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THE CHARACTERIZATIONS OF CURVES OF THE CONSTANT BREADTH VIA THE PARALLEL TRANSPORT FRAME OF THE TYPE-2 IN E_1^3

MURADIYE ÇIMDIKER[1]*, YASIN ÜNLÜTÜRK[2]

Abstract. In this purpose, we give some characterizations of space-like and time-like curves of the constant breadth via the parallel transport frame of the type-2 in Minkowski 3-space E_1^3.For these curves, we obtain some special cases. Moreover the differential equations for the space-like and time-like curves of the constant breadth via the parallel transport frame of the type-2 in E_1^3 are characterized.

1. Introduction

The structure of the parallel transport frame is defined by L. R. Bishop in [1]. This is why he defined this frame that curvature may vanish at some points on the curve. In this case, an alternative frame is a necessary the non-continuously differentiable curves on which parallel transport frame is well defined and constructed in Euclidean and its ambient spaces. A new version of the parallel transport frame which is called parallel transport frame of the type-2 examined by Yılmaz and Turgut [13]. By new version of the parallel transport frame, it means that the tangent vector N_1 and principal normal vector N_2 are considered as parallel transport plane while the binormal vector B remains fixed. According to the parallel transport trihedron type-2, classical differential geometry of curves studied by Özyılmaz [10]. Besides, for spacelike curves, the new version of the parallel transport frame obtained by Ünlütürk and Yılmaz [12].

L. Euler introduced the curves of the constant breadth [4]. Constant breadth curves are an important subject that keeps improving by its usage in many fields such as engineering sciences, cam designs This subject attracted attention by many geometers [2, 8]. A problem to determine whether there exist space curve of the constant breadth or not, and defined breadth for space curves and obtained these curves on a surfaces of constant breadth had been obtained by Fujıvara [5]. On this subject, some important publications published by Struik [11]. In Euclidean 3-space, Ö. Köse expressed some characterizations for space curves of the constant

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breadth [6]. In pseudo-Riemannian space, null curves of the constant breadth studied by Yılmaz and Turgut [14]. For a timelike curve, the new version of the parallel transport frame is constructed and Bishop plane is defined in E_1^3 [15].

In this purpose, we give some characterizations of curves of constant breadth via the parallel transport frame of the type-2 in Minkowski 3-space E_1^3. Moreover, the differential equations for these curves are characterized in E_1^3.

2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by $<,>= -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is rectangular coordinate system of E_1^3. Since $<,>$ is an indefinite metric recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters; it can be space-like if $< v, v > 0$ or $v = 0$, time like if $< v, v >< 0$ and null if $< v, v >= 0$ and $v \neq 0$. Similarly, an arbitrary curve $\sigma = \sigma(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vector σ' are space-like, time-like or null (light-like), respectively, for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $v \in E_1^3$, is given by $\|v\| = \sqrt{|< v, v >|}$. σ is called an unit speed curve if velocity vector σ' of all satisfies $\|\sigma'\| = \mp 1$. For vectors $u, w \in E_1^3$ are said to be orthogonal each other if and only if $< u, w >= 0$ [7].

The moving Serret-Frenet frame $\{T, N, B\}$ is denoted along curve σ in the space E_1^3. For an arbitrary space-like curve σ in the space E_1^3, the moving Serret-Frenet formulae are written as follows:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where $\epsilon = \mp 1$, and the functions $\kappa(s) = \sqrt{\langle T'(s), T'(s)\rangle}$, $\tau(s) = \langle N'(s), B(s)\rangle$ are, respectively, the first and second (torsion) curvature [15].

Theorem 2.1. Assume that $\sigma = \sigma(s)$ a space-like unit speed curve with a space-like principal normal. If $\{N_1, N_2, B\}$ is an adapted frame, then we have the parallel transport derivative formulae as

$$\begin{bmatrix} N_1' \\ N_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & -k_2 \\ -k_1 & -k_2 & 0 \end{bmatrix} . \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix} \tag{2.2}$$

[12].

