AMO-Advanced Modeling and Optimization, Volume 20, Number 2, 2018

Geodesic semilocal E-b-vex functions on Riemannian manifolds

S.K. Porwal
Department of Mathematics
SRMS-CET, Bareilly
UP-243202, India
skpmathsdstcims@gmail.com

Abstract

In this paper, we introduce a new class of functions, namely geodesic semilocal E-b-vex functions on Riemannina manifolds and discuss some of their properties and certain characterizations for this class of functions.

MSC: 52A01, 90C26, 90C30, 53B21

Keywords: Convexity, Semilocal convexity, E-convex function, b-vex function, quasi and pseudo convex functions, Geodesic.

1 Introduction

Convexity play vital role in the field of optimization, the notion of convexity does no longer suffice. To weaken the limitations of convexity various generalizations of convex sets and convex functions have been introduced in the literature. One of them is b-vex functions introduced by Bector and Singh [2] which shares many properties with convex functions.

A new type of generalize convexity, known as semilocal convexity which is introduced by Ewing [6] by reducing the width of the line segment. The Generalizations of semilocal convex functions and their properties have been studied by Kaul and Kaur [11, 12] and Kaur [10].

The concepts of E-convex sets and E-convex functions were introduced by Youness [25], which have some important applications in various branches of mathematical sciences [1, 17, 18]. The initial results of Youness [25] inspired a great deal of subsequent work, which has greatly expanded the role of E-convexity in optimization theory; see [4, 5, 7, 20, 24]. Syau and Lee [20] introduced the concept of E-quasiconvex functions and discussed some properties of E-convex and E-quasiconvex functions. Chen [3] introduced a new class of functions namely, semi E-convex functions and discussed some of its basic properties. Moreover, Syau *et al.* [21] introduced a class of functions called E-b-vex functions, which is the generalization of b-vex function and E-convex function.

AMO-Advanced Modeling and Optimization. ISSN: 1841-4311

Recently, Mishra *et al.* [15], introduced a new class of functions namely, semi E-b-vex functions and discussed some of its basic properties. For the recent developments and survey, we refer to Mishra *et al.* [16].

It is well-known that in linear topological spaces, the notion of convex sets rely on connecting any two points of the space by line segment. However,in several real-world applications, it is not possible to connect the points through line segment. A manifold is not a linear space and it is natural to extend the ideas from linear spaces to Riemannian manifolds. Udriste [22] and Rapcsak [19] proposed a generalization of the convexity notion by replacing the linear space by Riemannian manifolds, the line segments by a geodesic segments between any two points and the convex function by the positiveness of their Hessian.

Iqbal *et al.* [9] intoduced a new class of functions, known as geodesic semi E-convex function and discussed some of their properties. Iqbal *et al.* [8] introduced the class of geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds and derived certain characterizations for geodesic E-convex functions in terms of E-epigraph.

Inspired by the work of the papers [8, 9, 15, 23] and references therein, we derive some properties of geodesic semilocal E-b-vex functions on Riemannian manifolds.

2 Preliminaries and definitions

In this section, we gave some preliminary notations, about Riemannian manifolds and some definitions which will be used throughout this paper. For more about differential geometry, we can refer [13, 14].

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}_+$ be the set of nonnegative real numbers. Let M be a C^∞ smooth manifold together with a Riemannian metric $\langle .,. \rangle_p$ on the tangent space T_pM and corresponding norm is denoted by $\|.\|_p$, which yields the Riemannian manifold M. Let us recall that the length of a piecewise differentiable curve $\gamma : [a,b] \to M$ joining p to q such that $\gamma(a) = p$ and $\gamma(b) = q$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

Minimizing this length functional on the set of all piecewise differentiable curves joining p and q in M, we get a distance function $d(p,q)$. This distance function d induces the original topology on M. Let $\chi(M)$ denote the space of all vector fields on M. The metric induces a map $f \mapsto grad\ f \in \chi(M)$, which associates to each f its gradient via the rule $\langle df, X \rangle = df(X)$ for each $X \in \chi(M)$. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields $X, Y \in M$. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation

$$\nabla_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0.$$

Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic and it is called a minimal geodesic.

