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Stochastic Programming Models in Financial Optimization: A Survey[1]

Li-Yong YU   Xiao-Dong JI   Shou-Yang WANG[2]

Institute of Systems Science
Academy of Mathematics and Systems Sciences
Chinese Academy of sciences
Beijing 100080, People's Republic of China

Abstract: In this paper, we survey the stochastic programming models developed to deal with financial optimization problems. A few methods are introduced in details to generate reasonable scenarios which are of much importance for a successful model. Besides, computation aspect as well as some open problems in this area are addressed.

Keywords: Stochastic Programming, Financial Optimization, Risk Management, Scenario Generation

1 Introduction

Financial optimization is one of the most attracting areas in decision-making under uncertainty. Prominent examples include: 1)asset allocation for pension plans and insurance companies; 2)security selection for stock and bond portfolio managers; 3)currency hedging for multi-national corporations; 4)hedge fund strategies to capitalize on market conditions; 5)risk management for large public corporations. In these situations, time periods and uncertainties play important roles. For example, a pension plan must focus on both the long-term and short-term consequences of his investment strategy. He must attempt to minimize pension contribution expenses over time, while satisfying the needs of the retirees, and reducing risks. Besides, there are many uncertainties in financial planning problems, such as economic factors, prices of the securities considered, amount of cash flows, etc.. To capture both these aspects, multi-stage stochastic programming models are well suited to address significant practical issues.

Stochastic programming (SP) models have been proposed and well studied since late 1950s by Dantzig[1][2], Beale[3], Charnes and Cooper[4] and others. They proposed a stochastic view to replace the deterministic one, where the unknown coefficients or parameters are random with assumed probability distribution that is independent of the decision variables. Over these years, progress in computational methods is impressive and large scale problems can be efficiently solved with high reliability (Lustig et al.,1991[22]; Bixby et al., 1992[6]; Levkovitz and Mitra, 1993[7]; Mulvey et al.,1995[8]). It is these advances that have progressively made SP techniques applicable to real-world problems. Moreover, high frequency data are readily available on a global basis, and powerful computers are also easily found to conduct the optimization search. The obstacles for applying stochastic optimization models are quickly receding.

[1]Supported by MADIS, CAS and NSFC
[2]The corresponding author. Email address: sywang@iss02.iss.ac.cn

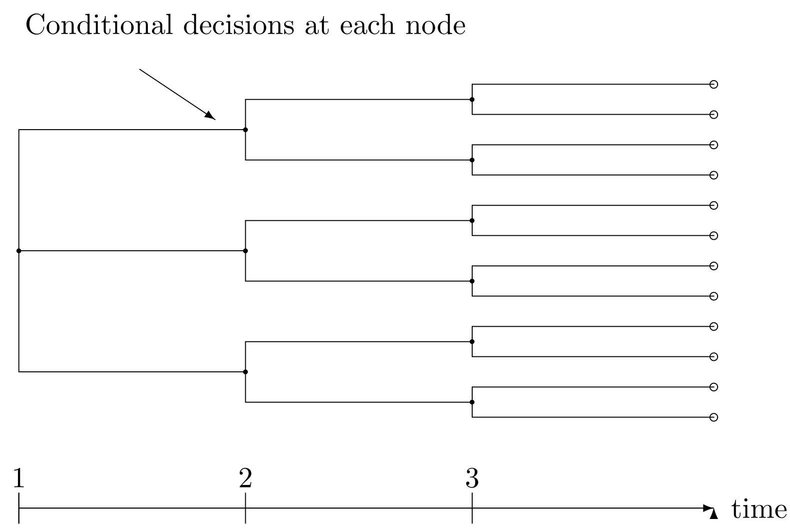

Figure 1: A scenario tree for a multi-stage stochastic program

Stochastic programming provides a general purpose-modelling framework, which captures the real-world features such as turnover constraints, transaction costs, risk aversion, limits on groups of assets and other consideration. However, the optimization model turns out to be intractable for the enormous number of decision variables, especially for the multi-stage problems. Figure 1 presents an example of a scenario tree whereby the decisions expand exponentially with time periods. Each node in this tree depicts a juncture for rendering decisions. A scenario, a complete path from the root node to a leaf, defines a single realization of the set of random variables.

The paper is organized as follows. Basic stochastic programming models and related applications in financial optimization are introduced in sections 2 and 3 respectively. Section 4 is dedicated to the scenario generation and computation aspect is addressed in section 5. The comparison with other methods is listed in section 6 and we conclude the paper with open problems in this field.

2 Basic Stochastic Programming Models

The anticipative and the adaptive models are special cases of stochastic programs. The combination of them makes a recoursive model which is widely applied in financial field.

2.1 Anticipative models

Anticipative model is also referred to as static model, for which the decision does not depend in any way on future observations of the environment. The prudent planning has to take into account all possible future realizations since there is no opportunity to adapt decisions later on, which may lead to overly conservative decisions.

In anticipative models feasibility is expressed in terms of *probabilistic (or chance) constraints*. For example, a *reliability* level α, where $0 < \alpha \leq 1$, is specified and constraints are expressed in the form

$$P\{\omega | f_j(x, \omega) = 0, j = 1, 2, ..., n\} \geq \alpha,$$

where x is the $m-$dimensional vector of decision variables and $f_j : \mathbb{R}^m \times \Omega \to \mathbb{R}, j = 1, 2, ..., n$. The objective function may also be of a reliability type, such as $P\{\omega | f_0(x, \omega) \leq \gamma\}$, where $f_0 : \mathbb{R}^m \times \Omega \to \mathbb{R} \bigcup \{+\infty\}$ and γ is a constant.

An anticipative model selects a policy that meets desirable characteristics of the constraints and the objective function. In the example above, it is desirable that the probability of a constraint violation is less than the prespecified threshold value $1-\alpha$. The precise value of α depends on the application at hand, the cost of constraint violation, and other similar considerations.

2.2 Adaptive models

In an adaptive model, information related to the uncertainty becomes partially available before decision making, so optimization takes place in a learning environment, which is the essential difference with an adaptive model. Let $\mathcal{A}$ be the collection of all the relevant information available through observation, which is a subfield of all possible events. The decision x depends on the events that can be observed, and x is termed $\mathcal{A}$-*adapted* or $\mathcal{A}$-*measurable*. An adaptive stochastic program can be formulated as:

$$\begin{array}{rll} \text{Minimize} & E[f_0(x(\omega), \omega) | \mathcal{A}] & \\ \text{subject to} & E[f_j(x(\omega), \omega) | \mathcal{A}] = 0 & j = 1, 2, ..., n \\ & x(\omega) \in X & \text{almost surely} \end{array} \tag{2.1}$$

The mapping $x : \Omega \to X$ is such that $x(\omega)$ is $\mathcal{A}$-measurable. This problem can be addressed by solving for every ω the following deterministic programs:

$$\text{Minimize} \quad E[f_0(x, \cdot) | \mathcal{A}](\omega) \tag{2.2}$$

$$\text{subject to} \quad E[f_j(x, \cdot) | \mathcal{A}](\omega) = 0 \quad j = 1, 2, ..., n \tag{2.3}$$

$$x \in X \tag{2.4}$$

The two extreme cases, complete information and no information at all, deserve special mention. The latter reduces the model to the anticipative form while the former is known as *distribution model*, which characterizes the distribution of the optimal objective value. However, the most interesting and valuable situation arises when partial information is available, which is what we will discuss below.

2.3 Recourse models

The recoursive model combines the former two models in a common mathematical framework, which seeks a policy that not only anticipates future observations but also takes into account temporarily available information to make recourse decisions. For example, a portfolio manager considers both future movements of stock prices (anticipation) as well as rebalancing the portfolio positions as prices change (adaptation).