Theorem 2.2. Assume that $\sigma = \sigma(s)$ a time-like unit speed curve with a space-like principal normal. If $\{N_1, N_2, B\}$ is an adapted frame, then we have

$$\begin{bmatrix} N_1' \\ N_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ k_1 & k_2 & 0 \end{bmatrix} . \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}, \tag{2.3}$$

where $\langle N_1, N_1\rangle = -1$, $\langle N_2, N_2\rangle = \langle B, B\rangle = 1$ [15].

Theorem 2.3. Let $M \subset E^3$ is a curve given by the chart (I, φ). Then M is an inclined curve if and only if $H(s) = \dfrac{k_1(s)}{k_2(s)}$ is constant for all $s \in I$ [3].

3. Main Results

3.1. **The Space-Like Curves of the Constant Breadth Via the Parallel Transport Frame of the Type-2 in E_1^3.** Suppose that $\sigma = \sigma(s)$ an ordinary

closed space-like curve in E_1^3. These curves will be determined by (C). The normal plane at every point P on the space-like curve corresponds the space-like curve at a single point Q other than P, where the point Q is the opposite point of P. Consider a space-like curve in the class Γ which have parallel tangents T and T^* opposite directions at opposite points σ and $\sigma^*$of the curves. An ordinary closed space-like curve of the constant breadth which have parallel tangents in opposite directions is defined by

$$\sigma^*(s) \quad = \quad \sigma(s) + \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 B \tag{3.1}$$

where $\lambda_i(s)$, $1 \leq i \leq 3$ are arbitrary functions, σ and σ^* are opposite points and $\{N_1, N_2, B\}$ denote the parallel transport frame of the type-2 in E_1^3.

If N_1 is taken instead of tangent vector and differentiating both sides of the expression (3.1), then we have

$$\begin{aligned} \frac{d\sigma^*}{ds} = \frac{d\sigma^*}{ds^*}\frac{ds^*}{ds} = N_1^*\frac{ds^*}{ds} = & \left(1 + \frac{d\lambda_1}{ds} - \lambda_3 k_1\right) N_1 + \\ & \left(\frac{d\lambda_2}{ds} - \lambda_3 k_2\right) N_2 + \left(\frac{d\lambda_3}{ds} + \lambda_1 k_1 - \lambda_2 k_2\right) B, \end{aligned} \tag{3.2}$$

where k_1, k_2 are the first and the second curvatures of the space-like curve, respectively.

Since $N_1^* = -N_1$, we get

$$\begin{aligned} & \frac{ds^*}{ds} + \frac{d\lambda_1}{ds} - \lambda_3 k_1 + 1 = 0, \\ & \frac{d\lambda_2}{ds} - \lambda_3 k_2 = 0, \\ & \frac{d\lambda_3}{ds} + \lambda_1 k_1 - \lambda_2 k_2 = 0. \end{aligned} \tag{3.3}$$

Taking the angle θ between the tangent of the space-like curve (C) at the point $\sigma(s)$ with a given direction and considering $\dfrac{d\theta}{ds} = k_1$, then the equation (3.3) can be rewritten as:

$$\begin{aligned} \frac{d\lambda_1}{d\theta} &= \lambda_3 - f(\theta), \\ \frac{d\lambda_2}{d\theta} &= \rho k_2 \lambda_3, \\ \frac{d\lambda_3}{d\theta} &= \rho k_2 \lambda_2 - \lambda_1, \end{aligned} \tag{3.4}$$

In here, the radius of curvature $\rho = \dfrac{1}{k_1}$, $\rho^* = \dfrac{1}{k_1^*}$ at the points σ and σ^* are denoted respectively, such that $f(\theta) = \rho + \rho^*$.

Using the differential system (3.4), the following differential equation due to λ_1 is obtained as

$$\begin{aligned} &\left(\frac{k_1}{k_2}\right)\frac{d^3\lambda_1}{d\theta^3} + \left(\frac{k_1}{k_2}\right)' \frac{d^2\lambda_1}{d\theta^2} + \left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\frac{d\lambda_1}{d\theta} + \left(\frac{k_1}{k_2}\right)' \lambda_1 \\ &+ \left(\frac{k_1}{k_2}\right) f''(\theta) + \left(\frac{k_1}{k_2}\right)' f'(\theta) - \left(\frac{k_2}{k_1}\right) f(\theta) = 0. \end{aligned} \tag{3.5}$$

Hence, the equation (3.5) is a characterization for σ^*.

