^0) < \tau$.*

begin
  $x := x^0;\ y = y^0;\ s = s^0;$
  $\mu := \mu^0;$
  while $x^T s > \epsilon$ **do begin**
    $\mu := (1-\theta)\mu;$
    Substitute $\varphi(t) = \sqrt{t}$ *in (5) and compute* $(\Delta x, \Delta y, \Delta s)$
    $x := x + \Delta x;$
    $y := y + \Delta y;$
    $s := s + \Delta s;$
  end
end.

In the next section we shall prove that this algorithm is well defined, thus feasibility is maintained strictly and the condition $\sigma(xs, \mu) < \tau$ is satisfied throughout the algorithm. We shall obtain that this algorithm solves the linear optimization problem in polynomial time.

5.4 Convergence Analysis

In the following lemma we give a condition which guarantees the feasibility of the full Newton step. Let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$ be the vectors obtained after a full Newton step.

Lemma 5.1 *Let $\sigma = \sigma(xs, \mu) < 1$. Then*

$$x_+ > 0 \quad \text{and} \quad s_+ > 0,$$

thus the full Newton step is strictly feasible. ■

In the next lemma we analyse under which circumstances the Newton process is quadratically convergent.

Lemma 5.2 *Let $\sigma = \sigma(xs, \mu) < 1$. Then*

$$\sigma(x_+ s_+, \mu) \leq \frac{\sigma^2}{1 + \sqrt{1-\sigma^2}}.$$

Thus the full Newton step provides local quadratic convergence of the proximity measure. ■

In the following lemma we investigate the effect of the full Newton step on the duality gap.

Lemma 5.3 *Let $\sigma = \sigma(xs, \mu)$ and suppose that the vectors x_+ and s_+ are obtained after a full Newton step, thus $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. We have*

$$(x_+)^T s_+ = \mu(n - \sigma^2),$$

hence $(x_+)^T s_+ \leq \mu n$. ■

In the next lemma we analyse the effect on the proximity measure of a Newton step followed by an update of the parameter μ. Suppose that μ is reduced by the factor $(1 - \theta)$ in every iteration.

Lemma 5.4 *Let $\sigma = \sigma(xs, \mu) < 1$ and $\mu_+ = (1 - \theta)\mu$, where $0 < \theta < 1$. Then*

$$\sigma(x_+ s_+, \mu_+) \leq \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}.$$

Moreover, if $\sigma < \frac{1}{2}$, $\theta = \frac{1}{2\sqrt{n}}$ and $n \geq 4$ then we have $\sigma(x_+ s_+, \mu_+) < \frac{1}{2}$. ■

A consequence of Lemma 5.4 is that the algorithm is well defined. Indeed, the conditions $(x, s) > 0$ and $\sigma(xs, \mu) < \frac{1}{2}$ are maintained throughout the algorithm. In the next lemma we analyse the question of the bound on the number of iterations.

Lemma 5.5 *Suppose that the pair (x^0, s^0) is strictly feasible, $\mu^0 = \frac{(x^0)^T s^0}{n}$ and $\sigma(x^0 s^0, \mu^0) < \frac{1}{2}$. Let x^k and s^k be the vectors obtained after k iterations. Then for*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil,$$

we have $(x^k)^T s^k \leq \epsilon$. ■

We know that using the self-dual embedding we can assume without loss of generality that $x^0 = s^0 = e$, hence $\mu^0 = 1$. In this case we obtain the following lemma.

Lemma 5.6 *Suppose that $x^0 = s^0 = e$. Then Algorithm 5.1 performs at most*

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil,$$

interior point iterations. ■

Now using the default value for θ we obtain the following theorem.

Theorem 5.7 *Suppose that $x^0 = s^0 = e$. Using the default values for θ and τ Algorithm 5.1 requires no more than*

$$O\left(\sqrt{n}\log\frac{n}{\epsilon}\right),$$

interior point iterations. The resulting vectors satisfy $x^T s \leq \epsilon$. ■

5.5 Implementation of the Algorithm

We have implemented the new algorithm using object oriented techniques in the C++ programming language. We obtained that if a starting strictly feasible solution is available, then the new algorithm is generally more efficient than the standard primal-dual algorithm.

