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Alternative Statistical Specifications of Commodity Price Distribution with Fat Tail

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Abstract: We investigate the modeling of commodity prices that exhibit "fat tails" in the empirical marginal distributions. Using electricity price data, we explore the goodness-of-fit of different classes of distributions with an emphasis on capturing the fat tails in the data. Specifically, we fit empirical marginal distributions of time series data to distributions with either quantile functions or probability density functions in closed-forms. The theoretical distributions under consideration all have rich tail behaviors that enable us to model the heavy tails in the commodity prices caused by jumps and stochastic volatility. The fact that the theoretical distributions are easy to simulate makes the models appealing since the tasks of parameter estimation and derivative pricing can be directly implemented based on observed market data.

Keywords: fat tail distribution, electricity price modeling, stochastic volatility, $\alpha-$stable distribution, risk management.

1 Introduction

Normal or log-normal distributions are classical statistical models for marginal distributions of financial time series data. However, it has been well documented that Normal distributions can not sufficiently represent the heavier-than-Normal tails present in the empirical distributions of equity returns (e.g. [13], [3], and [4] among others). Such inadequacy in modeling is more pronounced when it comes to model commodity prices.

Commodities such as electricity and bandwidth have erratic price behaviors due to their non-storable nature and dependence on capacitated physical networks. Ever since the moment that electricity became a traded commodity, its price has been displaying the highest level of volatility and the most complex features among all commodity prices. With the over-the-counter markets for bandwidth commodity shaping up in 2000, bandwidth poses as yet another commodity whose price behavior has the comparable level of complexity as that of electricity. Taking a look at the left panel of Figure 1, it plots the historical power price paths in two regions of the U.S., Northern California (Cal-PX) and Pennsylvania-New Jersey-Maryland (PJM), during the time period from April 1, 1998 to August 31, 2000. The price data exhibits salient features of mean-reverting, jumps, and spikes, which reflect the commodity nature of electricity and the unique characteristics of electricity such as non-storability, heavy reliance on the transmission networks,

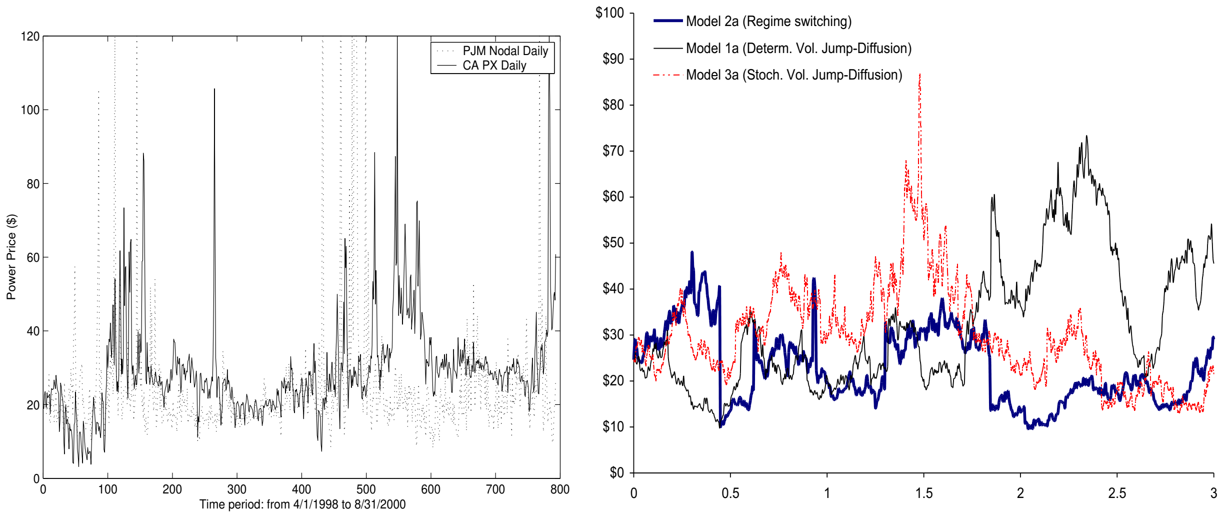

Figure 1: Historical Power Daily Spot Prices (left) vs. Simulated Daily Prices (right)

and characteristic steepness of the aggregate electricity supply function at high production levels in almost all regions of the U.S. Such volatile price behaviors could create dramatic market risks and operational risks. In risk management jargons, market risk refers to the financial impact of an adverse move in financial or commodity markets; and operational risk refers to the impact of a firm's lack of preparedness for market volatility. The market and operational risks associated with remarkably volatile commodity prices make it an indispensable task to accurately model commodity price dynamics for the purposes of risk management.