The two-stage stochastic programming problem with recourse can be written as follows:

$$\begin{array}{rl} \text{Minimize} & f(x) + E[\mathcal{Q}(x, \omega)] \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}_+^{m_0} \end{array} \tag{2.5}$$

where x is the first-stage anticipative decisions, which is made before the random variables are observed, and $\mathcal{Q}(x, \omega)$ is the optimal value, for any given Ω, of the following nonlinear program

$$\begin{array}{ll} \text{Minimize} & q(y, \omega) \\ \text{subject to} & W(\omega)y = h(\omega) - T(\omega)x \\ & y \in \mathbb{R}_+^{m_1} \end{array} \tag{2.6}$$

where y is the second-stage adaptive decisions, which depends on the realization of the first-stage random vector. $q(y, \omega)$ denotes the second-stage cost function, and $\{T(\omega), W(\omega), h(\omega)|\omega \in \Omega\}$ are model parameters with reasonable dimensions. Those parameters are functions of the random vector ω and are, therefore, random parameters. T is the *technology matrix* containing the technology coefficients that convert the first-stage decision x into resources for the second-stage problem. W is the *recourse matrix* and h is the second-stage resource vector.

Generally the two-stage recourse model can be formulated as follows:

$$\begin{array}{ll} \text{Minimize} & f(x) + E[\min_{y \in \mathbb{R}_+^{m_1}} \{q(y, \omega)|T(\omega)x + W(\omega)y = h(\omega)\}] \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}_+^{m_0} \end{array} \tag{2.7}$$

2.4 Deterministic equivalent formulation

We consider now the case where the random vector ω has a discrete and finite distribution, with support $\Omega = \{\omega^1, \omega^2, ..., \omega^N\}$. In this case the set Ω is called a *scenario set*. Denote by p^l the probability of realization of the lth scenario ω^l. It is assumed that $p^l > 0$ for all $\omega^l \in \Omega$, and that $\sum_{l=1}^N p^l = 1$.

The expected value of the second-stage optimization problem can be expressed as

$$E[\mathcal{Q}(x, \omega)] = \sum_{l=1}^{N} p^l \mathcal{Q}(x, \omega^l). \tag{2.8}$$

For each realization of the random vector $\omega^l \in \Omega$ a different second-stage decision is made, which is denoted by y^l. The resulting second-stage problems can then be written as:

$$\begin{array}{ll} \text{Minimize} & q(y^l, \omega^l) \\ \text{subject to} & W(\omega^l)y^l = h(\omega^l) - T(\omega^l)x, \\ & y^l \in \mathbb{R}_+^{m_1}. \end{array} \tag{2.9}$$

Combining now (2.8) and (2.9) we reformulate the stochastic nonlinear program (2.7) as the following *large-scale deterministic equivalent* nonlinear program:

$$\text{Minimize} \quad f(x) + \sum_{l=1}^{N} p^l q(y^l, \omega^l) \tag{2.10}$$

$$\text{subject to} \quad Ax = b, \tag{2.11}$$

$$T(\omega^l)x + W(\omega^l)y^l = h(\omega^l) \quad \text{for all} \quad \omega^l \in \Omega, \tag{2.12}$$

$$x \in \mathbb{R}_+^{m_0}, \tag{2.13}$$

$$y^l \in \mathbb{R}_+^{m_1}. \tag{2.14}$$

2.5 Multistage models

The recourse problem is not restricted to the two-stage formulation. It is possible that observations are made at T different stages and are captured in the information sets $\{\mathcal{A}_t\}_{t=1}^T$ with $\mathcal{A}_1 \subset \mathcal{A}_2 \cdots \subset \mathcal{A}_T$. Stages correspond to time instances when some information is revealed and a decision can be made. (Note that T is a time index, while $T(\omega)$ are matrices.)

A multistage stochastic program with recourse will have a recourse problem at stage τ conditioned on the information provided by $\mathcal{A}_\tau$, which includes all information provided by the information sets $\mathcal{A}_t$, for $t = 1, 2, ..., \tau$. The program also anticipates the information in $\mathcal{A}_t$, for $t = \tau + 1, ..., T$.

Let the random vector ω have support $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_T$, which is the product set of all individual support sets $\Omega_t,\ t = 1, 2, ..., T$. ω is written componentwise as $\omega = (\omega_1, ..., \omega_T)$. Denote the first-stage variable vector by y_0. For each stage $t = 1, 2, ..., T$, define the recourse variable vector $y_t \in \mathbb{R}^{m_t}$, the random cost function $q_t(y_t, \omega_t)$, and the random parameters $\{T_t(\omega_t), W_t(\omega_t), h_t(\omega_t) | \omega_t \in \Omega_t\}$.

The multistage program, which extends the two-stage model (2.7), is formulated as the following nested optimization problem

$$\begin{array}{rl} \text{Minimize} & f(y_0) + E\left[\min\limits_{y_1 \in \mathbb{R}_+^{m_1}} q_1(y_1, \omega_1) + \ldots E\left[\min\limits_{y_T \in \mathbb{R}_+^{m_T}} q_T(y_T, \omega_T)\right] \cdots\right] \\ \text{subject to} & T_1(\omega_1) y_0 + W_1(\omega_1) y_1 = h_1(\omega_1), \\ & \vdots \\ & T_T(\omega_T) y_{T-1} + W_T(\omega_T) y_T = h_K(\omega_T), \\ & y_0 \in \mathbb{R}_+^{m_0}. \end{array} \tag{2.15}$$

For the case of discrete and finitely distributed probability distributions it is again possible to formulate the multistage model into a deterministic equivalent large-scale nonlinear program.

3 Stochastic Programming Models in Financial Optimization

Lots of articles in the literature have illustrated that stochastic programming models are flexible tools to describe financial optimization problems under uncertainty with realistic market imperfections and trading restrictions. Bradley and Crane (1972)[9] and Kusy and Zeimba (1986)[10] describe stochastic linear programs for bank asset/liability management, Carino et al.(1994)[11] formulate the asset/liability management problem of a Japanese insurance company as a multiperiod stochastic linear program, Mulvey and Vladimirou(1992)[12] propose a multiperiod stochastic network model for the purpose of asset allocation, and Hiller and Eckstein(1993)[13], Zenios(1993)[36], and Golub et al.(1993)[15] describe stochastic programming models for fixed-income securities management.

In the following, we introduce some application of stochastic programming models in financial optimization.

3.1 Stochastic Programming Model for Asset Allocation Problem

Asset allocation problem can be viewed as multiperiod dynamic decision problems where transactions take place at discrete time points.To define the model, we divide the entire planning horizon T into two discrete time intervals T_1 and T_2, where $T_1 = 0, 1, ..., \tau$ and $T_2 = \tau + 1, ..., T$.

The former corresponds to periods in which investment decisions are made. Period τ defines the date of the planning horizon; we focus on the investor's position at the beginning of period τ. Decisions occur at the beginning of each time stage. Much flexibility exists. An active trader might see his time interval as short as minutes, whereas a pension plan advisor will be more concerned with much longer planning periods such as the dates between the annual Board of Director's meeting. It is possible for the steps to vary over time - short intervals at the beginning of planning period and longer intervals towards the end. T_2 handles the horizon at time τ by calculating economic and other factors beyond period τ up to period T. The investor cannot render any active decisions after the end of period τ.

Asset investment categories are defined by set $A = 1, 2, ..., I$, with category 1 representing cash. The remaining categories can include broad investment groupings such as stocks, bonds, and real estate. The categories should track well-defined market segments. Ideally, the co-movements between pairs of asset returns would be relatively low so that diversification can be done across the asset categories.

In the model, uncertainty is represented by a set of distinct realization $s \in S$. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period.

We assume that the portfolio is rebalanced at the beginning of each period. Alternatively, we could simply make no transaction except reinvest any dividend and interest—a buy and hold strategy. For convenience, we also assume that the cashflows are reinvested in the generating asset category and all the borrowing is done on a single period basis.