5.6 Conclusion

In this chapter we have developed a new class of search directions based on an equivalent form of the central path (2). The main idea was that we have introduced a function φ, and we have applied Newton's method for the system (4). We have shown that particularizing the function φ accordingly we obtain the directions defined in [95] and [96]. Using $\varphi(t) = \sqrt{t}$ we have defined a new primal-dual interior-point algorithm. We have proved, that this short-update algorithm has also the iteration bound $O(\sqrt{n}\log\frac{n}{\epsilon})$, the best known iteration bound for IPMs. We have implemented the new algorithm, and if a starting interior point was known, then generally we obtained better results than with the standard primal-dual algorithm.

6 A New Method for Solving Self-Dual Problems

6.1 Introduction

In the previous chapter, and in the paper [36] we have defined a new method for finding search directions for IPMs in LO. Using one particular member of the new family of search directions we have developed a new primal-dual interior point algorithm for LO. We have proved that this short-update algorithm has also the $O(\sqrt{n}\log\frac{n}{\epsilon})$ iteration bound, like the standard primal-dual interior point algorithm. In this chapter we describe a similar approach for self-dual LO problems. This method provides a starting interior feasible point for LO problems. We prove that the iteration bound is $O(\sqrt{n}\log\frac{n}{\epsilon})$ in this case too.

Let us consider the LO problem in canonical form

$$\begin{array}{rl} & min\ c^T\xi \\ s.t. & A\xi \geq b, \\ & \xi \geq 0, \end{array} \tag{CP}$$

where $A \in \Re^{m\times k}$ with $rank(A) = m$, $b \in \Re^m$ and $c \in \Re^k$. The dual of this problem is:

$$\begin{array}{rl} & max\ b^T\pi \\ s.t. & A^T\pi \leq c, \\ & \pi \geq 0. \end{array} \tag{CD}$$

It is well-known the following theorem.

Theorem 6.1 (strong duality) *Let $\xi \geq 0$ and $\pi \geq 0$ so that $A\xi \geq b$ and $A^T\pi \leq c$, in other words ξ is feasible for (CP) and π for (CD). Then ξ and π are optimal if and only if $c^T\xi = b^T\pi$.* ■

This theorem implies that if (CP) and (CD) have optimal solutions then

$$\begin{array}{rll} A\xi - z = b, & \xi \geq 0, & z \geq 0, \\ A^T\pi + w = c, & \pi \geq 0, & w \geq 0, \\ b^T\pi - c^T\xi = \rho, & \rho \geq 0 & \end{array} \tag{11}$$

has also a solution, where $z \in \Re^m$, $w \in \Re^k$ and $\rho \in \Re$ are slack variables. Furthermore, every solution of (11) provides optimal solutions of (CP) and (CD). Let us introduce the matrix $\bar{M}$ and the vectors $\bar{x}$ and $\bar{s}(\bar{x})$ as

$$\bar{M} = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \pi \\ \xi \\ \tau \end{bmatrix}, \quad \text{and} \quad \bar{s}(\bar{x}) = \begin{bmatrix} z \\ w \\ \rho \end{bmatrix},$$

where $\tau \in \Re$. Consider the following homogeneous system

$$\bar{s}(\bar{x}) = \bar{M}\bar{x}, \quad \bar{x} \geq 0, \quad \bar{s}(\bar{x}) \geq 0. \tag{12}$$

We mention that system (12) is the so-called *Goldman-Tucker model* [55, 117]. Let $\bar{n} = m+k+1$ and observe that the matrix $\bar{M} \in \Re^{\bar{n}\times\bar{n}}$ is skew-symmetric, i.e. $\bar{M}^T = -\bar{M}$. Now we can state the following theorem.

