Symmetry properties of distribution of Riemann Zeta Function values on critical axis

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Abstract

The question of the values of Hardy's Z-function at a discrete sequence of points on the critical axis is quite interesting. Some results in this direction can be proved rigorously. At the same time, the present state of Riemann zeta function theory gives limited information about the value distribution of $Z(t)$ at discrete sequences of points. Moreover, possibly these analogues could differ significantly from the continuous case. To investigate the question we perform a numerical study of the value distribution of Hardy's Z-function at generalized Gram points.

Keywords: Riemann zeta, Value Distribution, Symmetry, Hardy's function
Mathematics Subject Classification (MSC): 11M06, 11-04.

1 Introduction

In this work we discuss the symmetry properties of the distribution of Riemann zeta values on the critical axis. The study of the symmetry properties should give us insight into the theoretical underpinnings of the location of zeros of the Riemann Zeta Function. The new results presented in this work are:

1. An anti-symmetry relation Equation 2.6,
2. A symmetry relation Equation 2.7.

Given that these conjectures express fundamental properties of the behavior of the zeta function, it seems surprising that they have not been commented on in the literature. Our current knowledge of the theory of the Riemann zeta function doesn't seem to help us understand the theoretical basis for the symmetries. The new results are stated in Section 2.1, after we set up the required notation. There has been much study on the value distribution of the

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Riemann zeta function in the literature. In this work we will cite the references which have some bearing on the current results.

The paper is organized as follows. Section 2 establishes the required notation for the Riemann Zeta Function and presents the conjectured symmetry relations. Section 3 provides evidence for the symmetry relations. In Section 4 we discuss related work in the literature. Section 5 presents the conclusions. In the Appendix we discuss the nature of the probability distribution a little more in depth.

2 Notation for the Riemann zeta function and the symmetry properties

In this section we establish the required notation for the Riemann Zeta Function. We follow closely the treatment in Ref. [1]. For $\mathrm{Re}(s) > 1$ the Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in primes} \left(1 - p^{-s}\right)^{-1}. \tag{2.1}$$

$\zeta(s)$ can be continued to the complex plane. The mean spacing δ of the zeros at height t is $\delta = 2\pi(\ln(t/2\pi))^{-1}$. For numerical studies of the Riemann hypothesis one defines Hardy's function [6]

$$Z(t) = exp(i\theta(t))\zeta(1/2 + it) \tag{2.2}$$

where

$$\theta(t) = arg(\pi^{it/2}\Gamma(\frac{1}{4} + \frac{it}{2})). \tag{2.3}$$

The argument in Eq. 2.3 is defined by continuous variation of t starting with the value 0 at $t = 0$. $Z(t)$ is real valued for real t, and we have $|Z(t)| = |\zeta(1/2 + it)|$. Thus the zeros of $Z(t)$ are the imaginary part of the zeros of $\zeta(s)$ which lie on the critical line.

Many of the zeros are separated by the Gram points [8]. When $t \geq 7$, the θ function Eq.(2.3) is monotonic increasing. For $n \geq -1$, the n-th Gram point g_n is defined as the unique solution > 7 to $\theta(g_n) = n\pi$. A Gram interval is the interval $G_n = [g_n, g_{n+1})$. Ref. [1] studied empirically the distribution of $Z(t)$ values at Gram points. We will generalize the study to other points on the critical axis, as defined below. In analogy with Gram points, we can associate an angle ϕ with a point t on the critical axis as follows:

Definition 2.1. For $t \geq 7$, t is said to be a generalized Gram point with value phi if $\theta(t) = 2k\pi + \phi$, where $0 \leq \phi < 2\pi$.

We will study the distribution of $Z(t)$ values at a given value of ϕ, and investigate symmetry and anti-symmetry relations for the distributions at different values of ϕ. The sample space for all the distributions in this work is an interval at height t which is large compared to the Gram interval, but small enough that $\ln(t)$ can be considered to be effectively constant over the interval. The latter condition is not essential but is convenient, in that it simplifies the numerical work. The notation $\ln(t)$ stands for the natural logarithm of t. We assume the existence of a probability distribution function for $Z(t)$ at generalized Gram points, $p_\phi(y)$:

Definition 2.2.

$$\int_a^b p_\phi(y)dy \tag{2.4}$$

is the probability that $a < Z(t) < b$ when we consider the values of $Z(t)$ for a large number of generalized Gram points in the sample space.