The motivation of our work is to identify a commodity price model that is both realistic for option pricing and risk management purposes and feasible for parameter estimation given limited available market data. Enlightened by the fact that certain stochastic volatility processes can be conveniently characterized by their marginal distributions, we propose to model commodity prices using the marginal distributions of a couple of alternative stochastic volatility models.

In the rest of this paper, we take electricity price as an illustrative example to outline our modeling approach. The organization of our paper is as follows. We give a brief review on other approaches to modeling electricity price in Section 2. We then specify a stochastic process to model the power price dynamics and fit marginal distributions of power price to different theoretical distributions in Section 3. Some parameter estimation results based on electricity market price data will be presented in Section 4. Finally, we conclude and point out future research direction in Section 5.

2 Previous Work

Prices of non-storable commodities (e.g. electricity and bandwidth) cannot be adequately represented by the classic financial asset price models such as the geometric Brownian motion (GBM) model that underlies the Black-Scholes option pricing formula. The left panel of Figure 1 clearly demonstrates this point. Mean-reversion is a common feature in almost all commodity prices (e.g. [14]). Other features such as stochastic volatility and jump may be present in commodity prices as well. In a realistic electricity price model, for instance, mean-reversion, jump, and stochastic volatility are among the key characteristics that need to be captured (see [10] and [5]). [2] offers a one-factor price model that combines a mean-reverting process with a single jump process. [5] presents three mean-reversion jump-diffusion models including multi-factor affine jump-diffusion models that can incorporate regime-switching jumps as well as stochastic volatility. By introducing jumps into the continuous-time mean-reverting spot price models, one can better capture the abrupt market changes caused by unpredictable scenarios such as abnormal weather conditions and forced capacity outages. Power prices simulated from each of the three models in [5], as plotted in the right panel of Figure 1, reveal a strong similarity to historical power price curves shown in the left panel of the same figure. Unfortunately, all of these models, which adopt certain types of stochastic processes to model power prices, require a relatively large amount of market price data, in particular options prices, for carrying out rigorous parameter estimation. However, in the newly developed power markets, neither quality nor quantity of options price data can be guaranteed. In a discrete-time setup, [11] examines a Markov regime-switching model with a mean reverting stochastic process model as an alternative power price model. Using data from four electricity markets, they estimate the model by applying the method of maximum likelihood. [1] utilizes a frequency-domain method to separate out periodic price variations from random variations and then estimate the volatility and mean-reversion parameters associated with the random variation of power prices.

The marginal distributions and the quasi long range dependent structure in some financial time series are investigated in [3], [4], and [9]. Two of their conclusions are:

- The marginal distributions of those time series have tails which are heavier than Norm but lighter than Cauchy
- The so called quasi long range dependent structure in those time series can be fitted very well by the superposition of the Ornstein-Uhlenbeck processes which will be described in the next section.

3 Model Description

Before describing our model, let us give a brief introduction to the so called Ornstein-Uhlenbeck processes. A stochastic process $x(t)$ is said to be of Ornstein-Uhlenbeck type, or for short an OU process, if it is stationary and satisfies a stochastic differential equationof the form

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(\lambda t) \tag{1}$$

where $\lambda > 0$ and $z(\lambda t)$ is a Lévyprocess, which may depend on λ. We refer to $z(\lambda t)$ as the background driving Lévyprocess, abbreviated BDLP.

We now describe our model. Let $X_\Delta(t)$ be the difference of the electricity prices, measured in the length Δ, i.e., if we use $P(t)$ to denote the the price of the electricity, then $X_\Delta(t) = P(t+\Delta) - P(t)$. Since Δ is not really relevant in our modeling once we choose it, we denote $X_\Delta(t)$ by $X(t)$ below. To begin with, we choose two independent OU processes to model the serial correlation among electricity prices. In case we encounter a stronger series dependence structure in power or any other commodity price data, we can easily extend our model to capture it by superposing more than two independent OU processes. The following equations define our model.

$$\begin{aligned} X(t) &= X_1(t) + X_2(t) \qquad (2) \\ \mathrm{d}X_j(t) &= -\lambda_j X_j(t) + \mathrm{d}Z_j(t) \text{ for } j = 1, 2 \end{aligned}$$

where $X_1(t)$ and $X_2(t)$ are two independent OU processes, while $Z_1(t)$ and $Z_2(t)$, the BDLP's, are two independent Lévyprocesses. Notice that by modeling the price difference as an OU process, we can capture the effect of power prices possibly being negative due to the facts that there is no free-disposal of electricity and there are costs associated with starting up and shutting down a power plant. For other commodities which do not permit negative prices, the price return can be modeled as an OU process.