Theorem 6.2 *Consider the primal-dual pair (CP) and (CD). Then we have*

1. *If ξ and π are optimal solutions of (CP) and (CD) respectively, then for $\tau = 1$ and $\rho = 0$ we obtain that $\bar{x}$ is a solution of (12).*
2. *If $\bar{x}$ is a solution of (12), then we have $\tau = 0$ or $\rho = 0$, thus we cannot have $\tau\rho > 0$.*
3. *If $\bar{x}$ is a solution of (12) and $\tau > 0$, then $(\frac{\xi}{\tau}, \frac{\pi}{\tau})$ is an optimal solution of the primal-dual pair (CP)-(CD).*
4. *If $\bar{x}$ is a solution of (12) and $\rho > 0$, then at least one of the problems (CP) and (CD) are infeasible.*

■

In the next section we shall use the system (12) to accomplish the self-dual embedding of the primal-dual LO pair.

6.2 Self-Dual Embedding

In this section we investigate a generalized form of the system (12). Our approach follows the method proposed in [103]. Let us consider the LO problem

$$\begin{aligned} & \min \ \bar{q}^T\bar{x} \\ s.t. \quad & \bar{M}\bar{x} \geq -\bar{q}, \\ & \bar{x} \geq 0, \end{aligned} \qquad (\overline{SP})$$

where $\bar{M} \in \Re^{\bar{n}\times\bar{n}}$ is a skew-symmetric matrix, $\bar{q} \in \Re^{\bar{n}}$ and $\bar{q} \geq 0$. Moreover, let

$$\bar{s}(\bar{x}) = \bar{M}\bar{x} + \bar{q}.$$

We are going to solve $(\overline{SP})$ with an IPM, thus we need starting feasible solutions, so that $\bar{x} > 0$ and $\bar{s}(\bar{x}) > 0$. We say that in this case the problem $(\overline{SP})$ satisfies the *interior point condition* (IPC). Unfortunately such starting feasible solution for the problem $(\overline{SP})$ does not exist, but we can construct another problem equivalent to $(\overline{SP})$ which satisfies the IPC. For this purpose let

$$r = e - \bar{M}e \quad \text{and} \quad n = \bar{n} + 1,$$

where e denotes the all-one vector of length $\bar{n}$. Furthermore, introduce the notations

$$M = \begin{bmatrix} \bar{M} & r \\ -r^T & 0 \end{bmatrix}, \quad x = \begin{bmatrix} \bar{x} \\ \vartheta \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ n \end{bmatrix},$$

and consider the problem

$$\begin{aligned} &min\ q^T x \\ s.t.\quad &Mx \geq -q, \\ &x \geq 0. \end{aligned} \tag{SP}$$

Observe that the matrix M is also skew-symmetric, and problem (SP) satisfies the IPC. Indeed, we have

$$M \begin{bmatrix} e \\ 1 \end{bmatrix} + q = \begin{bmatrix} \bar{M} & r \\ -r^T & 0 \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} \bar{M}e + r \\ -r^T e + n \end{bmatrix} = \begin{bmatrix} e \\ 1 \end{bmatrix}.$$

We have used that the matrix $\bar{M}$ is skew-symmetric, thus $e^T \bar{M} e = 0$, and this equality yields

$$-r^T e + n = -(e - \bar{M}e)^T e + n = 1.$$

In order to solve the problem (SP) we use an IPM. Let

$$s = s(x) = Mx + q,$$

and consider the path of *analytic centers* [109], the primal-dual *central path*

$$\begin{aligned} Mx + q &= s, \\ xs &= \mu e, \end{aligned} \tag{13}$$

where $\mu > 0$, and xs is the coordinatewise product of the vectors x and s. It is well-known that if the IPC holds for the problem (SP), then the system (13) has a unique solution for each $\mu > 0$. IPMs generally follow the central path by using Newton's method. In the next section we are going to formulate an equivalent form of the central path, and we shall apply Newton's method to obtain new search directions.