The probability density $p_\phi(y)$ depends on the sample space (i.e., on the height t and on the size of the sample space). In practice the densities are not sensitive to the choice of the sample space as long as the height t is large enough and the length of the interval from which the sample is collected is large enough (but not too large on log scale). The emphasis in this work is on the empirical study of the distribution. There are important open theoretical questions about the distribution that we do not cover, and we mention those questions in the Appendix. In particular, Laurincikas [7] showed that $|\zeta(1/2 + it)|$ possesses a limiting distribution only if one does a power norming (see section 3.3 of Ref. [7]). This makes the regularities that we observe all the more remarkable (a symmetry which manifests itself very clearly, but which may be broken at a larger scale, leads to quite deep results, as the analogy with the weak interactions in physics illustrates nicely).

2.1 Conjectures

Our empirical study led us to three results, stated in Equations 2.5, 2.6 and 2.7. After the first draft of this work was written up, we discovered that Equation 2.5 had been stated and proved in Ref. [10]. Thus, our new results are Equations 2.6 and 2.7. Our first empirical result is that the mean value $\langle Z(t_\phi)\rangle$ is given by

$$\langle Z(t_\phi)\rangle = 2\cos(\phi). \tag{2.5}$$

Equation 2.5 was explicitly called out for Gram points ($\phi = 0$ and $\phi = \pi$) in Ref. [9]. It has been proved for all ϕ points in Ref. [10].

The second empirical result, and our first new conjecture, is the anti-symmetry condition

$$p_\phi(y) = p_{\phi+\pi}(-y). \tag{2.6}$$

Equation 2.6 is a generalization of the result in Ref. [1] for the anti-symmetry of the distribution of $Z(t)$ at even and odd Gram points. To relate this equation with the work in Ref. [1], we note that an odd Gram point has $\phi = \pi$ and an even Gram point has $\phi = 0$.

The third empirical result, and our second new conjecture, is the symmetry condition

$$p_{\phi}(y) = p_{2\pi-\phi}(y). \tag{2.7}$$

The relations are deceptively simple to state. However, to the best of the author's knowledge they do not seem to have been remarked upon in the literature. Understanding the cause for these properties will give us new knowledge about the zeta function theory. The empirical evidence for these generalizations is given in the next section.

3 Evidence for the symmetry relations

For the empirical validation of the mean value condition Equation 2.5, the anti-symmetry condition Equation 2.6 and the symmetry condition Equation 2.7 we have to evaluate $Z(t)$ at several points on the critical axis. We chose our sample space to be 10^6 Gram intervals starting at $t = 10^{12} + 243.777560$ (which is a Gram point with Gram index 3945951431271). We chose this sample space because Ref [11] has evaluated all the zeros in this range. We used the zeros from Ref [11] to check the accuracy of our zeta function calculations. Our evaluations of $Z(t)$ are accurate to better than 10^{-6}.

3.1 Numerical evaluation

Hardy's function $Z(t)$ is evaluated using the Riemann−Siegel series

$$Z(t) = 2\sum_{n=1}^{m} \frac{\cos(\theta(t) - t\ln(n))}{\sqrt{n}} + R(t), \tag{3.1}$$

where m is the integer part of $\sqrt{t/(2\pi)}$. $R(t)$ is a small remainder term which can be evaluated to the desired level of accuracy. We used the techniques in Refs. [12, 13, 14] to efficiently evaluate the zeta function at large t. The most important source for loss of accuracy at large heights is the cancellation between large numbers that occur in the arguments of the cos terms in Eq. (3.1). We use a high precision module to evaluate the arguments. The rest of the calculation is done using regular double precision accuracy.

3.2 Quantiles

Table 1 shows the quantiles and mean for $Z(t)$ when ϕ values are multiples of $\pi/4$. The column showing the mean of the distribution clearly validates the mean value condition Equation 2.5.

The anti-symmetry condition Equation 2.6 can be checked, for example, by validating that rows 2 and 6 of the table have anti-symmetric distribution parameters (row 2 has $\phi = \pi/4$ while row 6 has $\phi = 5\pi/4$).