If the distance between opposite points of (C) and (C^*) is constant, then we have

$$\|\sigma^* - \sigma\| = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 = l^2 = const. \tag{3.6}$$

From the equation (3.6), we can write

$$\lambda_1 \frac{d\lambda_1}{d\theta} + \lambda_2 \frac{d\lambda_2}{d\theta} - \lambda_3 \frac{d\lambda_3}{d\theta} = 0. \tag{3.7}$$

Considering the system (3.4), we obtain

$$\lambda_1 \left[\frac{d\lambda_1}{d\theta} - \lambda_3\right] = 0. \tag{3.8}$$

Thus, we can write $\lambda_1 = 0$ or $\dfrac{d\lambda_1}{d\theta} = \lambda_3$. Now, we shall study in the following subcase.

Case 3.1. If $\dfrac{d\lambda_1}{d\theta} = \lambda_3$, then $f(\theta) = 0$. So, (C^*) is formed by the position vector $u = \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 B$ of the space-like curve of (C).

Now, we shall investigate the solution of the equation (3.5), in some special cases.

Case 3.2. Consider $\sigma(s)$ as an inclined space-like curve. If the equation (3.5) is rearranged, then we obtain the following differential equation:

$$\frac{d^3\lambda_1}{d\theta^3} + \left(1 - \frac{k_2^2}{k_1^2}\right)\frac{d\lambda_1}{d\theta} = 0 \tag{3.9}$$

General solution of the expression (3.9) depends on

$$\lambda_1 = c_1 + c_2 \cos\sqrt{1 - \frac{k_2^2}{k_1^2}}\theta + c_3 \sin\sqrt{1 - \frac{k_2^2}{k_1^2}}\theta. \tag{3.10}$$

nnd so,

$$\lambda_2 = \frac{k_2}{k_1}\left(c_2 \cos\sqrt{1-\frac{k_2^2}{k_1^2}}\theta + c_3 \sin\sqrt{1-\frac{k_2^2}{k_1^2}}\theta\right), \tag{3.11}$$

$$\lambda_3 = \sqrt{1-\frac{k_2^2}{k_1^2}}\left(c_2 \sin\sqrt{1-\frac{k_2^2}{k_1^2}}\theta - c_3 \cos\sqrt{1-\frac{k_2^2}{k_1^2}}\theta\right), \tag{3.12}$$

where c_1, c_2 and c_3 are real numbers.

3.2. **The Time-Like Curves of the Constant Breadth Via the Parallel Transport Frame of the Type-2 in E_1^3.** Assume that $\sigma = \sigma(s)$ a simple ordinary closed time-like curve in E_1^3. These curves will be determined by (C). The normal plane at every point P on the time-like curve corresponds the time-like curve at a single point Q other than P. A simple ordinary time-like curve of the constant breadth which have parallel tangents in opposite directions is defined by

$$\sigma^*(s) \quad = \quad \sigma(s) + \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 B \tag{3.13}$$

where $\lambda_i(s)$, $1 \leq i \leq 3$ are arbitrary functions, σ and σ^* are opposite points and $\{N_1, N_2, B\}$ denote the parallel transport frame of the type-2 in E_1^3.

If N_1 is taken instead of tangent vector and differentiating both sides of the expression (3.13), then we have

$$\begin{aligned} \frac{d\sigma^*}{ds} = \frac{d\sigma^*}{ds^*}\frac{ds^*}{ds} = N_1^*\frac{ds^*}{ds} = \quad & \left(1 + \frac{d\lambda_1}{ds} + \lambda_3 k_1\right) N_1 + \\ & \left(\frac{d\lambda_2}{ds} + \lambda_3 k_2\right) N_2 + \left(\frac{d\lambda_3}{ds} + \lambda_1 k_1 + \lambda_2 k_2\right) B, \end{aligned} \tag{3.14}$$

where k_1, k_2 are the first and the second curvatures of the time-like curve, respectively.