In practice, one may specify an OU process in several ways: the BDLP modeling, first specifying the BDLP of an OU process then determining its stationary distribution or going the other way around, the so called the stationary distribution modeling. In both modelings, the likelihood functions based on some discretely observed data are not explicitly available, unless in some stable OU processes (see [9]), and the parameter estimation often requires some simulation-based methods, for instance, the method in [7], or the so called partner approach or the Q-Q estimation method, both proposed by [9]. In these simulation-based procedures, the most time-consuming part is to simulate the following form of the stochastic integral

$$\int_0^{\triangle} f(t)\mathrm{d}Z(t)$$

where $f(t)$ is a deterministic function and $Z(t)$ is a Lévy processwhich is determined by some parameters. A useful way to save the computational time is to determine the tail heaviness of the stationary distribution first. [9] uses a term "tail order" to describe it and finds that the tail order of an OU process is exactly equal to the tail order of its BDLP $Z(t)$. Using this fact, one can then effectively lower the dimension of parameter space by the following procedure: First, we fit the marginal distributions of $X(t)$ by our two new classes of distributions. We can then understand the tail order of the marginal distribution of $X(t)$, and this tail order is exactly the same of the BDLP of $X(t)$, namely $Z(t)$.

3.1 Marginal Distribution of $X(t)$

Our plan is to fit marginal distributions of power price difference process $X(t)$ to four different classes of distributions. For the first two classes, termed Class I and Class II distributions, we match the quantiles of the empirical distribution and a theoretical distribution out of two classes of distributions with properly constructed quantile functions. For the third class, we fit the probability density function of a mixture of Cauchy and double-exponential distributions. The forth class that we consider as fitting candidates are the α-stable distributions.

We start with Class I distributions since the distributions from the first class have closed-form formulas for probability densities, probability distribution functions, and quantile functions. Therefore the task of estimating parameters is relatively straightforward. A Class I quantile function is given below (see [9]).

$$q(y;\alpha,\beta,\delta,\mu) = \delta^{\frac{1}{\alpha}}\{log\frac{y^{\beta}}{1-y^{\beta}}\}^{(\frac{1}{\alpha})} + \mu \tag{3}$$

where $\alpha, \beta, \delta, \mu$ are parameters with $\delta, \alpha, \beta \in \mathbf{R}_+$, $\mu \in \mathbf{R}$, while the superscript '$(\alpha)'$ for $\alpha > 0$ represents the following operations:

$$x^{(\alpha)} = \begin{cases} x^{\alpha} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -(-x)^{\alpha} & \text{if } x < 0 \end{cases} \tag{4}$$

We note that all parameters in (3) have some intuitive interpretations. μ is a location parameter; δ is a scaling parameter; β acts like a tail balance adjuster: $\beta = 1$ means a balanced tail, and $\beta < (>)1$ means the left (right) tail is fatter than the right (left) tail; and α indicates the tail order - the smaller the α, the fatter the tail of a distribution.

We also employ Class II distributions to examine whether the unbalanced tail behavior in power prices can be better captured by this class of distributions than by Class I distributions.

A class II quantile function is given below (also see [9]).

$$\begin{aligned} q(y;\alpha_-,\alpha_+,\delta_-,\delta_+,\mu) &= -\delta_-^{\frac{1}{\alpha_-}}(\log\frac{1}{y})^{\frac{1}{\alpha_-}} \\ &\quad +\delta_+^{\frac{1}{\alpha_+}}(\log\frac{1}{1-y})^{\frac{1}{\alpha_+}} + \mu \end{aligned} \tag{5}$$

where $\alpha_-, \alpha_+, \delta_-, \delta_+$ and μ are parameters with $\alpha_-, \alpha_+, \delta_-, \delta_+ \in \mathbf{R}_+, \mu \in \mathbf{R}$. We are mainly interested in the cases where $\alpha_- \leq 1, \alpha_+ \leq 1$. Similar to a Class I distribution, α_- and α_+ measure the fatness of the tails of a Class II distribution.