6.3 A New Class of Directions

New search directions have been studied recently by Peng, Roos and Terlaky [96, 95, 94]. In a recent paper [36], and in the previous chapter we have proposed a different approach for defining a new class of directions for LO. In this section we propose a similar approach for the self-dual problem (SP). Thus, we introduce a new class of directions for the problem (SP). Let $\Re^+ = \{x \in \Re \mid x \geq 0\}$, and let us consider the function

$$\varphi \in C^1, \qquad \varphi : \Re^+ \to \Re^+,$$

and suppose that the inverse function φ^{-1} exists. Then the system of equations which defines the central path (13) is equivalent to

$$\begin{aligned} Mx + q &= s, \\ \varphi\left(\frac{xs}{\mu}\right) &= \varphi(e). \end{aligned} \tag{14}$$

Using Newton's method for the system (14) we obtain new search directions for the problem (SP). Denote

$$v = \sqrt{\frac{xs}{\mu}},$$

and assume that $(x, s) > 0$ and $Mx + q = s$, thus x is an interior feasible solution of the problem (SP). Applying Newton's method for the system (14) we get

$$M\Delta x \;=\; \Delta s, \tag{15a}$$

$$\frac{s}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta s \;=\; \varphi(e) - \varphi\left(\frac{xs}{\mu}\right) \tag{15b}$$

We introduce the notations

$$d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}.$$

We have

$$\mu v(d_x + d_s) = s\Delta x + x\Delta s, \tag{16}$$

and

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{17}$$

Consequently (15b) can be written in the following form

$$d_x + d_s = p_v, \tag{18}$$

where

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi\prime(v^2)}.$$

Now using that M is skew-symmetric we get

$$\Delta x^T \Delta s = \Delta x^T M \Delta x = -\Delta x^T M \Delta x,$$

hence $\Delta x^T \Delta s = 0$. Moreover, from (17) follows

$$d_x^T d_s = e^T(d_x d_s) = \frac{1}{\mu} e^T(\Delta x \Delta s) = \frac{1}{\mu} \Delta x^T \Delta s = 0,$$

thus d_x and d_s are orthogonal. We shall use this relation later in this chapter. We conclude that in this section we have defined a class of search directions for the problem (SP). For this purpose we have used a function φ to transform the system (13) in an equivalent form. In the next section we shall consider a particular member of this class of search directions. Thus we shall develop a new polynomial algorithm for the self-dual problem (SP).

6.4 The Algorithm

In the remaining part of this chapter we assume that $\varphi(x) = \sqrt{x}$. Using this function we present a new primal-dual interior-point algorithm for solving the problem (SP). Consequently, we obtain also a solution of (CP) and (CD). In this case applying Newton's method for the system (14) yields

$$\begin{aligned} M\Delta x &= \Delta s, \\ \sqrt{\frac{s}{\mu x}}\Delta x + \sqrt{\frac{x}{\mu s}}\Delta s &= 2\left(e - \sqrt{\frac{xs}{\mu}}\right). \end{aligned} \tag{19}$$

For $\varphi(x) = \sqrt{x}$ we have

$$p_v = 2(e - v), \tag{20}$$

and we can define a proximity measure to the central path by

$$\sigma(x, \mu) = \frac{\|p_v\|}{2} = \|e - v\| = \left\| e - \sqrt{\frac{xs}{\mu}} \right\|,$$

where $\| \cdot \|$ denotes the Euclidean norm (l_2 norm). Let us introduce the notation

$$q_v = d_x - d_s$$

Now using that the vectors d_x and d_s are orthogonal we obtain

$$\|p_v\| = \|q_v\|,$$

therefore the proximity measure can be written in the form

$$\sigma(x, \mu) = \frac{\|q_v\|}{2}.$$

Moreover, we have

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2} \quad \text{and} \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{21}$$

The algorithm can be defined as follows.

Algorithm 6.1 *Let* $\epsilon > 0$ *be the accuracy parameter and* $0 < \theta < 1$ *the update parameter (default* $\theta = \frac{1}{2\sqrt{n}}$*).*
begin
  $x := e$; $\mu := 1$;
  while $n\mu > \epsilon$ **do begin**
    $\mu := (1 - \theta)\mu$;
    Compute Δx *using (19);*
    $x := x + \Delta x$;
  end
end.

In the next section we shall prove that this algorithm solves the linear optimization problem in polynomial time.