Since $N_1^* = -N_1$, we have

$$\begin{aligned} &\frac{ds^*}{ds} + \frac{d\lambda_1}{ds} + \lambda_3 k_1 + 1 = 0, \\ &\frac{d\lambda_2}{ds} + \lambda_3 k_2 = 0, \\ &\frac{d\lambda_3}{ds} + \lambda_2 k_2 + \lambda_1 k_1 = 0. \end{aligned} \tag{3.15}$$

Taking the angle θ between the tangent of the time-like curve (C) at the point $\sigma(s)$ with a given direction and considering $\frac{d\theta}{ds} = k_1$, then the equation (3.15) can be rewritten as follow:

$$\frac{d\lambda_1}{ds} = -\lambda_3 - f(\theta),$$
$$\frac{d\lambda_2}{ds} = -\rho k_2 \lambda_3, \tag{3.16}$$
$$\frac{d\lambda_3}{ds} = -\rho k_2 \lambda_2 - \lambda_1,$$

where $\rho = \dfrac{1}{k_1}$, $\rho^* = \dfrac{1}{k_1^*}$ denote the radius of curvature at the points σ and σ^* respectively, such that $f(\theta) = \rho + \rho^*$.

If the differential system (3.16) is used, then the following differential equation due to λ_1 is obtained as

$$\left(\frac{k_1}{k_2}\right)\frac{d^3\lambda_1}{d\theta^3} + \left(\frac{k_1}{k_2}\right)'\frac{d^2\lambda_1}{d\theta^2} - \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\frac{d\lambda_1}{d\theta} - \left(\frac{k_1}{k_2}\right)'\lambda_1$$
$$+\left(\frac{k_1}{k_2}\right)f''(\theta) + \left(\frac{k_1}{k_2}\right)'f'(\theta) - \left(\frac{k_2}{k_1}\right)f(\theta) = 0. \tag{3.17}$$

Hence, the equation (3.17) is a characterization for σ^*.

If the distance between opposite points of (C) and (C^*) is constant, then we get

$$\|\sigma^* - \sigma\| = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = l^2 = const. \tag{3.18}$$

Differentiating the equation (3.18), then we can write

$$-\lambda_1\frac{d\lambda_1}{d\theta} + \lambda_2\frac{d\lambda_2}{d\theta} + \lambda_3\frac{d\lambda_3}{d\theta} = 0. \tag{3.19}$$

Considering the differential system (3.16), then we obtain

$$\lambda_1\left[\frac{d\lambda_1}{d\theta} + \lambda_3\right] = 0. \tag{3.20}$$

From here, we can write $\lambda_1 = 0$ or $\dfrac{d\lambda_1}{d\theta} = -\lambda_3$. Now, we shall study in the following subcase.

Case 3.3. If $\dfrac{d\lambda_1}{d\theta} = -\lambda_3$, then $f(\theta) = 0$. So, (C^*) is formed by the position vector $u = \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 B$ of the time-like curve of (C).

Now we shall examine the solution of the equation (3.17), in some special cases.

Case 3.4. Assume that $\sigma(s)$ as an inclined time-like curves, then the solution of differential equation (3.18) gives

$$\frac{d^3\lambda_1}{d\theta^3} - \left(1 + \frac{k_2^2}{k_1^2}\right)\frac{d\lambda_1}{d\theta} = 0. \tag{3.21}$$

General solution of the equality (3.21) depends on

$$\lambda_1 = c_1 + c_2\cosh\sqrt{1 + \frac{k_2^2}{k_1^2}}\,\theta + c_3\sinh\sqrt{1 + \frac{k_2^2}{k_1^2}}\,\theta. \tag{3.22}$$

And so, we have λ_2 and λ_3, respectively,

$$\lambda_2 = \frac{k_2}{k_1}\left(c_2 \cosh\sqrt{1+\frac{k_2^2}{k_1^2}}\theta - c_3 \sinh\sqrt{1+\frac{k_2^2}{k_1^2}}\theta\right), \tag{3.23}$$

$$\lambda_3 = \sqrt{1+\frac{k_2^2}{k_1^2}}\left(-c_2 \sinh\sqrt{1+\frac{k_2^2}{k_1^2}}\theta - c_3 \cosh\sqrt{1+\frac{k_2^2}{k_1^2}}\theta\right), \tag{3.24}$$

where c_1, c_2 and c_3 are real numbers.