Definition 2.1. *A set $S \subseteq M$ is said to be geodesic local E-b-vex if and only if there are the maps $E : M \to M, b : M \times M \times [0,1] \to \mathbb{R}_+$ and a positive maximal number $0 < a(x,y) \leq 1,$ with $\lambda b(x,y,\lambda) \in [0, d(x,y)]$, such that*

$$\gamma_{E(x),E(y)}(\lambda) \in S,$$

for each $x, y \in S$ and $\lambda \in [0, a(x,y)]$.

Definition 2.2. *A function $f : S \to \mathbb{R}$ is said to be geodesic local E-b-vex on a geodesic local E-b-vex set $S \subseteq M$ if and only if there are the maps $E : M \to M, b : M \times M \times [0,1] \to \mathbb{R}_+$ and a positive number $d(x,y) \leq a(x,y)$ with $\lambda b(x,y,\lambda) \in [0, d(x,y)]$, such that*

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x,y,\lambda) f(E(x)) + (1 - \lambda b(x,y,\lambda)) f(E(y)),$$

for each $x, y \in S$ and $\lambda \in [0, d(x,y)]$.

If the above inequality is strict for each $x, y \in S, x \neq y$ and $\lambda \in (0, d(x,y))$, then the function $f : S \to \mathbb{R}$ is called strictly geodesic local E-b-vex on S.

Definition 2.3. *A function $f : S \to \mathbb{R}$ is said to be geodesic semilocal E-b-vex on a geodesic local E-b-vex set S if and only if there are the maps $E : M \to M, b : M \times M \times [0,1] \to \mathbb{R}_+$ and a positive number $d(x,y) \leq a(x,y)$ with $\lambda b(x,y,\lambda) \in [0, d(x,y)]$ such that*

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y),$$

for each $x, y \in S$ and $\lambda \in [0, d(x,y)]$.

If the above inequality is strict for each $x, y \in S, x \neq y$ and $\lambda \in (0, d(x,y))$, then the function $f : S \to \mathbb{R}$ is called strictly geodesic semilocal E-b-vex on S.

Definition 2.4. *The mapping $f : S \to \mathbb{R}$ is said to be geodesic quasi-semilocal E-b-vex on a geodesic local E-b-vex set S, if*

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \max\{f(x), f(y)\},$$

for each $x, y \in S$ and $\lambda \in [0, d(x,y)]$.

If the above inequality is strict for each $x, y \in S, x \neq y$ and $\lambda \in (0, d(x,y))$, then the function $f : S \to \mathbb{R}$ is called strictly geodesic quasi-semilocal E-b-vex on S.

Definition 2.5. *A function $f : S \to \mathbb{R}$ is said to be geodesic pseudo-semilocal E-b-vex on geodesic local E-b-vex set S, if there exists a strictly positive function $e : S \times \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) < f(y) \Rightarrow f(\gamma_{E(x),E(y)}(\lambda)) \leq f(y) + \lambda b(x,y,\lambda)(\lambda - 1) e(x,y),$$

for all $x, y \in S$ and $\lambda \in (0, d(x,y))$.

Definition 2.6. *If $K \subseteq M \times \mathbb{R}$ then the set K is said to be geodesic local E-b-vex if $(x,\alpha), (y,\beta) \in K$ imply*

$$(\gamma_{E(x),E(y)}(\lambda), \lambda b \alpha + (1 - \lambda b)\beta) \in K, \ \lambda \in [0, a((x,\alpha),(y,\beta))].$$

Now, the E-epigraph $epi(f)$ of f is given by

$$epi(f) = \{(E(x), \alpha) : x \in S, \alpha \in \mathbb{R}, f(E(x)) \leq \alpha\}.$$

3 Properties of geodesic semilocal E-b-vex functions

In this section, we derive some properties of geodesic semilocal E-b-vex functions on Riemannian manifolds.

Theorem 3.1. *Suppose the function $f : S \to \mathbb{R}$ is geodesic local E-b-vex on a geodesic local E-b-vex set S. Then f is geodesic semilocal E-b-vex on S if and only if $f(E(x)) \leq f(x)$ for each $x \in S$.*

Proof Suppose f is geodesic semilocal E-b-vex function on a geodesic local E-b-vex set S, then for any $x, y \in S$ and $\lambda \in [0, b(x, y)]$. We have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y).$$

Thus, for $\lambda b(x, y, \lambda) = 1$, we get $f(E(x)) \leq f(x)$.

