ON BISECTOR RULED SURFACE OF GIVEN A SPECIAL CURVE IN MINKOWSKI 3-SPACE AND ITS APPLICATION

TALAT KÖRPINAR AND MUHAMMED TALAT SARIAYDIN

Abstract. In this paper presents bisector surfaces of some special curves in Minkowski space. Firstly, we give the Bisector surface generated a point and a Bertrand curve of a given space curve Then, we show that the Bisector surface is the ruled surface. Some applications is also given for these surfaces.

1. Introduction

The Bisector surface is a special surface because this surface is defined by any two objects in 2-dimensional or 3-dimensional space. These objects can be point-curve, curve-curve or surface-surface. Moreover the Bisector surface is the set of points which are equidistant from the two objects, [4]. Then, this surface is often used in scientific research from the past. For example, Horvath proved that all bisectors are topological images of a plane of the embedding Euclidean 3-space iff the shadow boundaries of the unit ball K are topological circles in [8], and Elber studied a new computational model in $E^3$ in [5]. The parallel curves are developed by Chrastinova in 2007. These curves are not easy to characterize in 3-dimensional space until this time. Additionally, he study parallel curves of a special curve as helices.

In this paper, we study Bisector surfaces of some special curves in Minkowski 3-space. The first, we give properties and the basic concepts of curves in $E^3_1$. Then, we construct bisector surface generated by pedal and parallel curves of given timelike curve in $E^3_1$. Moreover, we show how to generate the this surface. Finally, we give an example of these surfaces in $E^3_1$.

2. Preliminaires

Given a spatial curve $\xi : s \rightarrow \xi(s)$, which is parameterized by arc-length parameter $s$. For each point of $\xi(s)$, the set $\{t(s), n(s), b(s)\}$ is called the Frenet Frame along $\xi(s)$, where $t(s), n(s), b(s)$ are the unit tangent, principal normal,
and binormal vectors of the curve at the point $\xi(s)$, respectively. Derivative of the Frenet frame according to arc-length parameter is governed by the relations;

$$\begin{pmatrix}
    e'_1(s) \\
    e'_2(s) \\
    e'_3(s)
\end{pmatrix} = \begin{pmatrix}
    0 & \kappa_1(s) & 0 \\
    -\kappa_1(s) & 0 & \kappa_2(s) \\
    0 & -\kappa_2(s) & 0
\end{pmatrix} \begin{pmatrix}
    e_1(s) \\
    e_2(s) \\
    e_3(s)
\end{pmatrix},$$

where

$$\kappa_1(s) = \|\xi''(s)\|, \kappa_2(s) = \frac{\langle \xi'(s), \xi''(s), \xi'''(s) \rangle}{\|\xi''(s)\|^2}.$$ 

Assume that $\xi(s)$ is an arbitrary timelike curve in the space $\mathbb{E}^3_1$, then, the Frenet formulae of $\xi(s)$ are given by

$$\begin{pmatrix}
    e'_1(s) \\
    e'_2(s) \\
    e'_3(s)
\end{pmatrix} = \begin{pmatrix}
    0 & \kappa_1(s) & 0 \\
    \kappa_1(s) & 0 & \kappa_2(s) \\
    0 & -\kappa_2(s) & 0
\end{pmatrix} \begin{pmatrix}
    e_1(s) \\
    e_2(s) \\
    e_3(s)
\end{pmatrix},$$

where

$$\langle e_1, e_1 \rangle = -1, \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1.$$ 

Assume that Minkowski 3-space is consider $\mathbb{E}^3_1 = [\mathbb{E}^3_1, (-, +, +)]$ and the Lorentzian inner product and vector product, respectively, are

$$\langle X, Y \rangle = -x_1y_1 + x_2y_2 + x_3y_3,$n
$$X \times Y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{E}^3_1$, [25].

A surface in $\mathbb{E}^3_1$ is called a timelike surface if the normal vector on the surface is spacelike vector. A surface is called spacelike surface if the normal vector on surface is the timelike vector.

3. The Bisector Surface Obtained a Point and a Bertrand curve

In this paper, our goal is construct Bisector surface generated a point and a Bertrand curve in $\mathbb{E}^3_1$. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a timelike curve in $\mathbb{E}^3_1$ and $(\alpha, \beta)$ be Bertrand curve couple in Minkowski 3-space. Then, the curve $\beta$ can be written as

$$\beta(s) = \alpha(s) + \lambda(s) n_a(s),$$

where $\lambda(s)$ is a smooth function on $I$.