For each $i \in A$, $t \in T_1$, and $s \in S$, we define the following parameters and decision variables.

Parameters:

$r^s_{i,t}$ $=1 + \rho^s_{i,t}$, where $\rho^s_{i,t}$ is the percent return for asset i, time period t, under scenario s (projected by the stochastic scenario generator, for example, see Mulvey et al. 1999 [16]).

π_s Probability that scenario s occurs, $\sum_{s=1}^{S} \pi_s = 1$.

w_0 Wealth in the beginning of time period 0.

$\sigma_{i,t}$ Transaction costs incurred in rebalancing asset i at the beginning of time period t (symmetric transaction costs are assumed, i.e., cost of selling equals cost of buying).

β^s_t Borrowing rate in period t under scenario s.

Decision variables:

$x^s_{i,t}$ Amount of money for asset category i, in time period t, under scenario s, after rebalancing.

$v^s_{i,t}$ Amount of money in asset category i, in the beginning of time period t, under scenario s, before rebalancing.

w^s_t Wealth at the beginning of time period t, under scenario s.

$p^s_{i,t}$ Amount of asset purchased for rebalancing in period t, under scenario s.

$d^s_{i,t}$ Amount of asset i sold for rebalancing in period t, under scenario s.

b^s_t Amount of money borrowed in period t, under scenario s.

Given these definitions, we present a general stochastic programming model in financial optimization.

Model SP

$$\text{Maximize} \quad Z = \sum_{s=1}^{S} \pi_s f(w_\tau^s) \tag{3.1}$$

Subject to

$$\sum_i x_{i,0}^s = w_0 \qquad \forall s \in S, \tag{3.2}$$

$$\sum_i x_{i,\tau}^s = w_\tau^s \qquad \forall s \in S, \tag{3.3}$$

$$v_{i,t}^s = r_{i,t-1}^s x_{i,t-1}^s \qquad \forall s \in S, t = 1, ..., \tau, i \in A, \tag{3.4}$$

$$x_{i,t}^s = v_{i,t}^s + p_{i,t}^s(1 - \sigma_{i,t}) - d_{i,t}^s \qquad \forall s \in S, i \neq 1, t = 1, ..., \tau, \tag{3.5}$$

$$x_{1,t}^s = v_{1,t}^s + \sum_{i \neq 1} d_{i,t}^s(1 - \sigma_{i,t}) - \sum_{i \neq 1} p_{i,t}^s - b_{t-1}^s(1 + \beta_{t-1}^s) + b_t^s \qquad \forall s \in S, t = 1, ..., \tau, \tag{3.6}$$

$$x_{i,t}^s = x_{i,t}^{s'} \quad \text{for all scenarios } s \text{ and } s' \text{ with identical past up to time } t. \tag{3.7}$$

As with the single-period models, the nonlinear objective function (3.1) can take several different forms. If the classical return-risk function is employed, then (3.1) becomes Max $Z = \eta$ Mean$(w_\tau) - (1 - \eta)$Risk(w_τ), where Mean(w_τ) is the expected total wealth and Risk(w_τ) is the risk of the total wealth across the scenarios at the end of period τ. Parameter η indicates the relative importance of risk as compared with the expected value. This objective leads to an efficient frontier of wealth at period τ by allowing alternative values of η in the range [0,1]. An alternative to mean-risk is the von Neumann-Morgenstern expected utility of wealth at period τ. Here, the objective becomes Max$Z = \sum_{s=1}^{S} \pi_s \text{Utility}(w_\tau^s)$ where Utility(W) is the Von Meumann Morgenstern utility function. Other objective functions are possible, such as the one proposed by Zhao and Zeimba[17].

Constraint (3.2) guarantees that the total initial investment equals the initial wealth. Constraint (3.3) represents the total wealth at the beginning of period τ. This constraint can be modified to include assets, liabilities, and investment goals. The modified result is called the surplus wealth(Muvey,1989 [18]). Most investors render investment decisions without reference to liabilities or investment goals. Mulvey employs the notion of surplus to the mean-variance and the expected utility models to address liabilities in the context of asset allocation strategies. Constraint (3.4) depicts the wealth $v_{i,t}^s$ accumulated at the beginning of period t before rebalancing in asset i. The flow balance constraint for all assets except cash for all periods is given by constraint (3.5). This constraint guarantees that the amount invested in period t equals the net wealth for asset. Constraint (3.6) represents flow balancing constraint for cash. Non-anticipativity constraints are represented by (3.7). These constraints ensure that the scenarios with the same past will have identical decisions up to that period. While these constraints are numerous, solution algorithms take advantage of their simple structure.

Model (SP) is a split variable formulation of stochastic asset allocation problem. This formulation has proven successful for solving the model using techniques such as progressive hedging algorithm of Rockafellar and Wets(1991)[19] and the quadratic diagonal approximation of Mulvey and Ruszczynski(1995)[20] and Berger et al.(1994)[21]. The split variable formulation can be beneficial for direct solvers that use the interior point method (Lustig, Mulvey, and Carpenter, 1991[22].

By substituting constraint (3.7) back in constraint (3.2) to (3.6), we obtain a standard form of the stochastic allocation problem. Constraints for this formulation exhibit a dual block diagonal structure for two stage stochastic programs and a nested structure for general multi-stage problems. This formulation may be better some direct solvers. The standard form of the stochastic program possesses fewer decision variables than the split variable model and is the preferred structure by many researchers in the field. This model can be solved by means of decomposition methods, for example, the L-shaped method (a specialization of Benders algorithm). See Birge and Loveaux(1997)[23], Dantzig and Infanger(1993)[24], Dempster(1998)[25], Infanger(1994)[26], and Kall and Wallace(1994)[27] and their numerous references.

The multi-stage model can provide superior performance over single period models. See the references(Berger and Mulvey 1996[28], Carino et al.1994[11], Dempster 1998[25], Dert 1995[29], Holmer 1994[30], Klaassen 1994[31] and 1998[60], Mulvey and Zeimba 1995[33], Nielsen and Zenios 1996[34], Worzel et al. 1995[35], and Zenios 1993[36]).

3.2 Stochastic Programming Models for the Management of Fixed-income Securities

The new fixed-income securities, including high-yield bonds, mortgage-backed securities, callable bonds, and international bonds have been developing rapidly in recent years. Compared to the traditional portfolio optimization, the management of fixed-income securities have to cope with more uncertainties, so is more complex.

The management of fixed-income securities can be viewed as a multistage decision problem in which portfolio actions are taken at successive (discrete) points in time. At each decision period, the portfolio manager has an inventory of securities and/or cash on hand. Based upon present credit market conditions and his assessment of future interest rates and cash flows, the manager must decide which securities to hold in the portfolio over the next time period, which securities to sell, and which securities to purchase from the market. These decisions are made subject to a constraint on total portfolio size, which may be larger or smaller than the previous period's constraint depending upon whether a cash inflow or outflow occurred. At the next decision period, the portfolio manager faces a new set of interest rates and a new portfolio size constraint. He must then make another set of portfolio decisions which take into account the new information. This decision-making process, which is repeated over many time periods, is dynamic in the sense that the optimal first period decision depends upon the actions which will be taken in each future period for each uncertain event.

Any proposed model for the management of fixed-income securities should be able to cope with the uncertainties inherent in those securities. Bennett Golub et al.[37] pointed that there are two major sources of uncertainty in the new fixed-income markets, in addition to the usual interest rate risk. First, is the timing and amount of cash flows received from the securities. Second, is the spread over the risk-free rate that prices the security. This spread reflects premia on risk factor that are present in the specific fixed-income market, but are absent from Treasury securities. The timing of the cash flows depends on the embedded (or real) options of the security. For example, corporate bonds may be called prior to maturity. The loans backing a mortgage security may prepay or default.