Conversely, suppose $f(E(x)) \leq f(x)$, for each $x \in S$, then for any $x, y \in S$ and $\lambda \in [0, d(x, y)]$, we have

$$\begin{aligned} f(\gamma_{E(x),E(y)}(\lambda)) &\leq \lambda b(x, y, \lambda)f(E(x)) + (1 - \lambda b(x, y, \lambda))f(E(y)) \\ &\leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y). \end{aligned}$$

Hence, proved.

□

Theorem 3.2. *If $f : S \to \mathbb{R}$ is a geodesic semilocal E-b-vex function on a geodesic local E-b-vex set S, then for any real number $\alpha \in \mathbb{R}$, the α -level set $K_\alpha = \{x : x \in S, f(x) \leq \alpha\}$ is a geodesic local E-b- vex set.*

Proof For any $x, y \in K_\alpha$ and $0 \leq \lambda b(x, y, \lambda) \leq d(x, y)$, then $f(x) \leq \alpha, f(y) \leq \alpha$. Since f is geodesic semilocal E-b-vex on a geodesic local E-b-vex set S, then

$$\begin{aligned} f(\gamma_{E(x),E(y)}(\lambda)) &\leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y) \\ &\leq \lambda b(x, y, \lambda)\alpha + (1 - \lambda b(x, y, \lambda))\alpha \\ &= \alpha. \end{aligned}$$

Hence, K_α is geodesic local E-b-vex set.

□

Theorem 3.3. *Let $S \subseteq M$ is geodesic local E-b-vex set. Then the function $f : S \to \mathbb{R}$ is geodesic quasi-semilocal E-b-vex if and only if the level set $K_\alpha = \{x : x \in S, f(x) \leq \alpha\}$ is geodesic local E-b- vex for each $\alpha \in \mathbb{R}$.*

Proof Let f be geodesic quasi-semilocal E-b-vex on S. Thus for any $x, y \in K_\alpha$ and $\lambda \in [0, d(x, y)]$, $f(x) \leq \alpha, f(y) \leq \alpha$ we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \max\{f(x), f(y)\} \leq \alpha,$$

this implies that

$$\gamma_{E(x),E(y)}(\lambda) \in K_\alpha.$$

Thus, the set K_α is geodesic local E-b-vex.

Conversely, let $S \subseteq M$ be a geodesic local E-b-vex set and K_α is geodesic local E-b-vex for each $\alpha \in \mathbb{R}$, we have to show that f is geodesic quasi-semilocal E-b-vex. For each $x, y \in S$ and $\lambda \in [0, d(x,y)]$ such that

$$\gamma_{E(x),E(y)}(\lambda) \in S.$$

Let $\alpha = \max\{f(x), f(y)\}$, thus $x, y \in K_\alpha$, since K_α is geodesic local E-b-vex set, then $\gamma_{E(x),E(y)}(\lambda) \leq \alpha = \max\{f(x), f(y)\}$. Hence, f is geodesic quasi-semilocal E-b-vex.

□

Theorem 3.4. *Let $g_i : S \to \mathbb{R}$ is geodesic quasi-semilocal E-b-vex $i = 1, \dots, k$ on geodesic local E-b-vex set S. Then the set $M = \{x \in S : g_i(x) \leq 0, i = 1, \dots, k\}$ is geodesic local E-b-vex.*

Proof The proof follows from the previous Theorem 3.3.

□

Given a map $E : M \to M$, let the mapping $E \times I : M \times \mathbb{R} \to M \times \mathbb{R}$ is defined by

$$(E \times I)(x,t) = (E(x), t),\ \forall (x,t) \in M \times \mathbb{R}.$$

It is easy to show that $S \subseteq M$ is geodesic local E-b- vex, if and only if $M \times \mathbb{R}$ is geodesic local $E \times I$-b-vex.

Now, we give a characterization of geodesic semilocal E-b-vex functions in terms of its epigraph.