**Theorem 3.1** Suppose that $\mathcal{M} = (\omega_1, \omega_2, \omega_3)$ is a fixed point and $(\alpha, \beta)$ is called Bertrand curve couple. Then, the rational ruled bisector surface $\mathcal{F}(s, t)$ is

$$\mathcal{F}(s, t) = \mathcal{F}(s) + t \mathcal{N}(s); \text{ for } s, t \in \mathbb{R},$$

where

$$\mathcal{N}(s) = \mathbf{t}_3(\beta(s) - \mathbf{w})$$

$$= (\mathbf{t}_3^\beta(\beta_2 - \omega_2 - \beta_3 - \omega_3),$$

$$\mathbf{t}_3(\beta_1 - \omega_1) - \mathbf{t}_3^\beta(\beta_3 - \omega_3),$$

$$\mathbf{t}_3^\beta(\beta_2 - \omega_2) - \mathbf{t}_3^\beta(\beta_1 - \omega_1),$$
and

\[
\begin{align*}
\begin{bmatrix}
  t_1 \\
  t_2 \\
  t_3
\end{bmatrix} &= \frac{1}{3} 
\begin{bmatrix}
  k_1 & t_{\beta}^2 & t_{\beta}^3 \\
  k_2 & n_2 & n_3 \\
  k_3 & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{bmatrix}, \\
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix} &= \frac{1}{3} 
\begin{bmatrix}
  1 & t_{\beta}^1 & 0 \\
  -1 & 0 & 1 \\
  1 & 0 & 0 \\
  n_1 & n_2 & n_3 \\
  (\beta_1 - \omega_1) & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{bmatrix}.
\end{align*}
\]

where \( \beta \) is

\[
\begin{bmatrix}
  -t_{\beta}^1 & t_{\beta}^2 & t_{\beta}^3 \\
  -n_1 & n_2 & n_3 \\
  - (\beta_1 - \omega_1) & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{bmatrix}
\]

Proof. Taking the derivative of the eq (3.1), it is clear that

\[
\beta' = t_{\beta} = (1 + \lambda \kappa_1) t_\alpha + \lambda' n_\alpha + \lambda \kappa_2 b_\alpha.
\]

On the other hand, let \( \Sigma \) be a bisector point of \( \beta(s) \) and \( M \) with its foot points at \( \beta(s) \) and \( M \), respectively. Then, it is clear that the point \( \Sigma \) is contained both the normal plane \( H(s_0) \) and the bisector plane \( H_b(s_0) \). In that case \( H(s_0) \) and \( H_b(s_0) \) intersect in a line \( h(s_0) \).

Suppose that \( N(s) \) is the direction vector of \( h(t) \), then it is clear that \( N(s) \) is contained in both \( H(s) \) and \( H_b(s) \) and it is orthogonal to the normal vectors of \( H(s) \) and \( H_b(s) \). Therefore, the following equation can be written easily

\[
N(s) = t_\beta(s) \times (\beta(s) - M) = (t_{\beta}^3(\beta_2 - \omega_2) - t_{\beta}^2(\beta_3 - \omega_3), \\
- \beta_3^1(\beta_1 - \omega_1) + \beta_3^1(\beta_3 - \omega_3), \\
- \beta_3^1(\beta_2 - \omega_2) - \beta_3^2(\beta_1 - \omega_1)),
\]

which is a rational vector field.

An auxiliary plane \( AP \) \( H_a(s) \) is orthogonal to the intersection line \( h(s) \) and passes through the fixed point \( M \). So \( \Sigma(s) \) is the closest point of \( h(s) \) to \( M \), \( AP \) can be written as:

\[
H_a(s) : \langle \Sigma - M, N(s) \rangle = 0.
\]

If the above equations are considered together, we obtain the following equations for intersection point \( \Sigma \):

\[
\begin{align*}
\langle \Sigma, t_\beta(s) \rangle &= \langle \beta(s), t_\beta(s) \rangle, \\
\langle \Sigma, N(s) \rangle &= \langle M, N(s) \rangle, \\
\langle \Sigma, \beta(s) - M \rangle &= \frac{1}{2} \| \beta(s) \|^2 - \| M \|^2).
\end{align*}
\]

Then, we have the following matrix equation,

\[
(3.2) \quad \begin{bmatrix}
  -t_{\beta}^1 & t_{\beta}^2 & t_{\beta}^3 \\
  -n_1 & n_2 & n_3 \\
  - (\beta_1 - \omega_1) & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{bmatrix} \begin{bmatrix}
  t_1 \\
  t_2 \\
  t_3
\end{bmatrix} = \begin{bmatrix}
  k_1 \\
  k_2 \\
  k_3
\end{bmatrix},
\]
where

\[ k_1 = -\beta_1 t_\beta^1 + \beta_2 t_\beta^2 + \beta_3 t_\beta^3, \]
\[ k_2 = -\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3, \]
\[ k_3 = \frac{1}{2}(\|\beta(s)\|^2 - \|W\|^2). \]