In the absence of market imperfections, dynamic portfolio investment problems have been studied using continuous-time models, usually in the context of an individual who maximizes expected utility of future consumption(c.f. Cox and Huang 1989 [38]). These models can be solved analytically if the state variables are assumed to follow diffusion process and certain restrictions are imposed on the form of the utility function. In practice, the applications of these models are restricted for the number of risky and riskless assets to be analyzed is limited.

For portfolio investment problem of practical importance, one has no choice but to use stochastic programming models in which both time and the state space are discretized. An important contribution in multi-period dynamic models was made by Bradley and Crane [9], who proposed a stochastic programming models with recourse for bond management. Their model allows the full spectrum of interest rate and price changes, and permits portfolio rebalancing. After that, much work has been done in this field. For example, Zenios [39] proposed a model for managing portfolios of mortgage-backed securities. Bennett Golub [40] developed a multi-period dynamic portfolio optimization model to address the problem of management of fixed-income securities. Christiana Vassiadou-Zeniou and Stavros A. Zenios [41] presented a multi-stage stochastic program with recourse for the management of portfolios of callable bonds. In these models, stochastic programs are used to address the time and uncertainty in financial planning. To illustrate this in detail, we introduce the model of Andrea Beltratti et al.[42] for the management of international bond portfolios.

To the problem of portfolio management in the international markets, there are three aspects that need our consideration, interest rate risk in the local market, exchange rate volatility across markets, and decisions for hedging currency risk. In this model these decisions are integrated in a common framework, while in the past they were addressed separately. In order to get necessary data to realize the model, Monte Carlo simulation procedures are used to generate jointly scenarios of interest and exchange rates.

Now we give a detailed description of the problem. In this portfolio management problem, the aim is to manage a bond portfolio in a way that it tracks a broadly defined international market index. The indexation tracking strategy is widely used by insurance and pension fund companies, foundations, and money management firms. A bond index in each market $i = 1, 2, ..., m$, is constructed by creating a representative sample Φ_i of size N_i from the universe of eligible bonds Ω_i. For each security $j = 1, 2, ..., N_i$, in the representative set, the index specifies its relative weight ϕ_j^i which reflects the capitalization structure of the universe set Ω_i with bonds that have characteristics identical or similar to the jth bond. The global bond index is represented by a set Γ of country indices and the relative weights γ_i assigned to the bond index of each country based on the market value of the different indices. These weights are a measure of the share of the bond market of the ith country in the world bond market.

The manager of an international bond portfolio must determine the fraction of the portfolio value invested in each of the m markets, and to pick specific bonds from each market Ω_i to adjust the portfolio. These decisions are usually made in three steps. An asset allocation committee determines first the exposure of the portfolio to each market. Then traders identify mispriced bonds in each market and construct the country-specific portfolio. Finally, once the country -specific funds are constructed the currency exposure may be hedged.

Now we present the integrative model for tracking an international fixed income index.

The interesting contribution of the model is that it combines the following three aspects in an integrated fashion, i.e. asset allocation in different markets, bond picking in each market, and optimal currency hedging ratios.The model specifies optimal bond picking decisions in each of the m markets to track the global index.

Integrative model: First-stage constraints

The first stage (i.e. at time t_0) cashflow accounting equation of the model is:

$$c_0 + \sum_{i=1}^{m} e_{0_i} \sum_{j=1}^{N_i} \zeta_{0_j}^i Y_{0_j}^i = \sum_{i=1}^{m} e_{0_i} \sum_{j=1}^{N_i} (\zeta_{0_j}^i + \delta) X_{0_j}^i + v_0. \tag{3.8}$$

The inventory balance constraint is:

$$b_{0_j}^i + X_{0_j}^i = Y_{0_j}^i + Z_{0_j}^i \text{ for all } i \in \Gamma \text{ and } j \in \Phi_i. \tag{3.9}$$

For the sake of simplicity we assume here that all sales and purchases are made into and from the base currency (in which investor measures his return), thus avoiding the need to keep separate cashflow variables.

Integrative model: Time-staged constraints

Cashflow accounting constraints of the model at any time period t after $t = 0$ depend on the path l_t. These constraints limit the increase in holdings for each bond in each market.

There is one constraint for each path $l_t \in P_t$ (the arguments l_t are dropped from all variables and parameters below for simplicity of notation):

$$\rho_{t-1} v_{t-1} + \sum_{i=1}^{m} e_{t_i} \sum_{j=1}^{N_i} k_{t-1_j}^i Z_{t-1_j}^i + \sum_{i=1}^{m} e_{t_i} \sum_{j=1}^{N_i} \zeta_{t_j}^i Y_{t_j}^i$$

$$= \sum_{i=1}^{m} e_{t_i} \sum_{j=1}^{N_i} (\zeta_{t_j}^i + \delta) X_{t_j}^i + v_t. \tag{3.10}$$

Inventory balance equations constrain the amount of each bond sold or remaining in the portfolio to be equal to the outstanding amount at the end of the holding period, plus any additional amount purchased. There is one constraint for each bond and for each path $l_t \in P_t$:

$$Z_{t-1_j}^i + X_{t_j}^i = Y_{t_j}^i + Z_{t_j}^i \text{ for all } i \in \Gamma, j \in \Phi_i. \tag{3.11}$$

Integrative model: objective function

At the end of the planning horizon T and for each path $l_T \in P_T$ we calculate the return of the portfolio. This value depends on the composition of the portfolio and the value of the bonds at T and on any accrued cashflow from previous periods. The return of the portfolio is given by

$$R_p(l_T) \doteq R_p(Z_T(l_T)) = \frac{v_T + \sum_{i=1}^{m} e_{T_i} \sum_{j=1}^{N_i} \zeta_{T_j}^i Z_{T_j}^i - V_{p0}}{V_{p0}}, \tag{3.12}$$

where V_{p0} is the initial value of the portfolio.

For our aim is to track the market indexation, the objective function maximizes the expected utility of excess return of the portfolio over the index,

$$\text{Maximize} \sum_{l_T \in P_T} \pi_{l_T} \mathcal{U} \left(\frac{R_p(l_T)}{I_T(l_T)} \right), \tag{3.13}$$

where $\mathcal{U}(\cdot)$ denotes the utility function. Here we choose to maximize a utility function of excess return to allow for tradeoffs of growth versus security, which was addressed by MacLean, Ziemba and Blazenko (1992) [43]. Choosing for instance a logarithmic utility function we can implement in our model the investor's wish to follow a growth optimal strategy over the long run.

Optimal currency hedging ratios

Hedging decision in the optimization model can now be incorporated. Define a new variable $(H_{0_i})_{i=1}^m$ to denote the amount of each currency hedged at agreed upon 1-period forward rates $(f_{0_i})_{i=1}^m$ at period $t = 0$. Let also $(H_{t_i}(l_t))_{i=1}^m$ denote the amount hedged at forward rates $(f_{t_i}(l_t))_{i=1}^m$ at period t under path l_t.

The cashflow accounting constraints must be modified to account for the fact that at each period t an amount $H_{t_i}(l_t)$ of the i currency will be exchanged at rate $f_{t_i}(l_t)$ and any remaining amount will be exchanged at rate $e_{t_i}(l_t)$. Recall that there is one constraint for each path $l_t \in P_t$, and that the arguments l_t are dropped from all variables and parameters below for simplicity of notation.