A remarkable character of a Class II distribution is that it can have different tail orders at the two sides of the distribution density function: α_- at the left-hand side, and α_+ at the right-hand side. A very convenient way to carry out the parameter estimation in Class II distributions is to use the so called Q-Q estimation which minimizes certain kind of distance between a theoretical quantile and an empirical quantile (e.g. [9]).

Double exponential distributions and Cauchy distributions are popular candidates for fitting fat tail data. However, fitting power prices to a double exponential distribution or a Cauchy distribution alone does not yield satisfactory results. As a remedy, we consider a class of distributions being a mixture distribution of double exponential and Cauchy as the candidates for fitting power price differences. The probability density function of a mixture distribution is given by

$$\begin{aligned} p(x) &= wf_1(x) + (1-w)f_2(x) \\ &= \frac{we^{-\left|\frac{x-\mu}{\beta}\right|}}{2\beta} + \frac{1-w}{s\pi(1+(\frac{x-t}{s})^2)} \end{aligned} \tag{6}$$

where w and $1-w$ are the weights for a double exponential distribution and a Cauchy distribution, respectively. We estimate the parameter values by minimizing certain kind of distance between a mixture density function and the empirical density function.

Lastly, as a measure of quantifying the magnitude of fat tails in power price distributions, we fit power price differences to a class of α-stable distributions. For an α-stable distribution, we have the following characteristic function:

$$\varphi(t) = \begin{cases} \exp\{\lambda[it\gamma - |t|^\alpha + it(|t|^{\alpha-1} - 1)\beta \tan(\frac{\pi}{2})]\} & if \;\; \alpha \neq 1 \\ \exp\{\lambda(it\gamma - |t| - i\frac{2}{\pi}\beta \ln|t|)\} & if \;\; \alpha = 1 \end{cases} \tag{7}$$

where α is the tail index, which measures the "thickness" of tails, or the total probability in extreme tails. The smaller the α, the fatter the tails. In most cases, the explicit expression of an α-stable density function does not exist. For this model, we use three estimators to estimate the tail index α. They are Hill estimator, the unconditional Pickands estimator (UP), and the modified unconditional estimator (MUP) as described in [12].

3.2 Pricing of European-style Options

One immediate application of the marginal distribution modeling of the price process $X(t)$ is the pricing of European-style options. Suppose the electricity price is x at time 0. Let $C(x,T)$ denote the value of an European option with a terminal payoff of $f(P(T))$ at maturity time T where $P(T)$ is the electricity price at time T and f is a real-valued measurable function. Assuming there is no arbitrage opportunities, then there exists a risk-neutral probability distribution Q over all possible realizations of $P(T)$ so that

$$C(x,T) = E^Q[e^{-rT}f(P(T))] \tag{8}$$

$$= e^{-rT}\int_{-\infty}^{+\infty} f(s)dQ(s) \tag{9}$$

where r is the constant risk-free interest rate (e.g. [8]). Recall that $X(t)$ denote $P(t+T) - P(t)$. Let $\widetilde{Q}(s)$ denote the marginal distribution function of $X(t)$ under the risk-neutral measure. Then $P(T) \overset{distribution}{=} P(0) + X(0)$ and equation (8) becomes

$$C(x,T) = E^Q[e^{-rT}f(P(T))]$$

$$
\begin{aligned}
&= E^{\widetilde{Q}}[e^{-rT} f(P(0) + X(0))] \\
&= e^{-rT} \int_{-\infty}^{+\infty} f(x+s) d\widetilde{Q}(s). \qquad (10)
\end{aligned}
$$

Thus we can value any European-style option with payoff being a function of the terminal price $P(T)$ based on the marginal distribution of $X(0)$ under the risk-neutral probability measure.

4 Parameter Estimation

As an initial preliminary investigation, we fit the daily average prices at the California Power Exchange (Cal-PX) and the Pennsylvania-New Jersey-Maryland (PJM) market from April 1, 1998, to August 31, 2000, to Class I, Class II, mixture and α-stable distributions, respectively.

In particular, we take $\Delta = 1$ day and fit the marginal distribution of $X(t) = P(t) - P(t - \Delta)$ (namely, $X(t)$ is the price difference between the day-t price and the day-$(t-1)$ price) to the four classes of distributions. The fitting results are illustrated in Figure 2, Figure 3, and Figure 4.