6.5 Complexity analysis

In this section we are going to prove that Algorithm 6.1 solves the problem (SP) in polynomial time. In the first lemma we investigate under which conditions the feasibility of the full Newton step is assured. Let $x_+ = x + \Delta x$ and

$$s_+ = s(x_+) = M(x + \Delta x) + q = s + M\Delta x = s + \Delta s.$$

Using these notations we can state the lemma.

Lemma 6.3 *Let* $\sigma = \sigma(x, \mu) < 1$. *Then the full Newton step is strictly feasible, hence* $x_+ > 0$ *and* $s_+ > 0$. ■

In the following lemma we formulate a condition which guarantees the quadratic convergence of the Newton process. We mention that this requirement will be identical to that one used in Lemma 6.3, namely $\sigma(x, \mu) < 1$.

Lemma 6.4 *Let $\sigma = \sigma(x, \mu) < 1$. Then*

$$\sigma(x_+, \mu) \leq \frac{\sigma^2}{1 + \sqrt{1 - \sigma^2}}.$$

Hence, the full Newton step is quadratically convergent. ■

From the self-dual property of the problem (SP) follows that the duality gap is

$$2(q^T x) = 2(x^T s),$$

where x is a feasible solution of (SP), and $s = s(x)$ is the appropriate slack vector. For simplicity we also refer to $x^T s$ as the duality gap. In the following lemma we analyse the effect of the full Newton step on the duality gap.

Lemma 6.5 *Let $\sigma = \sigma(x, \mu)$ and introduce the vectors x_+ and s_+ such that $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. Then we have*

$$(x_+)^T s_+ = \mu(n - \sigma^2).$$

Thus $(x_+)^T s_+ \leq \mu n$. ■

In the following lemma we investigate the effect on the proximity measure of a full Newton step followed by an update of the parameter μ. Assume that μ is reduced by the factor $(1 - \theta)$ in each iteration.

Lemma 6.6 *Let $\sigma = \sigma(x, \mu) < 1$ and $\mu_+ = (1 - \theta)\mu$, where $0 < \theta < 1$. We have*

$$\sigma(x_+, \mu_+) \leq \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1 - \theta)(1 - \sigma^2)}}.$$

Furthermore, if $\sigma < \frac{1}{2}$ and $\theta = \frac{1}{2\sqrt{n}}$ then $\sigma(x_+, \mu_+) < \frac{1}{2}$. ■

From Lemma 6.6 we conclude that the algorithm is well defined. Indeed, the requirements $x > 0$ and $\sigma(x, \mu) < \frac{1}{2}$ are maintained at each iteration. In the following lemma we discuss the question of the bound on the number of iterations.

Lemma 6.7 *Let x^k be the k-th iterate of Algorithm 6.1, and let $s^k = s(x^k)$ be the appropriate slack vector. Then, for*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil,$$

we have $(x^k)^T s^k \leq \epsilon$. ■

For $\theta = \frac{1}{2\sqrt{n}}$ we obtain the following theorem.

Theorem 6.8 *Let* $\theta = \frac{1}{2\sqrt{n}}$. *Then Algorithm 6.1 requires at most*

$$O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$$

iterations. ■

6.6 Conclusion

In this chapter we have developed a new class of search directions for the self-dual linear optimization problem. For this purpose we have introduced a function φ, and we have used Newton's method to define new search directions. For $\varphi(x) = \sqrt{x}$ these results can be used to introduce a new primal-dual polynomial algorithm for solving (SP). We have proved that the complexity of this algorithm is $O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$.

7 Target-Following Methods

7.1 Introduction

In Chapter 5 and in the recent paper [36] we have introduced a new method for finding search directions for IPMs in LO, and we have developed a new polynomial algorithm for solving LO problems. It is well-known that using the self-dual embedding we can find a starting feasible solution, and this point will be on the central path. In the previous chapter we have proved that this initialization method can be applied for the new algorithm as well. However, practical implementations often don't use perfectly centered starting points. Therefore it is worth analysing the case when the starting point is not on the central path. In this chapter we develop a new weighted-path-following algorithm for solving LO problems. This algorithm has been introduced in [37]. We conclude that following the central path yields to the best iteration bound in this case as well.