Theorem 3.5. *Let S is geodesic local E-b-vex set, then f is geodesic semilocal E-b-vex function on S if and only if $epi(f)$ is geodesic local $E \times I$-b-vex on $S \times \mathbb{R}$.*

Proof Assume that f is geodesic semilocal E-b-vex on S, let $(x, \alpha), (y, \beta) \in epi(f), \lambda \in [0, d(x,y)]$, then, we have

$$\begin{aligned} f(\gamma_{E(x),E(y)}(\lambda)) &\leq \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y) \\ &\leq \lambda b(x,y,\lambda)\alpha + (1 - \lambda b(x,y,\lambda))\beta. \end{aligned}$$

Thus

$$(\gamma_{E(x),E(y)}(\lambda), \lambda b(x,y,\lambda)\alpha + (1 - \lambda b(x,y,\lambda))\beta) \in epi(f),$$

this implies that $epi(f)$ is geodesic local $E \times I$-b-vex on $S \times \mathbb{R}$.

Conversely, let $epi(f)$ is geodesic $E \times I$-b- vex on $S \times \mathbb{R}$. Let $x, y \in S, \lambda \in [0, d(x,y)]$ then $(x, f(x)), (y, f(y)) \in epi(f)$. Since $epi(f)$ is geodesic $E \times I$-b-vex on $S \times \mathbb{R}$, we have

$$(\gamma_{E(x),E(y)}(\lambda), \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y)) \in epi(f),$$

i.e.,

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y).$$

Thus, f is geodesic semilocal E-b-vex function on S.

□

Theorem 3.6. *If $f : S \to \mathbb{R}$ is geodesic semilocal E-b-vex on a geodesic local E-b-vex set S then f is geodesic pseudo-semilocal E-b-vex.*

Proof Since $f(x) < f(y)$ and f is geodesic semilocal E-b-vex on a geodesic local E-b-vex set S then for all $x, y \in S$ and $\lambda \in (0, d(x, y))$, we have

$$\begin{aligned} f(\gamma_{E(x),E(y)}(\lambda)) &\leq \lambda b(x,y,\lambda)f(x) + (1 - \lambda b(x,y,\lambda))f(y) \\ &= f(y) + \lambda b(x,y,\lambda)(f(x) - f(y)) \\ &< f(y) + \lambda b(x,y,\lambda)(1-\lambda)(f(x) - f(y)) \\ &= f(y) + \lambda b(x,y,\lambda)(\lambda - 1)(f(y) - f(x)) \\ &= f(y) + \lambda b(x,y,\lambda)(\lambda - 1)e(x,y), \end{aligned}$$

where $e(x, y) = f(y) - f(x) > 0$, hence f is geodesic pseudo-semilocal E-b-vex.

□

Theorem 3.7. *Let $f : S \to \mathbb{R}$ be a geodesic quasi-semilocal E-b-vex function on geodesic local E-b-vex set S, if $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing function, then the composition function of f and ϕ is a geodesic quasi-semilocal E-b-vex on S.*

Proof Since f is geodesic quasi-semilocal E-b-vex function, then for for all $x, y \in S$ and $\lambda \in (0, d(x, y))$, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \max\{f(x), f(y)\}.$$

Thus,

$$\phi\left[f(\gamma_{E(x),E(y)}(\lambda))\right] \leq \phi\left[\max\{f(x), f(y)\}\right],$$
$$\phi \circ f(\gamma_{E(x),E(y)}(\lambda)) \leq \max\{\phi \circ f(x), \phi \circ f(y)\},$$

from which it follows that the composite function $\phi \circ f$ is geodesic quasi-semilocal E-b-vex on S.

□

Now, we consider the following problem:

$$Minf(x), \ \textit{subject to } x \in S = \{x \in M : g_i(x) \leq 0, i = 1, 2, \ldots, m\},$$

where $f : M \to \mathbb{R}$ and $g_i : M \to \mathbb{R}, i = 1, 2, \ldots, m$, are real valued functions on M, S is a geodesic local E-b-vex set.

$$(P)$$

We also need the following problem in the sequel:

$$Min\ f(E(x)),$$

$$\textit{subject to } x \in S,$$

$$(P_E)$$

Theorem 3.8. *Assume S is a geodesic local E-b-vex set and $f(E(x)) \leq f(x)$ for each $x \in S$. If $\bar{x}$ is a solution of the problem (P_E), then $E(\bar{x})$ is a solution of the problem (P).*

Proof Let $E(\bar{x})$ is not a solution of the problem (P), then there exists $y \in S$ such that $f(y) < f(E(\bar{x}))$, then $f(E(y)) \leq f(y) < f(E(\bar{x}))$, which contradicts the optimality of $\bar{x}$ of problem (P_E). Hence, $E(\bar{x})$ is a solution of the problem (P).