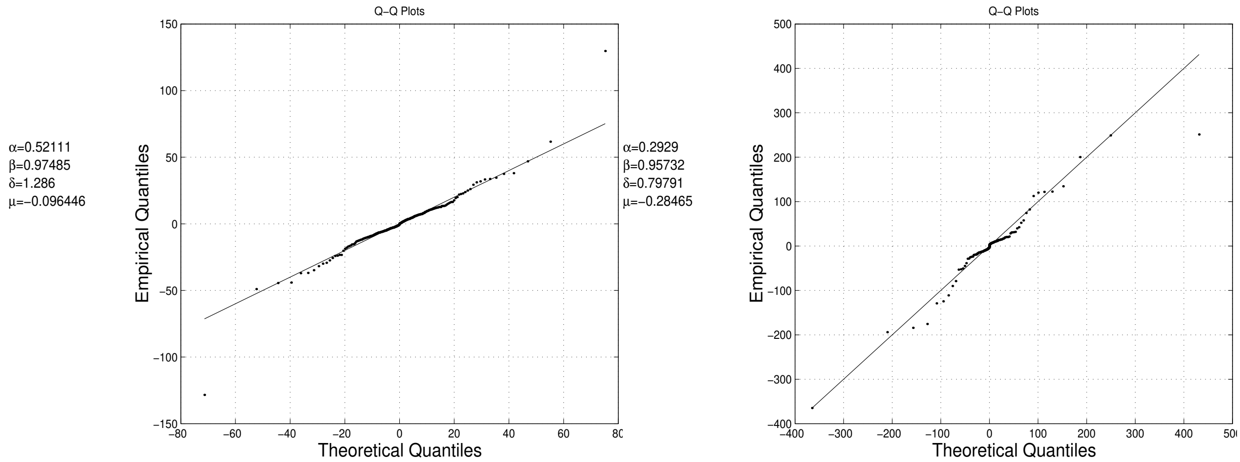

Figure 2: The Q-Q Plots of Quantile Functions for Class I Distributions: Cal-PX (left) vs. PJM Prices (right)

Figure 2 shows the Q-Q Plots of the empirical quantile functions of power price data versus the theoretical Class I quantile functions for Cal-PX and PJM market, respectively. (The estimated parameters are reported in the figure as well.) For Cal-PX data (left panel), we can see that the Q-Q plot forms a relatively straight line, which indicates a good fit between the empirical quantile function and the theoretical quantile function, except for several outliers. $\beta = 0.9749$ suggests that the left tail of the distribution of Cal-PX price differences is fatter than the right tail. On the other hand, the fitting of PJM prices is not as good as that of Cal-PX prices. The Q-Q plot of the empirical quantile function of PJM data versus the theoretical quantile function deviates slightly from the $45-$degree line (right panel). Similar to the Cal-PX data set, PJM price data also yields a less-than-one β value ($\beta = 0.9573$) indicating a fatter left tail of the price difference distribution in the PJM market. When comparing the two fitted Class I distributions at Cal-PX and PJM, we notice that Cal-PX has a larger α value than PJM does. This fact suggests that the distribution of PJM price differences has fatter tails than those of Cal-PX price differences.

We next use Class II distributions to fit Cal-PX and PJM daily average price differences. The estimated parameters are reported in the Q-Q plots in Figure 3. It seems that using a Class II distribution to fit the PJM price differences yields a better Q-Q plot (right panel) than using a Class I distribution. Moreover, α_- is less than α_+ for the PJM price data implying a fatter left tail than the right tail. This is consistent with what we observe when fitting PJM data to a Class I distribution. As for the Cal-PX price data set, it also generates a