It is well known that with every algorithm which follows the central path we can associate a target sequence on the central path. This observation led to the concept of *target-following* methods introduced by Jansen et al. [71]. A survey of target-following algorithms can be found in [103] and [68]. Weighted-path-following methods can be viewed as a particular case of target-following methods. These methods were studied by Ding and Li [41] for primal-dual linear complementarity problems, and by Roos and den Hertog [101] for primal problems. In this chapter we consider the LO problem in standard form, and we assume that the IPC holds. Using the self-dual embedding method a larger LO problem can be constructed in such a way that the IPC holds for that problem. Hence, the IPC can be assumed without loss of generality. Finding the optimal solutions of both the original problem and its dual, is equivalent to solving the following system

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & 0, & \end{array} \tag{22}$$

where xs denotes the coordinatewise product of the vectors x and s. The first and the second equations of system (22) serve for maintaining feasibility, hence we call them the *feasibility conditions*. The last relation is the *complementarity condition*, which in IPMs is generally replaced by a parameterized equation, thus we obtain

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & \mu e, & \end{array} \tag{23}$$

where $\mu > 0$, and e is the n-dimensional all-one vector, thus $e = [1, 1, \ldots, 1]^T$. If the IPC is satisfied, then for a fixed $\mu > 0$ the system (23) has a unique solution. This solution is called the *μ-center* (Sonnevend [109]), and the set of μ-centers for $\mu > 0$ formes the *central path*. The target-following approach starts from the observation that the system (23) can be generalized by replacing the vector μe with an arbitrary positive vector w^2. Thus we obtain the following system

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & w^2, & \end{array} \tag{24}$$

where $w > 0$. If the IPC holds then the system (24) has a unique solution. This feature was first proved by Kojima et al. [80]. Hence we can apply Newton's method for the system (24) to develop a primal-dual target-following algorithm. In the following section we present a new method for finding search directions by applying Newton's method for an equivalent form of system (24).

7.2 New Search-Directions

In this section we introduce a new method for constructing search directions by using the system (24). Let $\Re^+ = \{x \in \Re \mid x \geq 0\}$, and consider the function

$$\varphi \in C^1, \qquad \varphi : \Re^+ \to \Re^+.$$

Furthermore, suppose that the inverse function φ^{-1} exists. Then, the system (24) can be written in the following equivalent form

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ \varphi(xs) & = & \varphi(w^2), & \end{array} \tag{25}$$

and we can apply Newton's method for the system (25) to obtain a new class of search directions. We mention that a direct generalization of the approach defined in [36] would be the following variant. The system (24) is equivalent to

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ \varphi\left(\dfrac{xs}{w^2}\right) & = & \varphi(e), & \end{array} \tag{26}$$

and using Newton's method for the system (26) yields new search directions. For our purpose it is more convenient the first approach, hence in this chapter we use the system (25). Let us introduce the vectors

$$v = \sqrt{xs} \quad \text{and} \quad d = \sqrt{xs^{-1}},$$

and observe that these notations lead to

$$d^{-1}x = ds = v. \tag{27}$$

Suppose that we have $Ax = b$, and $A^T y + s = c$ for a triple (x, y, s) such that $x > 0$ and $s > 0$, hence x and s are strictly feasible. Applying Newton's method for the system (25) we obtain

$$\begin{array}{rcl} A\Delta x & = & 0, \\ A^T \Delta y + \Delta s & = & 0, \\ s\varphi\prime\,(xs)\,\Delta x + x\varphi\prime\,(xs)\,\Delta s & = & \varphi(w^2) - \varphi\,(xs)\,. \end{array} \tag{28}$$