The total cashflow in the ith currency—inflows from coupon payments and security sales and outflows from security purchase—at period t under scenario l_t is given by

$$w_{t_i} = \sum_{j=1}^{N_i} k_{t-1_j}^i Z_{t-1_j}^i + \sum_{j=1}^{N_i} \zeta_{t_j}^i Y_{t_j}^i - \sum_{j=1}^{N_i} (\zeta_{t_j}^i + \delta) X_{t_j}^i \tag{3.14}$$

The cashflow accounting equation (13) is rewritten to incorporate hedging as:

$$\rho_{t-1} v_{t-1} + \sum_{i=1}^{m} (f_{t-1_i} H_{t-1_i} + e_{t_i}(w_{t_i} - H_{t-1_i})) = v_t. \tag{3.15}$$

At the end of the planning horizon the return of the portfolio—in the base currency—will be a function of the amount hedged and the forward and current exchange rates. The return calculation takes the form:

$$R_p(l_T) \doteq R_p(Z_T(l_T)) = \frac{v_T + \sum_{i=1}^m \left(f_{T-1_i} H_{T-1_i} + e_{T_i} \left(\sum_{j=1}^{N_i} \zeta_j^i Z_{T_j}^i - H_{T-1_i} \right) \right) - V_{p0}}{V_{p0}} \tag{3.16}$$

In order to implement the portfolio optimization models we need a simulation procedure to generate interest rate and exchange rate scenarios. Once such scenarios are generated the calculation of bond price conditioned on the observed interest rates is straightforward; see , e.g., Mulvey and Zenios (1994) [44].

3.3 Stochastic Programming Models for Asset/Liability Management

Asset/Liability management(ALM) addresses the problem of an investor who faces a sequence of liability payments in the future, and wants to construct a portfolio of securities that allows him to meet these liabilities under a variety of plausible scenarios. From all feasible portfolios he wants to choose the one that optimizes some optimality criterium (e.g.,minimum cost). Both the size of the liability payments and the security returns may depend on the state of the world in the future. This problem can be modelled as a multi-period stochastic linear program, which explicitly includes the possibility of portfolio rebalancing at future points in time as response to new information that becomes available. As we are interested in presenting a modelling framework for realistic ALM problems, market imperfections and trading restrictions are taken into account.

In recent years the number of publications about stochastic programming for asset liability management has risen drastically, probably inspired by the rapid increase of efficiency and accessibility of computer systems. Some of the applications are very successful. For example, the insurance ALM model of Carino and Zeimba[45] has been extensively used by the Frank Rusell company in consulting ALM managers in insurance and pension fund. Their work with The Yasuda Fire and Marine Insurance Company was a finalist at the Franz Edelman Competition for Management Science Achievements. Similar acclaim was achieved by the Towers Perrin-Tillinghast model of Mulvey, Gould and Morgan [46]. Stochastic programming models for Dutch pension funds were developed by Dert [47], a general ALM model for insurers by Consigli and Dempster [48].

Now we introduce the model presented by Klaassen [60].

In asset/liability management one generally faces a trade-off between the initial cost of the asset portfolio of which the payoffs must be sufficient to meet the liabilities, and the value of the portfolio that is left at the end of the model horizon. Although we have assumed that the investor can borrow money at intermediate trading dates, we require that the final portfolio value must be nonnegative in all scenarios. The trade-off between the initial investment and the value of the portfolio at the end of the model horizon is captured in the objective function: the initial portfolio investment is minimized, but any positive final portfolio value is credited to the objective using a concave utility function $\mathcal{U}(\cdot)$. We assume that this utility function satisfies the expected utility property.

The ALM problem can now be formulated as the following multiperiod stochastic program.

$$
\begin{aligned}
\text{minimize} \qquad & (1+c)\sum_{i=1}^{I} S_{i,0}xb_{i,0} - (1-c)\sum_{i=1}^{I} S_{i,0}xs_{i,0} \\
& + P_0 y_0 - e^{-\kappa\Delta} P_0 z_0 - \sum_{s\in\mathcal{S}_T} \eta_T^s \mathcal{U}(y_T^s)
\end{aligned}
\tag{3.17}
$$

subject to

$$
-xs_{i,0} + xb_{i,0} - xh_{i,0} = -\bar{x}_{i,0} \qquad \forall i = 1, ..., I, \tag{3.18}
$$

$$
\begin{aligned}
& xh_{i,t-1}^{s^-} - xs_{i,t}^s + xb_{i,t}^s - xh_{i,t}^s = 0 \\
& \qquad \forall i = 1, ..., I, s \in \mathcal{S}_t, t = 1, ..., T-1,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
& \sum_{i=1}^{I} D_{i,t}^{n(s)} xh_{i,t-1}^{s^-} + y_{t-1}^{s^-} - z_{t-1}^{s^-} + (1-c)\sum_{i=1}^{I} S_{i,t}^{n(s)} xs_{i,t}^s \\
& -(1+c)\sum_{i=1}^{I} S_{i,t}^{n(s)} xb_{i,t}^s - P_t^{n(s)} y_t^s + e^{-\kappa\Delta} P_t^{n(s)} z_t^s = L_t^{n(s)} \\
& \qquad \forall s \in \mathcal{S}_t, t = 1, ..., T-1,
\end{aligned}
\tag{3.20}
$$

$$\sum_{i=1}^{I}(D_{i,T}^{n(s)}+S_{i,T}^{n(s)})xh_{i,T-1}^{s^-}+y_{T-1}^{s^-}-z_{T-1}^{s^-}-y_T^s=L_T^{n(s)}$$
$$\forall s\in\mathcal{S}_T, \tag{3.21}$$
$$xs_{i,t}^s, xb_{i,t}^s, xh_{i,t}^s\geq 0$$
$$\forall i=1,...,I, s\in\mathcal{S}_t, t=0,...,T-1, \tag{3.22}$$
$$y_t^s\geq 0 \quad \forall s\in\mathcal{S}_t, \quad t=0,...,T, \tag{3.23}$$
$$0\leq z_t^s\leq \bar{Z}_t^{n(s)} \quad \forall s\in\mathcal{S}_t, \quad t=0,...,T-1. \tag{3.24}$$

We will refer to this formulation as the ALM model.

The first four terms in the objective function represent the net cost of *additional* investments at time 0. These additional investments consist of asset purchases (including transaction costs) and investment in the riskless one period security, while revenues from the sale of assets (net of transaction costs) and borrowing are subtracted. The last term in the objective is the expected utility of a final portfolio surplus.

We distinguish between three types of constraints in the model: *portfolio-balance* constraints, *cash-balance* constraints and *borrowing* constraints. The portfolio-balance constraints link portfolio holdings between successive periods (ie., before and after rebalancing) in each scenario and for each asset. The portfolio-balance constraints are given by (3.18) for all assets at time 0, and by (3.19) for all assets in each scenario after time 0.

The cash-balance constraints make sure that sufficient cash is generated to meet the liability payment in each scenario at each time. For each scenario at time $t < T$, this constraint is given by (3.20). At the end of a period , the investor receives dividend payments on his asset holdings and the return on his investment in the one period riskless security but has to repay the amount borrowed in the previous period plus interest (represented by the first three terms on the left-hand side of (3.20)). The next two terms reflect rebalancing of the portfolio: revenues are generated by selling assets, and money can be invested by buying assets, where both are adjusted for transaction costs. The final two terms on the left-hand side are the investment in the riskless one period security and the amount borrowed, respectively, during the next period.

The cash-balance constraints (3.21) at time T define the final portfolio value in each scenario. The first three terms on the left-hand side determine the final portfolio value before meeting the liability: the portfolio holdings are converted at the current market prices, the return on the investment in the riskless one period security is added, and the amount due because of borrowing subtracted. The difference between this portfolio value and the liability payment in a scenario $s\in\mathcal{S}_T$ is the final portfolio surplus y_T^s.

The nonnegativity restriction on $xh_{i,t}^s$ prevent short sales of assets, while Equation (3.24) states the upper bounds on borrowing.

4 Scenario Generation

To operate the stochastic programming models, scenarios generation and constructing event trees are of very importance. We describe some methods that have been proposed to deal with these two problems. In constructing event trees, arbitrage-free condition should be noted.

4.1 Methods for Generating Scenarios

In this subsection we describe three specific methods for generating asset return scenarios with more detail: (i) bootstrapping historical data, (ii) statistical modelling with the Value-at-Risk approach, and (iii) modelling economic factors and asset returns with vector autoregressive models.