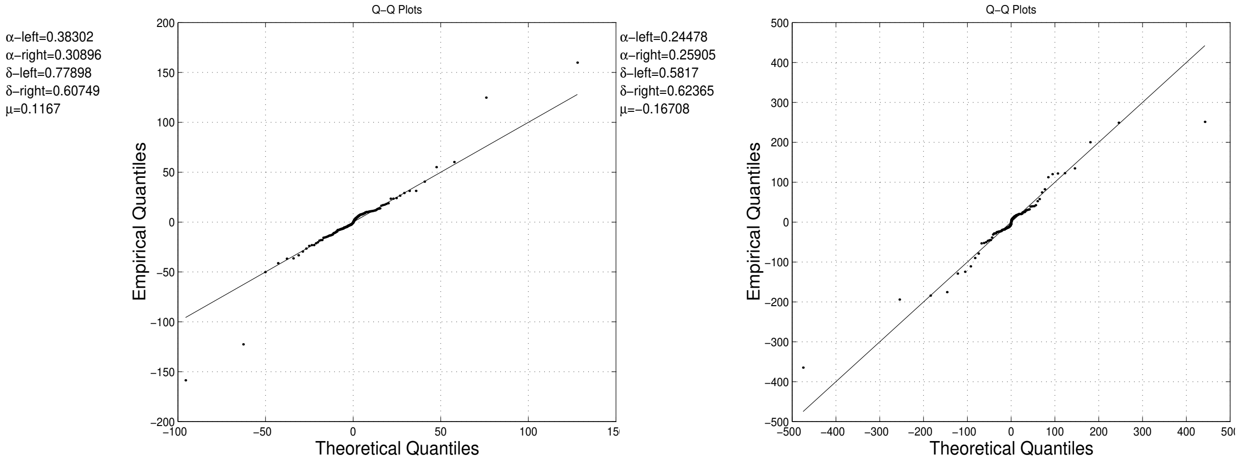

Figure 3: The Q-Q Plots of Quantile Functions for Class II Distributions: Cal-PX (left) vs. PJM Prices (right)

good Q-Q plot (right panel) when being fitted to a Class II distribution. However, the fact that α_- is greater than α_+ (see numbers in Figure 3) indicates a fatter right tail than the left tail in the distribution of Cal-PX price differences. This is however not consistent with our previous observation based on the fitting results to Class I distributions.

The fitted mixture distributions for Cal-PX and PJM data sets are plotted in Figure 4 along with the fitted distributions of Class I and Class II distributions for comparison purposes. In the left panel, we can see that the mixture distribution is at least as good as, if not better than, the Class II distribution in fitting Cal-PX price differences. Both the mixture distribution and the Class II distribution perform better than the Class I distribution. And, in the right panel for PJM data, it seems that the Class II distribution fits the empirical distribution best.

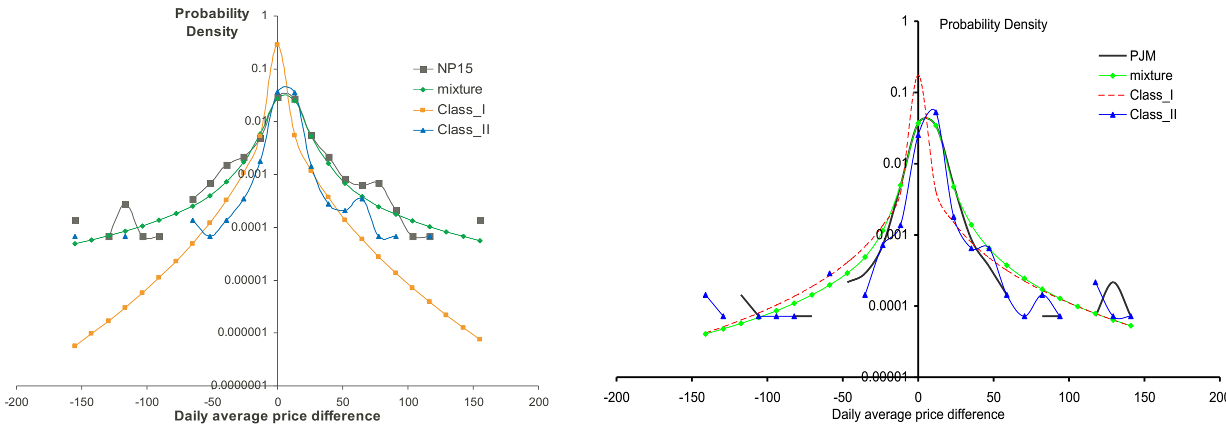

Figure 4: PDFs of Empirical Distributions and Fitted Distributions: Cal-PX (left) vs. PJM (right)

Finally, instead of reporting estimated tail index α's for the α-stable distributions, we note that Cal-PX's α's are smaller than PJM's. This implies that tails of Cal-PX are fatter. These results appear to be reasonable since there are more regulations and price caps in the PJM market resulting in a less volatile price behavior than that of the Cal-PX market. Moreover, estimated α values based on asymmetric α-stable distributions indicate that the left tails of power prices are fatter than the right tails.

5 Conclusion

By examining different models for marginal distributions using power prices as an illustrative example, we obtain some satisfactory fitting of the daily average commodity price series. Since the distributions that we employ for modeling marginal distributions have rich tail behaviors, our approach can effectively model the heavy tail behavior of commodity prices caused by jumps and stochastic volatility. The fact that these distributions are easy to simulate enables us to perform empirical parameter estimation. As for future research, we plan to investigate how we can, based on marginal distributions, infer the other essential parameters, λ_j's, of the underlying process (2) for dynamic hedging and other risk management applications.

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