Furthermore, denote

$$d_x = d^{-1}\Delta x, \qquad d_s = d\Delta s,$$

and observe that we have

$$v(d_x + d_s) = s\Delta x + x\Delta s, \tag{29}$$

and

$$d_x d_s = \Delta x \Delta s. \tag{30}$$

Hence the linear system (28) can be written in the following equivalent form

$$\begin{array}{rcl} \bar{A}d_x & = & 0, \\ \bar{A}^T \Delta y + d_s & = & 0, \\ d_x + d_s & = & p_v, \end{array} \tag{31}$$

where

$$p_v = \frac{\varphi(w^2) - \varphi(v^2)}{v\varphi\prime(v^2)}, \tag{32}$$

and $\bar{A} = Adiag(d)$. We also used the notation

$$diag(\xi) = \left[\begin{array}{cccc} \xi_1 & 0 & \dots & 0 \\ 0 & \xi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \xi_n \end{array} \right],$$

for any vector ξ. In the following section we will develop a new primal-dual weighted-path-following algorithm based on one particular search direction.

7.3 The Algorithm

In this section we let $\varphi(x) = \sqrt{x}$, and we develop a new primal-dual weighted-path-following algorithm based on the appropriate search directions. Thus, making the substitution $\varphi(x) = \sqrt{x}$ in (32) we get

$$p_v = 2(w - v). \tag{33}$$

Now for any positive vector v, we define the folowing proximity measure

$$\sigma(v, w) = \frac{\|p_v\|}{2\min(w)} = \frac{\|w - v\|}{\min(w)}, \tag{34}$$

where $\|\cdot\|$ is the Euclidean norm (l_2 norm), and for every vector ξ we denote $\min(\xi) = \min\{\xi_i \mid 1 \leq i \leq n\}$. We introduce another measure

$$\sigma_c(w) = \frac{max(w^2)}{min(w^2)},$$

where for any vector ξ we denote $\max(\xi) = \max\{\xi_i \mid 1 \le i \le n\}$. Observe that $\sigma_c(w)$ can be used to measure the distance of w^2 to the central path. Furthermore, let us introduce the notation

$$q_v = d_x - d_s,$$

observe that from (31) we get $d_x^T d_s = 0$, hence the vectors d_x and d_s are orthogonal, and thus we find that

$$\|p_v\| = \|q_v\|.$$

Consequently, the proximity measure can be written in the following form

$$\sigma(v, w) = \frac{\|q_v\|}{2\min(w)}, \tag{35}$$

thus we obtain

$$d_x = \frac{p_v + q_v}{2}, \qquad d_s = \frac{p_v - q_v}{2},$$

and

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{36}$$

Making the substitution $\varphi(x) = \sqrt{x}$ in (28) yields

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ \sqrt{\frac{s}{x}}\Delta x + \sqrt{\frac{x}{s}}\Delta s &= 2(w - \sqrt{xs}). \end{aligned} \tag{37}$$

Now we can define the algorithm.

Algorithm 7.1 *Suppose that for the triple* (x^0, y^0, s^0) *the interior point condition holds, and let* $w^0 = \sqrt{x^0 s^0}$*. Let* $\epsilon > 0$ *be the accuracy parameter, and* $0 < \theta < 1$ *the update parameter (default* $\theta = \frac{1}{5\sqrt{\sigma_c(w^0)n}}$*),*

begin
  $x := x^0;\ y := y^0;\ s := s^0;$
  $w := w^0;$
  while $x^T s > \epsilon$ **do begin**
    $w := (1-\theta)w;$
    Compute $(\Delta x, \Delta y, \Delta s)$ *from (37)*
    $x := x + \Delta x;$
    $y := y + \Delta y;$
    $s := s + \Delta s;$

end
end.

In the next section we shall prove that this algorithm is well defined for the default value of θ, and we will also give an upper bound for the number of iterations performed by the algorithm.

7.4 Convergence Analysis

In the first lemma of this section we prove that if the proximity measure is small enough, then the Newton process is strictly feasible. Denote $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$ the vectors obtained by a full Newton step, and let $v = \sqrt{xs}$ as usual.

Lemma 7.1 *Let $\sigma = \sigma(v, w) < 1$. Then $x_+ > 0$ and $s_+ > 0$, hence the full Newton step is strictly feasible.* ■

In the next lemma we prove that the same condition, namely $\sigma < 1$ is sufficient for the quadratic convergence of the Newton process.

