AMO-Advanced Modeling and Optimization, Vol.3, Number 1, 2001

On Generalized Convexity and Duality in Multiobjective Programming Problems

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Abstract. For a multiobjective programming problems which involved inequality constraints. Wolfe, Mond-Weir duals are formulated and the concept of efficiency (Pareto optimum) is used to state some duality results under generalized convexity conditions.

Key Words. Multiobjective programming, sufficient solutions, duality, weak strictly pseudoconvexity, strong pseudoconvexity.

1. Introduction

In this paper, Our aim is to use the concept of efficiency (Pareto optimum) to formuled some results of duality under generalized convexity assymption for the following class of multiobjective programming problems:

$$(P) \quad \begin{cases} \text{minimize} & f(x) = (f_1(x), ..., f_p(x)) \\ \text{subject to} & g(x) \leqq 0, \end{cases}$$

where $f : I\!R^n \longrightarrow I\!R^p$ and $g : I\!R^n \longrightarrow I\!R^m$ are assumed to be differentiable.

Mond-Weir [8] have used weak efficiency to establish some duality results under different assymptions of convexity (pseudo convexity, strictly pseudoconvexity and quasiconvexity). Where Wolfe and Mond-Weir duals are considered.

Following Mond-Weir [8] we consider for problem (P) the Wolfe vector dual and Mond-Weir vector dual. By considering the concept of efficiency a number of weak, strong and converse duality theorems are given under generalized convexity conditions.

2. Preliminaries

Throughout this paper the following conventions for vectors in $\mathbb{R}^n$ will be followed:

$$\begin{aligned} x \leqq y \quad & \text{if and only if} \quad x_i \leq y_i, \quad i = 1, ..., n; \\ x \leq y \quad & \text{if and only if} \quad x \leqq y, \quad \text{and} \quad x \neq y; \\ x < y \quad & \text{if and only if} \quad x_i < y_i, \quad i = 1, ..., n. \end{aligned}$$

for the problem (P), a point x^0 is said to be efficient if there exists no other point x for which $f(x) \leq f(x^0)$.

The following Fritz John theorem [6] will be needed:

Theorem 2.1. Let x^0 be an efficient solution for (P) at which the constraint qualification is satisfied. Then there exists $u \in \mathbb{R}^p$, $v \in \mathbb{R}^m$ such that

$$\nabla f(x^0)^t u + \nabla g(x^0)^t v = 0 \tag{1}$$

$$v^t g(x^0) = 0 \tag{2}$$

$$u \geq 0 \tag{3}$$

$$v \geqq 0 \tag{4}$$

Now we consider a generalized convexity, namely weak strictly pseudoconvexity, an slight extension of the vector pseudoconvexity defined by I. Marusciac [6] and strong pseudoconvexity defined by B. Aghezzaf and M. Hachimi [1, 2].

Definition 2.1. Let $X \subseteq\in \mathbb{R}^n$ be an open set and let $f : X \longrightarrow \mathbb{R}^p$ be differentiable at $x^0 \in X$. Then f is said to be weak strictly pseudoconvex at $x^0 \in X$ if for any $x \in X$, $x \neq x^0$

$$f(x) \leq f(x^0) \quad \Rightarrow \quad (\nabla f(x^0))(x - x^0) < 0. \tag{5}$$

This class of functions does not contain the class of convex functions, but does contain the class of strictly convex functions.

Definition 2.2. Let $X \subseteq\in \mathbb{R}^n$ be an open set and let $f : X \longrightarrow \mathbb{R}^p$ be differentiable at $x^0 \in X$. Then f is said to be strong pseudoconvex at $x^0 \in X$ if for any $x \in X$,

$$f(x) \leq f(x^0) \quad \Rightarrow \quad (\nabla f(x^0))(x - x^0) \leq 0. \tag{6}$$

Moreover, unlike the class of weak strictly pseudoconvex functions, the class of strong pseudoconvex functions does contain both the class of convex and weak strictly pseudoconvex functions.

3. Wolfe Vector Duality

In this section we consider weak and strong duality between (P) and the following Wolfe vector dual [9]

$$
\begin{aligned}
(WVD) \qquad & \text{maximize} f(y) + v^t g(y)e, \\
& \text{subject to } (\nabla f(y))^t u + (\nabla g(y))^t v = 0, && (7)\\
& v \geqq 0, && (8)\\
& u \geqq 0, && (9)\\
& u^t e = 1; && (10)
\end{aligned}
$$

where $e = (1, 1, ..., 1)^t \in I\!R^p$.