If eq. (3.2) can be solved, then following equation is obtained by

\[ t_1 = \frac{1}{3} \begin{vmatrix}
  k_1 & t_\beta^2 & t_\beta^3 \\
  k_2 & n_2 & n_3 \\
  k_3 & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{vmatrix}, \]
\[ t_2 = \frac{1}{3} \begin{vmatrix}
  -t_\beta^1 & k_1 & t_\beta^3 \\
  -n_1 & k_2 & n_3 \\
  - (\beta_1 - \omega_1) & k_3 & (\beta_3 - \omega_3)
\end{vmatrix}, \]
\[ t_3 = \frac{1}{3} \begin{vmatrix}
  -t_\beta^1 & t_\beta^2 & k_1 \\
  -n_1 & n_2 & n_3 \\
  - (\beta_1 - \omega_1) & (\beta_2 - \omega_2) & k_3
\end{vmatrix}, \]

where \( J \) is

\[ J = \begin{vmatrix}
  -t_\beta^1 & t_\beta^2 & t_\beta^3 \\
  -n_1 & n_2 & n_3 \\
  - (\beta_1 - \omega_1) & (\beta_2 - \omega_2) & (\beta_3 - \omega_3)
\end{vmatrix}. \]

The rational ruled bisector surface \( \mathcal{B}(s,t) \) can be constructed as follows:

\[ \mathcal{B}(s,t) = \mathcal{B}(s) + tN(s) \quad \text{for} \quad s, t \in \mathbb{R}. \]

Now, we will examine the degenerate case.

**Remark 3.2** Since the direction vector \( N(s) \) is cross product of \( t_\beta(s) \) and \( \beta(s) - W \), these vectors are linearly independent. On the contrary, we assume that these vectors \( \beta(s) \) and \( \beta(s) - W \) are linearly dependent. Since \( \beta(s) \) is regular, \( t_\beta(s) \neq 0 \) and so \( \beta(s) \) and \( \beta(s) - W \) parallel to each other. We can write

\[ W = \beta(s) = \mathcal{R}(s) t_\beta, \]

where \( \mathcal{R}(s) \in \mathbb{R} \). From eq. (3.3),

\[ W = \beta(s) + \mathcal{R}(s) t_\beta. \]

That is, the point \( W \) is on tangent of curve \( \beta(s) \) for all \( s \).

4. Application

Let us consider a unit speed timelike curve in \( \mathbb{E}_1^2 \) by

\[ \alpha = \alpha(s) = (\sqrt{2}s, \cos s, \sin s). \]

One can calculate its Frenet-Serret apparatus as the following, [9],

\[ t(s) = (\sqrt{2}, -\sin s, \cos s), \]
\[ n(s) = (0, -\cos s, -\sin s), \]
\[ b(s) = (-1, \sqrt{2}\sin s, -\sqrt{2}\cos s). \]
ON BISECTOR RULED SURFACE

Then, the curvatures of $\alpha$ is given by

\[ \kappa(s) = 1, \]
\[ \tau(s) = \sqrt{2}. \]

On the other hand, $P(s)$ parallel curve of a timelike $\alpha(s)$ curve with parametrized by arc-length in $E^3_1$ obtained as follows

\[ P(s) = (\sqrt{2}s - 2, 2\cos s + 2\sqrt{2}\sin s, 2\sin s - 2\sqrt{2}\cos s), \]

where we choose $t = \sqrt{5}$. Taking the derivative of the eq. (3.3.2), we can computed by

\[ P'(s) = (p_2s, 2\cos s + 2p_2\sin s, 2\sin s + 2p_2\cos s). \]

Now, if we choose $Q = (-2, 2, 2)$ be a fixed point in $E^3_1$, the direction vector $N(s)$ of the intersection line $l(t)$ between two planes $L(s)$ and $L_b(s)$ obtained by

\[ N(s) = (12 - (4 - 4\sqrt{2})\cos s - (4 + 4\sqrt{2})\sin s, \]
\[ 2\sqrt{2} + (4 + 2\sqrt{2}s)\cos s - (2\sqrt{2} - 4s)\sin s, \]
\[ -2\sqrt{2} + (2\sqrt{2} - 4s)\cos s + (4 + 2\sqrt{2}s)\sin s). \]