Lemma 7.2 *Let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$ be the vectors obtaind after a full Newton step, $v = \sqrt{xs}$ and $v_+ = \sqrt{x_+ s_+}$. Suppose $\sigma = \sigma(v, w) < 1$. Then*

$$\sigma(v_+, w) \leq \frac{\sigma^2}{1 + \sqrt{1 - \sigma^2}}.$$

Thus $\sigma(v_+, w) < \sigma^2$, which means quadratic convergence of the Newton step.
■

In the following lemma we give an upper bound for the duality gap obtained after a full Newton step.

Lemma 7.3 *Let $\sigma = \sigma(v, w)$. Moreover, let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. Then*

$$(x_+)^T s_+ = \|w\|^2 - \frac{\|q_v\|^2}{4},$$

hence $(x_+)^T s_+ \leq \|w\|^2$. ■

In the following lemma we discuss the influence on the proximity measure of the Newton process followed by a step along the weighted-path. We assume that each component of the vector w will be reduced by a constant factor $1 - \theta$.

Lemma 7.4 *Let $\sigma = \sigma(v, w) < 1$ and $w_+ = (1-\theta)w$, where $0 < \theta < 1$. Then*

$$\sigma(v_+, w_+) \leq \frac{\theta}{1-\theta}\sqrt{\sigma_c(w)n} + \frac{1}{1-\theta}\sigma(v_+, w).$$

Furthermore, if $\sigma \leq \frac{1}{2}$, $\theta = \frac{1}{5\sqrt{\sigma_c(w)n}}$ and $n \geq 4$ then we get $\sigma(v_+, w_+) \leq \frac{1}{2}$. ■

Observe that $\sigma_c(w) = \sigma_c(w^0)$ for all iterates produced by the algorithm. Thus, an immediate result of Lemma 7.4 is that for $\theta = \frac{1}{5\sqrt{\sigma_c(w^0)n}}$ the conditions $(x, s) > 0$ and $\sigma(v, w) \leq \frac{1}{2}$ are maintained throughout the algorithm. Hence the algorithm is well defined. In the next lemma we calculate an upper bound for the total number of iterations performed by the algorithm.

Lemma 7.5 *Assume that x^0 and s^0 are strictly feasible, an let $w^0 = \sqrt{x^0 s^0}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. Then, for*

$$k \geq \left\lceil \frac{1}{2\theta} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil,$$

the inequality $(x^k)^T s^k \leq \epsilon$ is satisfied. ■

For the default value of θ specified in Algorithm 7.1 we obtain the following theorem.

Theorem 7.6 *Suppose that the pair (x^0, s^0) is strictly feasible, an let $w^0 = \sqrt{x^0 s^0}$. If $\theta = \frac{1}{5\sqrt{\sigma_c(w^0)n}}$ then Algorithm 7.1 requires at most*

$$\left\lceil \frac{5}{2}\sqrt{\sigma_c(w^0)n} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil$$

iterations. For the resulting vectors we have $x^T s \leq \epsilon$. ■

7.5 Conclusion

In this chapter we have developed a new weighted-path-following algorithm for solving LO problems. Our approach is a generalization for weighted-paths of the results presented in Chapter 5. We have transformed the system (24) in an equivalent form by introducing a function φ. We have defined a new class of search directions by applying Newton's method for that form of the weighted-path. Using $\varphi(x) = \sqrt{x}$ we have developed a new primal-dual weighted-path-following algorithm, and we have proved that this algorithm performs no more than

$$\left\lceil \frac{5}{2}\sqrt{\sigma_c(w^0)n}\log\frac{(x^0)^T s^0}{\epsilon}\right\rceil$$

iterations. Observe, that this means that the best bound is obtained by following the central path. Indeed, we have $\sigma_c(w^0) = 1$ in this case, and we get the well-known iteration bound

$$O\left(\sqrt{n}\log\frac{(x^0)^T s^0}{\epsilon}\right).$$

If the starting point is not perfectly centered, then $\sigma_c(w^0) > 1$ and thus the iteration bound is worse.

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