Theorem 3.1 (Weak Duality) If for all (x, y, u, v),
a) $f(.) + v^t g(.)e$ is weak strictly pseudoconvex; or
b) $f(.) + v^t g(.)e$ is strong pseudoconvex,
Then, the following cannot hold:

$$f(x) \leq f(y) + v^t g(y)e. \tag{11}$$

Proof. a) Let x be feasible for (P) and (y, u, v) feasible for (WVD). Suppose $f(x) \leq f(y) + v^t g(y)e$ that

$$f_j(x) \leqq f_j(y) + v^t g(y), \quad \text{for all } j \tag{12}$$

$$f_i(x) < f_i(y) + v^t g(y), \quad \text{for some } i \tag{13}$$

then

$$f_j(x) + v^t g(x) \leqq f_j(y) + v^t g(y), \quad \text{for all } j \tag{14}$$

$$f_i(x) + v^t g(x) < f_i(y) + v^t g(y), \quad \text{for some } i \tag{15}$$

the weak strictly pseudoconvexity of $f(.) + v^t g(.)e$ implies that

$$(x - y)^t \nabla \{f(y) + v^t g(y)e\} < 0 \tag{16}$$

and hence

$$(x - y)^t \{\nabla f(y)^t u + \nabla g(y)^t v\} < 0 \tag{17}$$

which contradicts the constraint (7) of (WVD).
b) Let Let x be feasible for (P) and (y, u, v) feasible for (WVD). Suppose $f(x) \leq f(y) + v^t g(y)e$ that is

$$f_j(x) \leqq f_j(y) + v^t g(y), \quad \text{for all } j \tag{18}$$

$$f_i(x) < f_i(y) + v^t g(y), \quad \text{for some } i \tag{19}$$

then

$$f_j(x) + v^t g(x) \leqq f_j(y) + v^t g(y), \quad \text{for all } j \tag{20}$$

$$f_i(x) + v^t g(x) < f_i(y) + v^t g(y), \quad \text{for some } i \tag{21}$$

the strong pseudoconvexity of $f(.) + v^t g(.)e$ implies that

$$(x - y)^t \nabla \{f(y) + v^t g(y)e\} \leq 0 \tag{22}$$

and hence

$$(x - y)^t \{\nabla f(y)^t u + \nabla g(y)^t v\} < 0 \tag{23}$$

which contradicts the constraint (7) of (WVD) because $\nabla f(y))^t u + (\nabla g(y))^t v = 0$. □

Theorem 3.2 (Strong Duality) Let x^0 be an efficient solution for (P) at which the constraint qualification [6] is satisfied. Then there exist (u, v) such that (x^0, u, v) is feasible for (WVD) and the objective value of (P) and (WVD) are equal. If, also
a) $f(.) + v^t g(.)e$ is weak strictly pseudoconvex; or
b) $f(.) + v^t g(.)e$ is strong pseudoconvex,
then, (x^0, u, v) is an efficient solution for (WVD).

Proof. Since x^0 is an efficient solution for (P) at which the constrait qualification [6] is satisfied, then by theorem 2.1, there exist $u \geq$, $v \geqq 0$ such that

$$\nabla f(x^0)^t u + \nabla g(x^0)^t v = 0, \quad v^t g(x^0) = 0.$$

The vector u may be normalized according $u^t e = 1$, $u \geqq 0$, thus (x^0, u, v) is feasible for (WVD) and clearly the objective values of (P) and (WVD) are equal.
If (x^0, u, v) is not efficient for (WVD) then there exists feasible $(\bar{x}, \bar{u}, \bar{v})$ for (WVD) such that

$$f(\bar{x}) + \bar{v}^t g(\bar{x})e \geq f(x^0) + \bar{v}^t g(x^0)e$$

a) Since $f(.) + v^t g(.)e$ is weak strictly pseudoconvex, $(x^0 - \bar{x})^t \nabla \{f(\bar{x}) + \bar{v}^t g(\bar{x})e\} < 0$ thus $(x^0 - \bar{x})^t \{\nabla f(\bar{x})^t \bar{u} + \nabla g(\bar{x})^t \bar{v}\} < 0$ contradicting the feasibility of $(\bar{x}, \bar{u}, \bar{v})$. Thus (x^0, u, v) is an efficient solution for (WVD).
b) Since $f(.) + v^t g(.)e$ is strong pseudoconvex, it follows that
$(x^0 - \bar{x})^t \nabla \{f(\bar{x}) + \bar{v}^t g(\bar{x})e\} \leq 0$ thus $(x^0 - \bar{x})^t \{\nabla f(\bar{x})^t \bar{u} + \nabla g(\bar{x})^t \bar{v}\} < 0$ contradicting the feasibility of $(\bar{x}, \bar{u}, \bar{v})$. Hence (x^0, u, v) is efficient for (WVD). □

4. Mond-Weir Vector Duality

Here we establish various duality theorems for Mond-Weir dual [7, 8] for problem (P) suggested by Egudo [4] .