The symmetry condition Equation 2.7 can be checked by verifying, for example, that rows 2 and 8 of the table have the same distribution parameters

ϕ	$Min.$	$1st$ $Quantile$	$Median$	$Mean$	$3rd$ $Quantile.$	$Max.$
0	−69	0.1134	0.8517	2.0001	2.5403	165
$\pi/4$	−97	−0.1352	0.5916	1.4143	2.1226	159
$\pi/2$	−121	−1.082	0.0017	0.0001	1.089	137
$3\pi/4$	−148	−2.1121	−0.5852	−1.4141	0.1364	103
π	−161	−2.5277	−0.8467	−1.9999	−0.1122	69
$5\pi/4$	−153	−2.1045	−0.5891	−1.4141	0.1365	105
$3\pi/2$	−129	−1.083	−0.001	0.0001	1.084	138
$7\pi/4$	−93	−0.1336	0.5883	1.4143	2.1077	160

Table 1: Quantiles and mean for $Z(t)$ when ϕ values are multiples of $\pi/4$.

ϕ	$Min.$	$1st$ $Quantile$	$Median$	$Mean$	$3rd$ $Quantile.$	$Max.$
0	−69	0.1134	0.8517	2.0001	2.5403	165
$\pi/6$	−88	0.0167	0.7318	1.7321	2.3515	162
$\pi/3$	−105	−0.3877	0.4108	1.0001	1.8151	154
$\pi/2$	−121	−1.082	0.0017	0.0001	1.0888	137
$2\pi/3$	−140	−1.81	−0.4053	−0.9999	0.3904	115
$5\pi/6$	−155	−2.338	−0.7258	−1.732	−0.0160	91
π	−161	−2.5277	−0.8467	−1.9999	−0.1122	69
$7\pi/6$	−158	−2.334	−0.7294	−1.732	−0.0161	93
$4\pi/3$	−147	−1.8034	−0.4097	−0.9999	0.388	117
$3\pi/2$	−129	−1.083	−0.001	0.0001	1.0842	138
$5\pi/3$	−105	−0.3855	0.4086	1.0001	1.8103	154
$11\pi/6$	−80	0.0184	0.7317	1.7322	2.3393	164

Table 2: Quantiles and mean for $Z(t)$ when ϕ values are multiples of $\pi/6$.

ϕ	skewness	kurtosis
0	4.95	63.57
$\pi/4$	3.52	56.25
$\pi/2$	0.11	47.07
$3\pi/4$	-3.28	51.73
π	-4.68	56.93
$5\pi/4$	-3.26	49.18
$3\pi/2$	0.16	43.68
$7\pi/4$	3.57	54.10

Table 3: Skewness and kurtosis when ϕ values are multiples of $\pi/4$.

ϕ	skewness	kurtosis
0	4.95	63.57
$\pi/6$	4.29	60.15
$\pi/3$	2.52	52.13
$\pi/2$	0.11	47.07
$2\pi/3$	-2.29	48.96
$5\pi/6$	-4.04	54.56
π	-4.68	56.93
$7\pi/6$	-4.03	52.74
$4\pi/3$	-2.26	45.88
$3\pi/2$	0.16	43.68
$5\pi/3$	2.57	49.40
$11\pi/6$	4.33	58.69

Table 4: Skewness and kurtosis when ϕ values are multiples of $\pi/6$.

(row 2 has $\phi = \pi/4$ while row 8 has $\phi = 7\pi/4$). Another check of Equation 2.7 is that rows 4 and 6 of the table have the same distribution parameters (row 4 has $\phi = 3\pi/4$ while row 6 has $\phi = 5\pi/4$).

Table 2 shows the quantiles and mean for $Z(t)$ when ϕ values are multiples of $\pi/6$. These values also bear out the validity of Equations 2.5, 2.6 and 2.7. Repeated application of Equation 2.6 and Equation 2.7 gives us the relation between the distributions at ϕ, $\pi - \phi$, $\pi + \phi$ and $2\pi - \phi$. A mnemonic for remembering the relations is that the distributions are symmetric for reflections around Gram points ($\phi = 0$ and $\phi = \pi$) and anti-symmetric for reflections around the mid-points of Gram intervals ($\phi = \pi/2$ and $\phi = 3\pi/2$).

3.3 Skewness and kurtosis

Table 3 shows the skewness and kurtosis for $Z(t)$ when ϕ values are multiples of $\pi/4$. Table 4 shows the skewness and kurtosis for $Z(t)$ when ϕ values are multiples of $\pi/6$.