4.1.1 Bootstrapping historical data

The simplest approach for generating scenarios use only the available data without any mathematical modelling. It bootstraps a set of historical records. Each scenario is a sample of assets returns which is obtained by sampling returns that were observed in the past. Dates from the available historical records are selected randomly and for each date in the sample we read the returns of all asset classes or risk factors during the month prior to that date. These are scenarios of monthly returns. If we want to generate scenarios of returns for a long horizon—say 1 year—we sample 12 monthly returns from different points in time. The compounded return of the sampled series is the 1-year return. Note that with this approach the correlations among asset classes are preserved.

4.1.2 Statistical models from the Value-at-Risk literature

Time series analysis of historical data can be used to estimate volatilities and correlation matrices among asset classes of interest. These correlation matrices are used to measure risk exposure of a position through the Value-at-Risk (VaR) methodology.

Denote the random variables by the K-dimensional random vector ω. The dimension of ω is equal to the number of risk factors we want to model. Assuming that the random variables are jointly normally distributed we can define their probability density function of ω by

$$f(\omega) = (2\pi)^{-p/2}|Q|^{-1/2} \exp\left[-\frac{1}{2}(\omega - \bar{\omega})' Q^{-1}(\omega - \bar{\omega})\right], \tag{4.1}$$

where $\bar{\omega}$ is the expected value of ω and Q is the covariance matrix and they can be calculated from historical data. (It is typically the case in financial time series to assume that the logarithms of the changes of the random variables have the above probability density function, so that the variables themselves follow a lognormal distribution.)

Once the parameters of the multivariate normal distribution are estimated we can use it in Monte Carlo simulations, using either the standard Cholesky factorization approach or scenario generation procedures based on principal component analysis discussed in Jamshidian and Zhu (1997)[50].

The simulation can be applied repeatedly at different states of an event tree. However, we may want to condition the generated random values on the values obtained by some of the random variables. For instance, users may have views on some of the variables, or a more detailed model may be used in the simulation hierarchy to estimate some of the variables. This information can be incorporated when sampling the multivariate distribution.

The conditional sampling of multivariate normal variables proceeds as follows. Variable ω is partitioned into two subvectors ω_1 and ω_2, where ω_1 is the vector of dimension K_1 of random variables for which some additional information is available and ω_2 is the vector of dimension $K_2 = K - K_1$ of the remaining variables. The expected value vector and covariance matrix are partitioned similarly as

$$\bar{\omega} = \begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{4.2}$$

The marginal probability density function of ω_2 given $\omega_1 = \omega_1{}^*$ is given by

$$f(\omega_2|\omega_1 = \omega_1{}^*) =$$
$$(2\pi)^{-p_2/2}|Q_{22.1}|^{-1/2} \exp\left[-\frac{1}{2}(\omega_2 - \bar{\omega}_{2.1})' Q_{22.1}^{-1}(\omega_2 - \bar{\omega}_{2.1})\right], \tag{4.3}$$

where the conditional expected value and covariance matrix are given by

$$\bar{\omega}_{2.1}(\omega_1^*) = (\bar{\omega}_2 - Q_{21}Q_{11}^{-1}\mu_1) + Q_{21}Q_{11}^{-1}\omega_1^*, \tag{4.4}$$

and

$$Q_{22.1} = Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}, \tag{4.5}$$

respectively. Scenarios of ω_2 for period t conditioned on values of ω_1 given by ω_1^* can be generated from the multivariate normal variables from (4.3) through the expression

$$\omega_{2i}^t = \omega_{2i}^0 \exp\left[\sigma_i \sqrt{t}\omega_{2i}\right],$$

where ω_{2i}^0 is today's value and σ_i is the single-period volatility of the ith component of the random variable ω_2.

Consiglio and Zenios (2001)[51] use the Riskmetrics methodology in conjunction with discrete lattice models to generate joint scenarios of term-structure and exchange rates. Interest rate differences among two countries are key determinants of the exchange rate between the currencies. Hence, exchange rate scenarios are conditioned on the interest rates of the two currencies, the base currency and the foreign currency. The standard assumption applies that the logarithms of the ratios of exchange rates at period t to period $t-1$, and the logarithms of the ratios of spot interest rates at period t to period $t-1$ follow a multivariate normal distribution. Daily and weekly rates do not follow normal distributions but there is lack of empirical evidence against normality for monthly data such as those used by Consiglio and Zenios.

4.1.3 Scenario generation using vector autoregressive models

Vector autoregressive models are often used to generate scenarios. To illustrate this, we consider an ALM simulation system for Dutch pension funds as an example (see Boender 1997[52]). As the scope of ALM systems for Dutch pension funds is often limited to long term strategic decisions, the investment model only considers a small set of broad asset classes: deposits, bonds, real estate and stocks. Apart from the returns on these assets, each scenario should contain information about future wage growth in order to calculate the future values of the pension liabilities.

In order to generate asset returns and the wage growth rate a vector autoregressive model is applied by Boender (1997):

$$R_t = c + Vh_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, Q), \quad t = 1, 2, ..., T, \tag{4.6}$$
$$R_{it} = ln(1 + r_{it}), \quad i = 1, 2, ..., m, \quad t = 1, 2, ..., T, \tag{4.7}$$

where m is the number of asset time series, r_{it} is the discrete rate of change of variable i in year t, R_t is an m-dimensional vector of continuously compounded rates, c is the m-dimensional vector of coefficients, V is an $m \times m$ matrix of coefficients, ϵ_t is the m-dimensional vector of error terms and Q is the $m \times m$ covariance matrix.

The specification of the vector autoregressive model should be chosen carefully. Although some inter-temporal relationships between the returns might be weakly significant based on historical data, that does not imply that these relationships are also useful for generating scenarios for a financial optimization model with a long time horizon. To avoid any problems with unstable and spurious predictability of returns, we do not use lagged variables for explaining the returns of bonds, real estate, and stocks in the vector autoregressive model. The time series of the return on deposits and the increase of the wage level on the other hand are known to have some memory, so we model them by a first order autoregressive process.

There are many ways to estimate vector autoregressive models, see, e.g., Judge et al. (1988)[53]. After the vector autoregressive model has been used to generate scenarios of asset returns and wage growth, the liability values can be added to each scenario in a consistent manner by applying appropriate actuarial rules or financial valuation principles (Embrechts 2000)[54].

4.2 Constructing Event Trees

A stochastic programming model is based upon an event tree for the key random variables. Each node of the event tree has multiple successors, in order to model the process of information being revealed progressively through time. The stochastic programming approach will determine an optimal decision for each node of the event tree, given the information available at that point. As there are multiple succeeding nodes the optimal decisions will be determined without exploiting hindsight. If a stochastic programming model is formulated then the optimal policy will be tailor-made to fit the condition of the state of financial institution and the economy in each node, while anticipating the optimal adjustment of the policy later on as the tree evolves and more information is revealed.

A key issue for the successful application of stochastic programming in financial optimization is the construction of event trees with asset returns. The underlying return distributions have to be discretized with a small number of nodes in the event tree, otherwise the computational effort for solving a multi-stage stochastic programming model can easily explode. Clearly, a small number of nodes describing the return distribution at every stage of the event tree might lead to some approximation error. An important question is to which extent the approximation error in the event tree will bias the optimal solutions of the model.

In this section we will consider three different methods to construct event trees for stochastic programming models: (i) random sampling, (ii) adjusted random sampling, and (iii) tree fitting. In order to compare these methods we will apply them to construct trees with asset and liability returns for the estimated vector autoregressive model.