□

Theorem 3.9. *Let $f : M \to \mathbb{R}$ be a geodesic semilocal E-b-vex function on geodesic local E-b-vex set $S \subseteq M$ and $\bar{x}$ is a solution of the problem (P_E), then $E(\bar{x})$ is a solution of the problem (P).*

Proof The proof follows from previous Theorem 3.8.

□

Theorem 3.10. *If $x^\circ = E(z^\circ) \in E(A)$ is local minimum of the problem (P), on a geodesic local E-b-vex set $S, f : M \to \mathbb{R}$ is geodesic local E-b-vex on the set S and $f(E(x)) \leq f(x)$ for each $x \in S$, then x° is global minimum of the problem (P) on S.*

Proof Suppose $x^\circ = E(z^\circ) \in E(A)$ is a nonglobal minimum of the problem (P) on S, then there is $y \in S$ such that $f(y) < f(x^\circ) = f(E(z^\circ))$, since function $f : M \to \mathbb{R}$ is geodesic local E-b-vex and $f(E(x)) \leq f(x)$ for each $x \in S$, it implies that

$$\begin{aligned} f(\gamma_{E(z^\circ),E(y)}(\lambda)) &\leq \lambda b(x,y,\lambda) f(E(z^\circ)) + (1 - \lambda b(x,y,\lambda)) f(E(y)) \\ &\leq \lambda b(x,y,\lambda) f(x^\circ) + (1 - \lambda b(x,y,\lambda)) f(y) \\ &\leq f(x^\circ), \end{aligned}$$

for any small $\lambda \in (0, d(x,y))$, which contradicts the local optimality of x° for problem (P). Hence, x° is a global minimum of the problem (P) on S.

□

Theorem 3.11. *Let $f : M \to \mathbb{R}$ be strictly geodesic semilocal E-b-vex on a geodesic local E-b-vex set S, then the global optimal solutions of problem (P) is unique.*

Proof Let $x, y \in S$ be two different global optimal solutions of problem (P). Then, $f(x) = f(y)$. Since f is strictly geodesic semilocal E-b-vex on S, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y) = f(x),$$

for each $\lambda \in (0, d(x,y))$. Which is contradicts the optimality of x of problem (P). Hence, the global optimal solution of problem (P) is unique.

□

Theorem 3.12. *Let $f : M \to \mathbb{R}$ be geodesic quasi-semilocal E-b- vex on a geodesic local E-b-vex set $S \subseteq M$, and $\alpha = \underset{x\in S}{min} f(x)$. Then, the set $X = \{x \in S : f(x) = \alpha\}$ of optimal solutions of problem (P) is geodesic local E-b-vex. If f is strictly geodesic quasi-semilocal E-b-vex on S, then the set X is a singleton.*

Proof Let $x, y \in S$ be two different global solutions of problem (P). and $\lambda b(x, y, \lambda) \in [0, d(x, y)]$, then, $f(x) = \alpha$ and $f(y) = \alpha$. Since $f : M \to \mathbb{R}$ is geodesic quasi-semilocal E-b-vex on S, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq max\{f(x), f(y)\} = \alpha,$$

which implies that $\gamma_{E(x),E(y)}(\lambda) \in X$, this implies that X is geodesic local E-b-vex.

Now we need to show that X is singleton, on contrary we suppose that $x, y \in X, x \neq y$, for $\lambda \in (0, a(x, y))$, $\gamma_{E(x),E(y)}(\lambda) \in S$. Futher, since f is strictly geodesic quasi-semilocal E-b-vex on S, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) < max\{f(x), f(y)\} = \alpha.$$

Which contradicts that $\alpha = \underset{x\in S}{min} f(x)$, and hence X must be a singleton.