The tables show that the distributions are symmetric for reflections around Gram points ($\phi = 0$ and $\phi = \pi$) and anti-symmetric for reflections around the mid-points of Gram intervals ($\phi = \pi/2$ and $\phi = 3\pi/2$). The anti-symmetry condition Equation 2.6 can be checked, for example, by validating that rows 2 and 6 ($\phi = \pi/4$ and $\phi = 5\pi/4$) of Table 3 have anti-symmetric distribution parameters to within the statistical precision. The symmetry condition Equation 2.7 can be checked by verifying, for example, that rows 2 and 8 of the table ($\phi = \pi/4$ and $\phi = 7\pi/4$) have the same distribution parameters within statistical precision. Rows 3 and 7 ($\phi = \pi/2$ and $\phi = 3\pi/2$) of Table 3 are interesting, in that they are related by both Equation 2.7 and Equation 2.6.

The large values of kurtosis show that the distributions tend to have heavy tails, or outliers. Because of these heavy tails, the higher moments of the distribution function are likely to diverge logarithmically (we already saw that the mean is finite and well-behaved). The distributions also have large skew, except for $\phi = \pi/2$ and $\phi = 3\pi/2$, where the distributions are symmetric. For the symmetric case the odd moments of the distribution function will be zero.

3.4 Histograms

Ref. [1] presented figures for the histograms at even and odd Gram points, which are the histograms for $\phi = 0$ and $\phi = \pi$ respectively. Figs. 1 and 2 show the histograms for $\phi = \pi/2$ and $\phi = 3\pi/2$ respectively. Equation 2.7 predicts that the histograms have the same distribution. The figures validate this prediction to within the statistical precision.

Further, Equation 2.6 predicts that the histograms are mirror images of each other. For both relations to be valid, the distributions have to be symmetric. The figures also validate this fact. While the histograms are symmetric, the distribution is not normal. This can be seen from the kurtosis values in Table 3 for $\phi = \pi/2$ and $\phi = 3\pi/2$.

4 Related work

The value distribution of the Riemann zeta function has been well studied in the literature. We cannot cover all of this in the current work. We refer to Ivić's monograph [15] for a comprehensive survey of the field. Selberg in unpublished work showed that at large t $\log(\zeta(\frac{1}{2}+it))$ is approximately normally distributed with a standard deviation of order $\sqrt{log\ log\ t}$ (see Ref. [16]). He showed a similar result [17, 18] for $\log(|Z(t)|)$. Laurincikas [7] used probabilistic number theory to prove various results about the distribution of the Riemann zeta function.

Regarding the value distribution at specific points along the Gram interval, we have already mentioned the work of Titchmarsh [9] and Kalpokas and Steuding [10] pertaining to the mean value of the Riemann zeta function. Lester's [19] Ph. D. thesis also considers the distribution of $\log(|\zeta(\frac{1}{2}+it)|)$ for specific points along the Gram interval. The anti-symmetry condition Equation 2.6 and the symmetry condition Equation 2.7 of the current manuscript seem to be new.

5 Conclusions

In this work we defined generalized Gram points on the critical axis, in analogy with Gram points. We used this definition to formulate two new conjectures about the distribution of $Z(t)$ on the critical axis. We empirically observed the mean value condition Equation 2.5. This was stated and proved for Gram points by Ref. [9] and, as we discovered later, for all ϕ in Ref. [10]. Our two new results are the anti-symmetry condition Equation 2.6 and the symmetry condition Equation 2.7. Equation 2.6 is a generalization of Conjecture 1 in Ref. [1] to generalized Gram points. The relations seem to hint at profound new

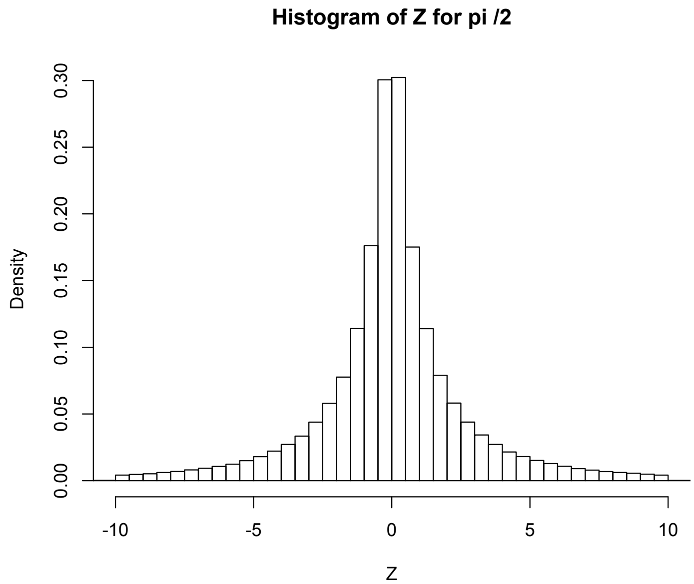

Figure 1: Distribution of zeta values at $\phi = \pi/2$.