(DP) maximize$f(y)$,

$$\begin{aligned} \text{subject to } \ (\nabla f(y))^t u + (\nabla g(y))^t v &= 0, && (24)\\ v^t g(y) &\geqq 0, && (25)\\ v &\geqq 0, && (26)\\ u &\geqq 0, && (27)\\ u^t e &= 1; && (28) \end{aligned}$$

where $e = (1, 1, ..., 1)^t \in I\!R^p$.

Theorem 4.1. If, for all feasible (x, y, u, v)
a) f is weak strictly pseudoconvex and $v^t g$ is quasiconvex ; or
b) f is strong pseudoconvex and $v^t g$ is strictly pseudoconvex,
then the following cannot hold:

$$f(x) \leq f(y). \tag{29}$$

Proof. a) Let x be feasible for (P) and (y, u, v) feasible for (DP). Suppose $f(x) \leq f(y)$. By weak strictly pseudoconvexity of f

$$(x - y)^t \nabla f(y) < 0.$$

Since $u \geq 0$,

$$(x - y)^t \nabla f(y)^t u < 0. \tag{30}$$

Since $v^t g(x) - v^t g(y) \leqq 0$, the quasiconvexity of $v^t g$ implies that

$$(x - y)^t \nabla g(y)^t v \leqq 0. \tag{31}$$

$$\tag{32}$$

Combining (30) and (32) gives

$$(x - y)^t \{\nabla f(y)^t u + \nabla g(y)^t v\} < 0,$$

which contradicts the constraint (24) of (DP).
b) Let x be feasible for (P) and (y, u, v) feasible for (DP). Suppose $f(x) \leq f(y)$. The strong pseudoconvexity of f implies that

$$(x - y)^t \nabla f(y) \leq 0.$$

Since $u \geq 0$,

$$(x - y)^t \nabla f(y)^t u \leqq 0.$$

By (24)

$$(x - y)^t \nabla v^t g(y) \geqq 0. \tag{33}$$

and since $v^t g$ is strictly pseudoconvex

$$v^t g(x) > v^t g(y),$$

which is a contradiction since $v^t g(x) \leqq 0$ and $v^t g(y) \geqq 0$ □

Theorem 4.2 (Strong Duality) Let x^0 be an efficient solution for (P) at which the constraint qualification [6] is satisfied. Then there exist (u, v) such that (x^0, u, v) is feasible for (DP) and the objective values of (P) and (DP) are equal. If, also
a) f is weak strictly pseudoconvex and $v^t g$ is quasiconvex; or
b) f is strong pseudoconvex and $v^t g$ is strictly pseudoconvex,
then, (x^0, u, v) is an efficient solution for (DP).

Proof. Since x^0 is an efficient solution for (P) at which the constrait qualification [6] is satisfied, then by theorem 2.1, there exist $u \geq$, $v \geqq 0$ such that

$$\nabla f(x^0)^t u + \nabla g(x^0)^t v = 0, \quad v^t g(x^0) = 0.$$

The vector u may be normalized according $u^t e = 1$, $u \geqq 0$, thus (x^0, u, v) is feasible for (DP) and clearly the objective values of (P) and (DP) are equal.
If (x^0, u, v) is not efficient for (DP) then there exists feasible $(\bar{x}, \bar{u}, \bar{v})$ for (DP) such that

$$f(\bar{x}) \geq f(x^0)$$

a) Since f is weak strictly pseudoconvex, $(x^0 - \bar{x})^t \nabla f(\bar{x}) < 0$. Thus, since $\bar{u} \geq 0$,

$$(x^0 - \bar{x})^t \nabla f(\bar{x})^t \bar{u} < 0. \tag{34}$$

Also $\bar{v}^t g(x^0) - \bar{v}^t g(\bar{x}) \leqq 0$ and since $\bar{v}^t g$ is quasiconvex

$$(x^0 - \bar{x})^t \nabla \bar{v}^t g(\bar{x})^t \leqq 0. \tag{35}$$

Combining (34) and (35) gives

$$(x^0 - \bar{x})^t \{\nabla f(\bar{x})^t \bar{u} + \nabla g(\bar{x})^t \bar{v}\} < 0$$

which contradicts the feasibility of $(\bar{x}, \bar{u}, \bar{v})$.
b) Since f is strong pseudoconvex,

$$(x^0 - \bar{x})^t \nabla f(\bar{x}) \leq 0$$

Thus, since $\bar{u} \geq 0$,

$$(x^0 - \bar{x})^t \nabla f(\bar{x})^t \bar{u} \leqq 0.$$

By (24)

$$(x^0 - \bar{x})^t \nabla \bar{v}^t g(\bar{x})^t \leqq 0,$$

and since $\bar{v}^t g$ is strictly pseudoconvex

$$\bar{v}^t g(x^0) > \bar{v}^t g(\bar{x}).$$

This is a contradiction as $\bar{v}^t g(x^0) \leqq 0$ and $\bar{v}^t g(\bar{x}) \geqq 0$. □