□

Theorem 3.13. *If $f : M \to \mathbb{R}$ is geodesic semilocal E-b-vex on a geodesic local E-b-vex set $S \subseteq M$, then the set of optimal solutions of problem (P) is geodesic local E-b-vex.*

Proof Let x^* be optimal solution of problem (P), and let $\alpha = f(x^*)$. Let X be set of optimal solutions for problem (P) as follows:

$$X = \{x \in S : f(x) \leq \alpha\},$$

for any $x, y \in X, x \neq y$, and $\lambda \in (0, d(x, y))$. Since $f : M \to \mathbb{R}$ is geodesic semilocal E-b-vex on a geodesic local E-b-vex set S, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y) \leq \alpha.$$

Thus,$\gamma_{E(x),E(y)}(\lambda) \in X$, this implies that X is a geodesic local E-b-vex.

□

Theorem 3.14. *If $f : M \to \mathbb{R}$ and $g_i : M \to \mathbb{R}, i = 1, 2, \ldots, m$ are geodesic quasi-semilocal E-b-vex on M. Then the set of optimal solutions of problem (P) is geodesic local E-b-vex.*

Proof From Theorem 3.4, it follows that $S \subseteq M$ is geodesic local E-b-vex set. By Theorem 3.12, the set $X = \{x \in S : f(x) = \alpha\}$ of optimal solutions of problem (P) is a geodesic local E-b-vex.

□

Theorem 3.15. *Let $f : M \to \mathbb{R}$ be geodesic semilocal E-b-vex on a geodesic local E-b-vex set S, let $\phi : \mathbb{R} \to \mathbb{R}$ be positively homogeneous non-decreasing function then the composite function $\phi \circ f$ is a geodesic semilocal E-b-vex on S.*

Proof Since f is geodesic semilocal E-b-vex on a geodesic local E-b-vex set S, we have for all $x, y \in S$ and $\lambda \in (0, d(x,y))$,

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y)$$

Thus,

$$\begin{aligned} \phi \circ f(\gamma_{E(x),E(y)}(\lambda)) &\leq \phi \circ [\lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y)] \\ &\leq \lambda b(x,y,\lambda) \phi \circ f(x) + (1 - \lambda b(x,y,\lambda)) \phi \circ f(y), \end{aligned}$$

from which it follows that $\phi \circ f$ is geodesic semilocal E-b-vex on S.

□

Theorem 3.16. *Let $f : M \to \mathbb{R}$ is differentiable geodesic semilocal E-b-vex on a geodesic local E-b-vex set $S \subseteq M$ and $u \in S$ is a fixed point of the map E, i.e., $E(u) = u$. Then $u \in S$ is the minimum of f on S if and only if*

$$\langle f'(E(u)), E(\nu) - E(u) \rangle \geq 0, \ \forall \nu \in M,$$

where f' is the differential of f at $E(u)$.

Proof Let $u \in S$ be a minimum of f on S. Then

$$f(E(u)) \leq f(E(\nu)), \ \forall \nu \in S.$$

Since S is geodesic local E-b-vex set, we have

$$f(\gamma_{E(\nu),E(u)}(\lambda)) \in S, \ \forall u, \nu \in S.$$

Since f is differentiable on S, we have

$$f(E(u)) \leq f(\gamma_{E(\nu),E(u)}(\lambda)) = f(E(u) + \lambda b(x,y,\lambda)(E(\nu) - E(u))),$$

let $\lambda b(x,y,\lambda) = \lambda'$, then

$$f(E(u)) \leq f(E(u) + \lambda b(x,y,\lambda)(E(\nu) - E(u))) = f(E(u)) + \lambda' \langle f'(E(u)), E(\nu) - E(u) \rangle + O(\lambda').$$

Dividing the above inequality by λ' and taking $\lambda' \to 0$, we have

$$\langle f'(E(u)), E(\nu) - E(u) \rangle \geq 0,$$

which is the required result. The converse part is easy to establish.