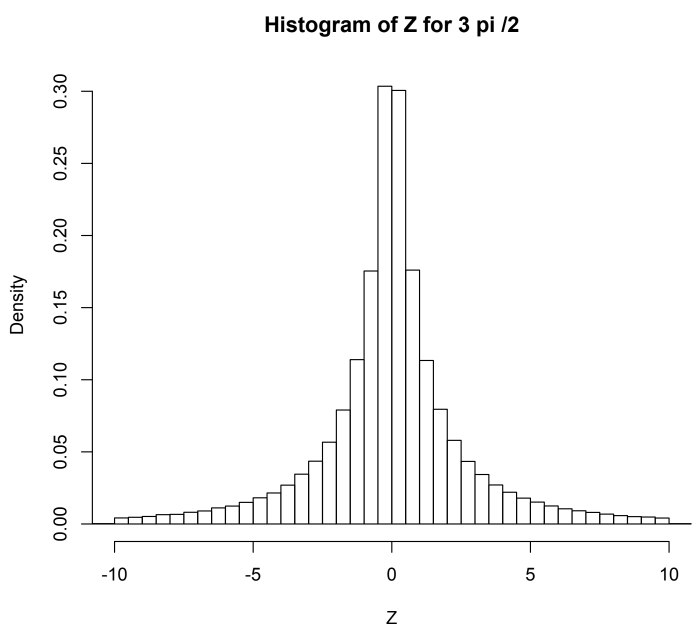

Figure 2: Distribution of zeta values at $\phi = 3\pi/2$.

ϕ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$
Standard deviation	4.998	5.002	5.004	5.004	5.000	4.996	4.991	4.987	4.985	4.986	4.989	4.993

Table 5: Standard deviation for $Z(t)$ when ϕ values are multiples of $\pi/6$.

aspects of the behavior of the zeta function which have not yet been discovered. Theoretical study of the relations presented here needs to be done.

Appendix

A more precise definition for the function $p_\phi(y)$ is provided. We consider the interval along the critical axis specified by (T_1, T_2). While empirical studies necessarily use large but finite T_1, T_2, we are interested in the limit $T_1 \to \infty, T_2 \to \infty, T_2 - T_1 \to \infty$, however

$$T_2 - T_1 \ll T_1. \tag{5.1}$$

Because of Equation 5.1, we can consider $\ln(t)$ to be effectively constant over the interval. A probability space W is defined by a sample space of elementary events Ω, a σ algebra of all considered events $\mathcal{F}$, and a probability measure P. Our space of elementary events Ω is the set of all generalized Gram points with value phi in the interval. This is easily seen to be a discrete space. If we denote $\theta(T_1) = 2k_1\pi + \phi$, $\theta(T_2) = 2k_2\pi + \phi$, then we can index the set of generalized Gram points in the interval by the integer k, where k lies in the interval (k_1, k_2). Since the mean spacing δ of the zeros at height t is $\delta = 2\pi(\ln(t/2\pi))^{-1}$, the cardinality of this set is $(T_2 - T_1) * (\ln(T_1/2\pi))/(2\pi)$. Since we are dealing with a discrete space, we follow standard practice and choose $\mathcal{F}$, the σ algebra, to be the collection of all subsets of Ω . Finally, the random variable whose probability distribution we wish to study is the value of $Z(t)$ at the generalized Gram point t.