The intersection point $B = (b_1, b_2, b_3)$ of three planes: $L(s)$, $L_n(s)$, and $L_b(s)$ can be computed by solving the following simultaneous linear equations in $B$:

\[ \langle B, t_P(s) \rangle = r_1, \]
\[ \langle B, N(s) \rangle = r_2, \]
\[ \langle B, P(s) - Q \rangle = r_3, \]

where

\[ r_1 = \langle P(s), t_P(s) \rangle \]
\[ r_2 = \langle Q, N(s) \rangle, \]
\[ r_3 = \frac{1}{2}(||P(s)||^2 - ||Q||^2). \]

Then, by Cramer’s rule, eq. (3.3.3) can be solved as follows:

\[ b_1 = \frac{1}{3}[r_2(12 + (-4 + 4\sqrt{2})\cos s + (-4 + 4\sqrt{2})\sin s) \]
\[ + n_2((-10 + 4\sqrt{2}s - 4\sqrt{2} + 2s)\cos s + (2\sqrt{2} \]
\[ + 2\sqrt{2}s - 4s - 8)\sin s + 4s - 4\sqrt{2}) + n_3((4s + 8 \]
\[ - 2\sqrt{2} - 2\sqrt{2}s)\cos s + (4\sqrt{2}s - 10 - 4\sqrt{2} + 2s)\sin s \]
\[ - 4s + 4\sqrt{2})], \]

\[ b_2 = \frac{1}{3}[n_1((-10 - 4\sqrt{2} + 2s + 4\sqrt{2}s)\cos s - (8 - 2\sqrt{2} \]
\[ + 4s - 2\sqrt{2}s)\sin s - 4\sqrt{2} + 4s) + r_2((4 + 2\sqrt{2}s)\cos s \]
\[ - (2\sqrt{2} - 4s)\sin s + 2\sqrt{2}) + n_3(4 + \sqrt{2} - 4s - \sqrt{2}s \]
\[ + 2\sqrt{2}s^2)], \]
\[ b_3 = \frac{1}{3} [ \mu_1 ((8 - 2\sqrt{2} + 4s - 2\sqrt{2}s) \cos s - (10 + 4\sqrt{2} - 2s) \cos s - 4\sqrt{2}s \sin s - 4s + 4\sqrt{2} + \mu_2 (-4 - \sqrt{2} + 4s + \sqrt{2}s - 2\sqrt{2}s^3) + r_2 ((2\sqrt{2} - 4s) \cos s + (4 + 2\sqrt{2}s) \sin s - 2\sqrt{2})], \]

where

\[ \begin{align*}
J &= \left[-12 + (4 - 4\sqrt{2}) \cos s + (4 + 4\sqrt{2}) \sin s \right]^2 \\
&\quad + [2\sqrt{2} + (4 + 2\sqrt{2}s) \cos s - (2\sqrt{2} - 4s) \sin s]^2 \\
&\quad + [-2\sqrt{2} + (2\sqrt{2} - 4s) \cos s + (4 + 2\sqrt{2}s) \sin s]^2, \\
r_2 &= 24 + (12\sqrt{2} - 8s + 4\sqrt{2}s) \cos s + (-12\sqrt{2} + 8s + 4\sqrt{2}s) \sin s, \\
n_1 &= 12 - (4 - 4\sqrt{2}) \cos s - (4 + 4\sqrt{2}) \sin s, \\
n_2 &= 2\sqrt{2} + (4 + 2\sqrt{2}s) \cos s + (-2\sqrt{2} + 4s) \sin s, \\
n_3 &= -2\sqrt{2} + (2\sqrt{2} - 4s) \cos s + (4 + 2\sqrt{2}s) \sin s.
\end{align*} \]

Then, the rational ruled bisector surface \( \mathfrak{B} (s, t) \) can be constructed as follows:

\[ \mathfrak{B} (s, t) = \mathfrak{B} (s) + t \mathcal{N} (s); \text{ for } s, t \in \mathbb{R}. \]

**Acknowledgment**

This work was supported by Muş Alparslan University Scientific Research Coordination Unit. Project Number: MŞU16-FEF-G01

**References**


ON BISECTOR RULED SURFACE


MUŞ ALPARSLAN UNIVERSITY, DEPARTMENT OF MATHEMATICS, 49250, MUŞ, TURKEY, SELÇUK UNIVERSITY, DEPARTMENT OF MATHEMATICS, 42130, KONYA, TURKEY

E-mail address: talatkorpinar@gmail.com