4.2.1 Random sampling

First we introduce random sampling from the error distribution of the vector autoregressive model. Given the estimated coefficients and the estimated covariance matrix of the vector autoregressive model, we can draw one random vector of yearly returns for bonds, real estate, stocks, deposits and wage growth. If we would like to construct an event tree with ten nodes after one year (we assume that the duration of each stage is one year), we can simply repeat this procedure ten times, sampling independent vectors of returns for each node. The nodes at stage two in the event tree can also be sampled randomly, however the conditional distribution from stage one to stage two depends on the outcomes at the first stage. For example, wage growth follows an autoregressive process, so the expected wage growth from year one to year two depends on the realized wage growth rate in the previous period.

An entire event tree for the stochastic program can be created by applying random sampling recursively, from stage to stage, while adjusting the conditional expectations of wage growth and deposits in each node based on previous outcomes.

The random sampling procedure for constructing a sparse multi-period event tree apparently leads to unstable investment strategies. An obvious way to deal with this problem is to increase the number of nodes in the randomly sampled event tree, in order to reduce the approximation error relative to the vector autoregressive model. However, the stochastic program might become computationally intractable if we increase the number of nodes at each stage, due to the exponential growth rate of the tree. Alternatively, the switching of asset weights might be bounded by adding constraints to the model or enforcing robustness through the choice of an objective function (Mulvey, Vanderbei and Zenios 1995 [55]). Although we might get a more stable solution in this case, the underlying problem remains the same: the optimal decisions are based on an erroneous representation of the return distributions in the event tree.

4.2.2 Adjusted random sampling

An adjusted random sampling technique for constructing event trees can resolve some of the problems of the simple random sampling method. First, assuming an even number of nodes, we apply antithetic sampling in order to fit every odd moment of the underlying distribution. For example, if there are ten succeeding nodes at each stage then we sample five vectors of error terms from the vector autoregressive model. The error terms for the five remaining nodes are identical but with opposite signs. As a result we match every odd moment of the underlying error distributions (note that the errors have a mean of zero). Second, we rescale the sampled values in order to fit the variance. This can be achieved by multiplying the set of sampled returns for each particular asset class by an amount proportional to their distance from the mean. In this way the sampled errors are shifted away from their mean value, thus changing the variance until the target value is achieved. The adjusted values for the error terms are substituted in the estimated equations of the vector autoregressive model to generate a set of nodes for the event tree.

Using adjusted random sampling to match the mean and the variance, we substantially reduce useless trading. The additional computational effort for adjusting the random samples is negligible.

4.2.3 Fitting the mean and the covariance matrix

A third method for constructing event trees is to estimate returns that match the first few moments of the underlying return distributions. This can be achieved by solving a non-linear optimization model following Hoyland and Wallace (1999)[56]. The decision variables in the optimization model are the returns and the probabilities of the event tree, while the objective function and the constraints enforce the desired statistical properties. The probabilities and returns in all nodes of the event tree can be estimated simultaneously. However, with this approach it might take longer to construct a desirable event tree than to solve the stochastic programming model for ALM itself. The tree fiting problem can be simplified by applying the method at each stage recursively as suggested in Kouwenberg (1998)[65]. This requires the assumption that the return distributions are not path dependent. This assumption is valid for long term asset and liability management with broad asset classes, but fails when modelling money management problems with path-dependent securities.

To illustrate the concepts, we write down the tree fitting equations to estimate a set of perturbations that will fit the mean and the residual covariance matrix of the vector autoregressive process. The probabilities are assumed uniform in order to ease comparison with random sampling. Let $i = 1, 2, ..., m$, denote the random time series that are modelled by the vector autoregressive process. In our example these are the returns on stocks, bonds, deposits, real estate and the wage growth rate. Suppose that a total of M succeeding nodes at stage $t+1$ are available to describe the conditional distribution of these random variables in a particular node at stage t. We define the perturbation ϵ_{ti}^{l} as the realization in node l for the ith element of the vector ϵ_t.

A tree fitting model that matches the mean of zero and the estimated covariance of vector autoregressive model (4.6)-(4.7) estimates the perturbations by solving equations (4.8) and (4.9). Equation (4.8) specifies that the average of the perturbations should be zero, while equation (4.9) states that they should have a covariance matrix equal to Q:

$$\frac{1}{M}\sum_{l=1}^{M}\epsilon_{ti}^{l} = 0, \quad \text{for all} \quad i = 1, 2, ..., m, \tag{4.8}$$

$$\frac{1}{(M-1)}\sum_{l=1}^{M}\epsilon_{ti}^{l}\epsilon_{tj}^{l} = Q_{ij}, \quad \text{for all} \quad i = 1, 2, ..., m, \; j = 1, 2, ..., m. \tag{4.9}$$

Obtaining a solution of the non-linear system (4.8)-(4.9) can be difficult, specially when higher order moments like skewness and kurtosis are also included as additional restrictions. Instead of solving a system of nonlinear equations we may solve instead a non-linear optimization model that penalizes deviations from the desired moments in the objective function. Good starting points for this optimization can be obtained using the adjusted random sampling method of the previous subsection, which is computationally very efficient. After solving the non-linear fitting model, we can substitute the optimal set of perturbations in the estimated equations of the vector autoregressive model to generate conditional return distributions. By applying this procedure recursively, from node to node and from stage to stage, we generate an event tree that fits the time varying conditional expectation and the covariance matrix of the underlying return distributions.

Finally, for completeness we would like to mention some other promising methods for constructing event trees from the stochastic programming literature. Mulvey and Zenios (1994)[44] discusses simulation techniques to generate scenarios of returns for fixed-income portfolio models, based on a underlying fine-grained interest rate lattice. Pflug and Swietanowski (1998)[57] derive promising theoretical results for optimal scenario generation for multiperiod financial optimization. Shtilman and Zenios (1993)[58] derive theoretical results for the optimal sampling from lattice models. Further theoretical and empirical research in this area is important, as the event trees used as input are crucial for the effectiveness of the stochastic programming approach to finanial planning problem.

4.3 Arbitrage Free Event Tree

The absence of arbitrage opportunities is an important property for event trees of asset returns that are used as input for stochastic programming models. If there is an arbitrage opportunity in the event tree, then the optimal solution of the stochastic programming model will exploit it. An arbitrage strategy creates profits without taking risk, and hence it will increase the objective value of nearly any financial planning model. So the presence of arbitrage opportunities in a portfolio optimization model can lead to substantial biases in the optimal solution that are due to profit opportunities which exist in the model. Even though the pofit opportunities are often unlikely to marerialize in reality, it is prudent for long term financial planning applications to generate scenarios that do not allow for arbitrage.

A potential problem for stochastic programming models in financial optimization are arbitrage opportunities in the event tree that are due to approximation errors. Klaassen (1997)[59] was the first to address this issue. Arbitrage opportunities might arise because the underlying return distributions are sometimes approximated poorly with a small number of nodes in the event tree. If the application only involves broad asset classes such as a stock index, a bond index and real estate index, then arbitrage opportunities are unlikely to occur unless the errors in the event tree are very big. However, applications that involve options, multiple bonds or other interest rate derivative securities can be quite vulnerable to these problems. For example, the prices of European call and put options with equal strike price should satisfy put- call parity in each node of the event tree. If this relationship is violated because of a small approximation error, then the event tree contains an arbitrage opportunity and hence a source of spurious profits for the stochastic programming model.

To deal with the arbitrage free problem, Klaassen (1998)[60] proposes an aggregation method. It starts with a very fine-grained event tree of asset prices without arbitrage opportunities and then reduce it to a smaller tree, while preserving the property of no-arbitrage. Recursively, a combination of nodes at a particular time period can be replaced by one agregated node, while preserving the no-arbitrage property. If a node has only one particular successor remaining at the next time, then the intermediate period can be eliminated. This method can reduce the recombining lattice to a much smaller event tree with less trading dates, while meeting the arbitrage free condition. Another method for reducing a fine-grained lattice of security prices to a sparse event tree without arbitrage is discussed in Gondzio, Kouwenberg and Vorst (1999)[61]. They apply their method to an option hedging problem with two sources of uncertainty: the stock price and stochastic volatility. First a three-dimensional fine-grained grid of time versus stock price and volatilty is constructed to calculate option prices. Second, the points on the grid are partitioned into groups at a small number of trading dates, corresponding to the decision stages in the stochastic programming model. Each groups of points on the grid is represented by a single aggregated node in the event tree of the stochastic programming model. If the prices in each aggregated node are calculated as a conditional expectation under the risk neutral measure of the prices in the corresponding partition on the grid, then the aggregated event tree will not contain arbitrage opportunities.