It is not known whether the limiting probability distribution exists for $Z(t)$ at generalized Gram points. However, we can get some insight from the studies of Kalpokas and Steuding [10]. They show that for ϕ_1 in the range $[0, \pi)$

$$\sum_{0<t<T, \zeta(\frac{1}{2}+it) \in e^{i\phi_1}\mathbb{R}} \zeta(\frac{1}{2} + it) = (2e^{i\phi_1} \cos(\phi_1)) \frac{T}{2\pi} \ln(\frac{T}{2\pi}) + O_\epsilon(T^{(\frac{1}{2}+\epsilon}). \tag{5.2}$$

From Equation 5.2 and the cardinality of our sample space, it follows that a finite mean can be defined for the value distribution in our sample space. This gives support to the existence of a limiting distribution. We now consider the possibility of defining higher order moments for the value distribution. Kalpokas and Steuding also show that

$$\sum_{0<t<T, \zeta(\frac{1}{2}+it) \in e^{i\phi_1}\mathbb{R}} |\zeta(\frac{1}{2}+it)|^2 = \frac{T}{2\pi}(\ln(\frac{T}{2\pi}))^2 + (2\gamma + 2\cos(2\phi_1))\frac{T}{2\pi}\ln(\frac{T}{2\pi}) + \frac{T}{2\pi} + O_\epsilon(T^{(\frac{1}{2}+\epsilon}), \tag{5.3}$$

where γ is Euler's constant. From Equation 5.3 and the cardinality of our sample space it seems that higher order moments of the value distribution diverge logarithmically. There is a lot of cancellation between positive and negative values of $Z(T)$ for odd moments, so odd moments will behave better than even moments. It seems likely that a limiting value distribution exists, it has a finite well-defined mean, and higher moments diverge logarithmically. Distributions which do not have all moments defined are used in applications, for example, the Cauchy-Lorentz distribution [20]. Equation 5.2 and 5.3 predict that the standard deviation should be independent of ϕ. This is verified in Table 5.

References

[1] O. Shanker, "Good to Bad Gram Point Ratio For Riemann Zeta Function", *Experimental Mathematics* **doi:10.1080/10586458.2018.1492474**(2018)

[2] B. Riemann, "Über die Anzahl der Primzahlen uter Einer Gegebenen Gröbe," *Montasb. der Berliner Akad.*, (1858), 671-680

[3] B. Riemann, "Gesammelte Werke", Teubner, Leipzig, (1892)

[4] E. Titchmarsh, "The Theory of the Riemann Zeta Function," Oxford University Press, Second Edition, (1986)

[5] H. M. Edwards, "Riemann's Zeta Function," Academic Press, (1974)

[6] G. H. Hardy and J. E. Littlewood, "Contributions to the theory of the Riemann zeta-function and the distribution of primes", *Acta Math.* **41**(1918), 119-196

[7] A. Laurincikas, "Limit Theorems for the Riemann Zeta-Function", Kluwer, Dordrecht, 1996

[8] J. P. Gram, "Sur les Zeros de la Fonction $\zeta(s)$ de Riemann", *Acta Math.* **27**(1903), 289-304

[9] E. C. Titchmarsh, "On van der Corput's method and the zeta-function of Riemann (IV)", *Quart. J. Math. Oxford Ser.* **5**(1934), 98-105

[10] J. Kalpokas and J. Steuding, "On the value-distribution of the Riemann zeta-function on the critical line", *Moscow Journal of Combinatorics and Number Theory* **1**(2011), 26-42

[11] G. A. Hiary, "An amortized-complexity method to compute the Riemann zeta function", *Mathematics of Computation* **80**(2011), 1785-1796

[12] A. Odlyzko, "The 10^{20}-th Zero of the Riemann Zeta Function and 175 Million of its Neighbors", report, `http://www.dtc.umn.edu/~odlyzko/unpublished/zeta.10to20.1992.pdf`, (1992)

[13] G. A. Hiary, "Fast methods to compute the Riemann zeta function", *Annals of Mathematics* **63**(1987), 891-946

[14] Xavier Gourdon, "The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height", report, `http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf`, (2004)

[15] Aleksandar Ivić, "The Theory of Hardy's Z-Function," Cambridge University Press, (2013)

[16] D. A. Hejhal, "On a result of Selberg concerning zeros of linear combinations of L-functions", *Int. Math. Res. Not.* **11**(2000), 551-577

[17] A. Selberg, "Selected Papers, Vol. I," Springer Verlag, (1989)

[18] A. Selberg, "Selected Papers, Vol. II," Springer Verlag, (1991)

[19] Stephen J. Lester, "The Distribution of Values of the Riemann Zeta-Function," Ph. D. dissertation, University of Rochester, (2013)

[20] William Feller, "An Introduction to Probability Theory and Its Applications, Volume II (2 ed.)", John Wiley and Sons, New York, 1971