□

4 Conclusion

In this paper, we have introduced a new class of functions, namely geodesic semilocal E-b-vex functions on Riemannian manifolds and establish certain characterizations for them. The novelty of this paper is that using geodesic semilocal E-b-vex functions and their generalization, we derive optimality conditions for the first time on Riemannian manifolds, these have not been studied extensively on Riemannian manifolds. The results of the paper generalize naturally, some earlier results from Mishra *et al.* [15], Iqbal *et al.* [8, 9], Upadhyay *et al.* [23] and references therein.

References

[1] I.A. Abou-Tair and W.T. Sulaiman, Inequalities via convex functions, Int. J. Math. Math. Sci. 22 (1999) 543-546.

[2] C.R. Bector and C. Singh, B-vex functions, Journal of Optimization Theory and Applications 71 (1991) 237-253.

[3] X. Chen, Some properties of semi E-convex functions, Journal of Mathematical Analysis and Applications 275 (2002) 251-262.

[4] D.I. Duca, E. Duca, L. Lupsa and R. Blaga, E-convex functions, Bull. Appl. Comput. Math. 43 (2000) 93-102.

[5] D.I. Duca and L. Lupsa, On the E-epigraph of an E-convex function, J. Optim. Theory Appl. 120 (2006) 341-348.

[6] G. M. Ewing, Sufficient conditions for global minima of suitably convex functionals from variational and control theory, SIAM Review, vol. 19, no. 2(1977) 202-220 .

[7] C. Fulga and V. Preda, Nonlinear programming with E-preinvex and local E-preinvex functions, European J. Oper. Res. 192 (2009) 737-743.

[8] A. Iqbal, S. Ali and I. Ahmad, On geodesic E-convex sets, geodesic E-convex functions and E-epigraphs, J. Optim Theory Appl., 155 (2012) 239-251.

[9] A. Iqbal, I. Ahmad and S. Ali, Some properties of geodesic semi E-convex functions, Nonlinear Analysis, 74(2011) 6805-6813.

[10] S. Kaur, Theoretical studies in mathematical programming, Ph.D. Thesis, University of Delhi, New Delhi, India, 1983.

[11] R. N. Kaul and S. Kaur, Generalizations of convex and related functions, European Journal of Operational Research, vol. 9, no. 4(1982) 369-377 .

[12] R. N. Kaul and S. Kaur, Sufficient optimality conditions using generalized convex functions, Opsearch, vol. 19, no. 4(1982) 212-224.

[13] W. Klingenberg, Riemannian geometry, Walter de Gruyter Studies in Mathematics, Walter de Gruyter, Berlin, 1982.

[14] S. Lang, Fundamentals of Differential Geometry, Graduate Texts in Mathematics, Springer-Verlag, New York, 1999.

[15] S.K. Mishra, R.N. Mohapatra and E.A. Youness, Some properties of semi E-b-vex functions, Applied Mathematics and Computation, 217(2011) 5525-5530.

[16] S.K. Mishra, S.Y. Wang and K.K. Lai, Generalized Convexity and Vector Optimization, Springer-Verlag, Berlin, Heidelberg, 2009.

[17] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217-229.

[18] M.A. Noor, Fuzzy preinvex functions, Fuzzy Sets and Systems 64 (1994) 95-104.

[19] T. Rapcsak, Smooth Nonlinear Optimization in $\mathbb{R}^n$, Kluwer Academic Publishers, Dordrecht, 1997.

[20] Yu-Ru Syau and E.S. Lee, Some properties of E-convex functions, Appl. Math. Lett. 18 (2005) 1074-1080.

[21] Yu-Ru Syau, L. Jia and E.S. Lee, Generalizations of E-convex and B-vex functions, Computers and Mathematics with Applications 58 (2009) 711-716.

[22] C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, Kluwer Academic Publishers, Dordrecht, Netherlands, 1994.

[23] B.B. Upadhyay, S.K. Mishra and S.K. Porwal, Explicitly geodesic b-preinvex functions on Riemannian manifolds, *Transactions on Mathematical Programming and Applications,* 3(2015) 1-14.

[24] X.M Yang, On E-convex sets, E-convex functions and E-convex programming, *J. Optim. Theory Appl.* 109 (2001) 699-704.

[25] E .A. Youness, E-convex sets, E-convex functions and E-convex programming. *J. Optim Theory Appl.* 102(1999) 439-450.