Although the absence of arbitrage opportunities is important for financial stochastic programs with derivative securities, one should keep in mind that it is only a minimal requirement for the event tree. The fact that the stochastic program can not generate riskless profits from arbitrage opportunities does not imply that the event tree is also a good approximation of the underlying return process. We still have to take care that the conditional return distributions of the assets are represented properly in each node of the event tree. In order to avoid computational problems that arise if the tree becomes too big, one could reduce the number of stages of the stochastic program. In this way more nodes are available to describe the return distributions accurately. It is also important to include more nodes for the earlier stages, while larger errors in the later stages will have a small effect on the first-stage decisions which are the decisions implemented today by the decision makers. End effects of stochastic programming models for financial applications are studied by Carino and Ziemba (1998)[45] and Carino, Myers and Ziemba (1998)[62].

5 Algorithm and implementations

Stochastic programming is one of the possible approaches that can be used to model real-life financial planning problems. However, taking into account both uncertainty and the dynamic structure of decision problem inevitably leads to an explosion of dimensionality in stochastic programming models. The models grow in size very quickly with the number of stages and the number of scenarios at each stage. In practice they have to be solved numerically. A very rich literature is devoted to designing algorithms for them.

The first group of methods are variants of the simplex methods which take advantage of the structure of the constraint matrix to construct compact representations of the basis inverse and to improve pivotal strategies (see [63]). The special block-angular structure of the constraint matrix of stochastic programs has prompted the development of specialized algorithms. Modern implementations of the simplex method, such as IBM's OSL or CPLEX by Ilog, incorporate many theoretical results of research on this problem, making commercially available two excellent versions of this algorithm for the solution of stochastic programming problems.

The second group are linear decomposition methods coming down from the famous decomposition principle of Dantzig and Wolfe[64]. Special purpose decomposition algorithms break up the deterministic equivalent formulation into smaller problems, which can be solved either serially or in parallel. In any case they are much smaller than the original problem hence solution times are substantially improved. OSL supports some decomposition methods. However, most software implementations of decomposition methods are supported by academic researchers. In general such systems are very efficient and quite robust, but they are not of industrial quality.

Except for these, Kouwenberg(1998)[65] solved the deterministic equivalent linear program of the stochastic programming model using an interior point algorithm that exploits the sparse block-angular structure . The fixed-income models and the asset allocation models can also be represented as network flow problems and can be solved using special purpose network optimization algorithms (Mulvey and Vladimirou, 1992[12], Nielsen and Zenios,1996[34]).

Work on the solution of stochastic programs has also focused on the intelligent sampling and pruning of the event tree. Clearly not all events on an event tree will have an effect on the optimal solution. It is important to sample only those events that have the most impact on the solution. Importance sampling (Dantzig and Infanger 1991[66]) and EVPI (expected value of perfect information, Dempster and Gassmann 1991[67]) have appeared as promising avenues for restricting the tree size, and structuring problems of moderate size. For a discussion of solution techniques and an extensive list of references see Censor and Zenios [68](1997, Ch. 13).

There have been numerous applications of stochastic programming models in various areas. Actual and potential applications in finance are particularly rich. They are surveyed in Dupacova(1991)[69], Mulvey and Zeimba(1995)[70], and Zeimba and Mulvey(1998)[71].

6 Conclusion

The literature on financial optimization models is vast and dates back to the seminal contribution of Markowitz (1952) [72]. Four alternative modelling approaches have emerged as suitable frameworks for representing financial optimization problems.They are mean-variance models, discrete-time multi-period models, continuous-time models, and stochastic programming. The mean-variance framework of Markowitz (1952) is widely considered as the starting point for modern research about portfolio optimization. Although this kind of model gets profound insight into the problem, it is of limited use in practice because of two major drawbacks. Firstly, variance is not always a good risk measure for investors; Secondly, a single-period model might be inappropriate for multi-period investment problems with long horizons. Continuous-time models and discrete-time models solved with dynamic programming and optimal control can provide good qualitative insights about fundamental issues in investments and ALM. However, their practical use as a tool for decision making is limited by the many simplifying assumptions that are needed to derive the solutions in a reasonable amount of time.

The stochastic programming approach for financial optimization can be considered as a practical multi-period extension of the normative investment approach of Markowitz (1952). The advantage of stochastic programming models for multi-period investment and ALM problems is that important practical issues such as transaction costs, multiple state variables, market incompleteness, taxes and trading limits, regulatory restrictions and corporate policy requirements can be handled simultaneously within the framework.

Of course this flexibility comes at a price and stochastic programming also has a drawback. The computational work explodes as the number of decision stages increases. When implementing a stochastic programming model, we are therefore often forced to make a trade off between the number of decision stages in the model and the number of nodes in the event tree that are used to approximate the underlying returns distributions.

Because stochastic programming can deal simultaneously with all important aspects of financial optimization problem, much work has been done in this area. But there are still a lot of interesting topics for further investigations. Some of them are mentioned as follows.

To character the realistic problems, it is very important to set up stochastic programming models that incorporate more consideration of uncertainties. For example, to the portfolio selection problem, in addition to the usual interest rate changes, uncertainty in the timing and amount of cashflows, changes in the default and other risk premia and so on should be considered. To the ALM, effort should be given to expand the applicability of stochastic programming to address enterprise-wide risk management problems.

When generating scenarios, an important issue is how to measure the approximation error of the returns in the event tree compared to the true underlying distribution. Once appropriate measures have been identified, one could try to develop methods for constructing event trees that minimize the approximation error (assuming the size of the event tree is fixed). A promising first step in this direction is made by Pflug and Swietanowski (1998)[57].

When the computational side of stochastic programming is considered, there seems to be a need for flexible and efficient model generation tools. Specialized optimization algorithms and the ever increasing computational power of computers make it feasible to solve large scale multi-stage financial optimization models with millions of variables and constraints on desktop computers nowadays. However, most commercial mathematical modelling languages are not capable of generating the data of these huge problems efficiently. Moreover, if the modelling language does not exploit the special structure of the stochastic program, it can easily run into memory problems that could be avoided. Model generation seems to have become the bottleneck that limits the size of multi-stage stochastic programming models applied to financial optimization.

Besides, the present boom of large-scale real-life applications has brought new challenging questions. An important task is an adequate reflection of the dynamic aspects, including further development of tractable numerical approaches. Additional problems are related with the fact that the probability distribution P is rarely known completely and/or that it has to be approximated for reasons of numerically tractability. Because of this one mostly solves an approximate stochastic program instead of the underlying true decision problem. The task is to generate the required input, i.e., to approximate P bearing in mind the required type of the problem; see e.g. [73]. Moreover, without additional analysis, the obtained output(the optimal value and optimal solutions of the approximate stochastic program) should not be used to replace the sought solution of the true problem; see [74]for discussion of suitable *output analysis methods*. These methods have to be tailored to the structure of the problem and they should also reflect the source, character and precision of the input data.

In current applications, the methods of output analysis address mainly the two-stage (multiperiod) stochastic programs. The reason is that the structure of multistage problems is much more involved and one cannot rely on intuitive straightforward generalizations. At the same time validation experiments, e.g. [75], provide an evidence that even three-stage stochastic programs may outperform significantly the existing static models. Hence, an extensive research in multistage stochastic programming is an important complex task of the day.

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