5. Converse Duality

Theorem 5.1. Let (x^0, u^0, v^0) be an efficient solution for (DP). Assume the $n \times n$ Hessian matrix $\nabla^2[u^{0t}f(x^0) + v^{0t}g(x^0)]$ is negative-definite and $\nabla v^{0t}g(x^0) \neq 0$. If for all (x, y, u, v)
a) f is weak strictly pseudoconvex and $v^t g$ is quasiconvex; or
b) f is strong pseudoconvex and $v^t g$ is strictly pseudoconvex,
then, x^0 is an efficient solution for (P).

Proof. Since (x^0, u^0, v^0) is an efficient solution for (DP), then by Fitz John theorem [3] there exists $\tau \in I\!R^m$, $\lambda \in I\!R^n$, $\alpha \in I\!R$, $\nu \in I\!R^m$, and $\mu \in I\!R^p$, such that

$$
\begin{aligned}
-(\nabla f(x^0))^t\tau + \nabla^2[u^{0t}f(x^0) + v^{0t}g(x^0)]\lambda & \\
-\alpha\nabla v^{0t}g(x^0) = 0, & (36)\\
\nabla f(x^0)\lambda - \mu = 0, & (37)\\
\nabla g(x^0)\lambda - \alpha g(x^0) - \nu = 0, & (38)\\
\alpha v^{0t}g(x^0) = 0, & (39)\\
\nu^t v^0 = 0, & (40)\\
\mu^t u^0 = 0, & (41)\\
(\tau, \alpha, \nu, \mu) \geqq 0. & (42)\\
(\tau, \lambda, \alpha, \nu, \mu) \neq 0. & (43)
\end{aligned}
$$

Multiplying (37) by u^{0t} and using (41), multiplying (38) by v^{0t} and using (40)and (39), we obtain

$$
\begin{aligned}
u^{0t}(\nabla f(x^0))^t\lambda = 0, & (44)\\
v^{0t}(\nabla g(x^0))^t\lambda = 0. & (45)
\end{aligned}
$$

Premultiplying (36) by λ^t and using (45), we have

$$
-\lambda^t(\nabla f(x^0))^t\tau + \lambda^t\nabla^2[u^{0t}f(x^0) + v^{0t}g(x^0)] = 0, \quad (46)
$$

or

$$
-\mu^t\tau + \lambda^t\nabla^2[u^{0t}f(x^0) + v^{0t}g(x^0)]\lambda = 0. \quad (47)
$$

We now claim that

$$
\tau \neq 0. \quad (48)
$$

Otherwise, from (47), we have

$$
\lambda^t\nabla^2[u^{0t}f(x^0) + v^{0t}g(x^0)]\lambda = 0. \quad (49)
$$

Since $\nabla^2[u^{0t}f(x^0)+v^{0t}g(x^0)]$ is assumed negative-definite, $\lambda=0$. Therefore, from (36) we have

$$-\alpha\nabla v^{0t}g(x^0)=0. \tag{50}$$

Using the fact that $\nabla v^{0t}g(x^0)\neq 0$, from (50) we obtain $\alpha=0$, from (37), $\mu=0$ and from (38), $\nu=0$ contradicting (43). Hence, (48) holds.
From (47) and using (42) we obtain

$$\lambda^t\nabla^2[u^{0t}f(x^0)+v^{0t}g(x^0)]\lambda=\mu^t\tau\geqq 0. \tag{51}$$

Since $\nabla^2[u^{0t}f(x^0)+v^{0t}g(x^0)]$ is assumed negative-definite, $\mu^t\tau=0$. Hence $\lambda=0$; therefore, from (36) we have

$$(\nabla f(x^0))^t\tau=-\alpha(\nabla g(x^0))^t v^0, \tag{52}$$

From (24), we have

$$-(\nabla f(x^0))^t u^0=(\nabla g(x^0))^t v^0. \tag{53}$$

From (52) and (53), we get

$$(\tau^t-\alpha u^{0^t})(\nabla g(x^0))^t v^0=0. \tag{54}$$

Using the hypothesis that, the vector $\nabla v^{0t}g(x^0)\neq 0$, from (54) we get

$$\tau=\alpha u^0. \tag{55}$$

$$\tag{56}$$

Hence $\alpha\neq 0$, because $\tau\neq 0$. Using (55) with $\lambda=0$ in (38), we obtain

$$g(x^0)\leqq 0, \tag{57}$$

$$\tag{58}$$

which shows that x^0 is feasible for (P). Therefore, using theorem 4.1 and the hypothesis a) , b), we have that x^0 is efficient solution for (P). □

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