Corollary 3.5. For space-like and time-like curves, the position vector of σ^* can be translated by the equations (3.10), (3.11), (3.12) and (3.22), (3.23), (3.24) respectively. Also, the curvature of σ^* is obtained as $k_1^* = -k_1$.

Case 3.6. Suppose that $\lambda_1 = 0$. Considering the equations (3.5) and (3.17), then we get

$$\left(\frac{k_1}{k_2}\right) f''(\theta) + \left(\frac{k_1}{k_2}\right)' f'(\theta) - \left(\frac{k_2}{k_1}\right) f(\theta) = 0. \tag{3.25}$$

Case 3.7. Let $\sigma(s)$ be an inclined space-like or time-like curve. The equation (3.25) can be rewritten as

$$f''(\theta) - \frac{k_2^2}{k_1^2} f(\theta) = 0. \tag{3.26}$$

So, the solutions of the above differential equation are

$$f_1(\theta) = C_1 \cosh\frac{k_2}{k_1} + C_2 \sinh\frac{k_2}{k_1}, \tag{3.27}$$

where C_1 and C_2 are real numbers.

Using the equation (3.27), we obtain

$$f_2(\theta) = C_1 \sinh\frac{k_2}{k_1} - C_2 \cosh\frac{k_2}{k_1}$$
$$f_3(\theta) = -C_1 \cosh\frac{k_2}{k_1} - C_2 \sinh\frac{k_2}{k_1}.$$

References

[1] L. R. Bishop, There is More Than One Way to Frame A Curve. Amer Math Month 82, (1975), 246-251.

[2] W. Blaschke, Konvexe Bereiche Gegebener Konstanter Breite and Kleinsten Inhalts. Mathematische Annalen, 76 (4), (1915), 504-513.

[3] H. G. Bozok, S. A. Sepet and M. Ergüt, Curves of Constant Breadth According to Type-2 Bishop Frame in E^3. Communications Faculty of. Sciences University of Ankara. Series A1 Mathematics and Statistics, 66 (1), (2017), 206-212.

[4] L. Euler, De Curvis Trangularibus. Acta Academiae Petropolitanae, (1870), 3-30.

[5] M. Fujivara, On Space Curves of Constant Breadth. Tohoku Mathematical Journal, 5, (1914), 179-184.

[6] Ö. Köse, On Space Curves of Constant Breadth. Doğa Mathematical Journal, 10 (1), (1986), 11-14.

[7] R. Lopez, Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space. Int Elect Journ Geom, 3 (2) (2010), 67-101.

[8] A. P. Mellish, Notes on Differential Geometry. Annals of Mathematics, 32 (1), (1931), 181-190.

[9] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity. New York, USA: Academic Press, 1983.

[10] E. Ozyılmaz, Classical Differential Geometry of Curves According to Type-2 Bishop Trihedra. Mathematical and Computational Applications, 16 (4), (2011), 858-867.

[11] D. J.Struik, Differential Geometry in The Large. Bulletin of the American Mathematical Society, 37 (2), (1931), 49-62.

[12] Y. Ünlütürk and S. Yılmaz, A New Version of Bishop Frame And Its Application to Smarandache Curves of A Spacelike Curve in Minkowski 3-Space. International Journal of Mathematical Combinatorics, 4, (2016), 29-43.

[13] S. Yılmaz and M. Turgut, On the Time-Like Curves of Constant Breadth in Minkowski 3-Space. International Journal of Mathematical Combinatorics, Vol. 3, 34-39, (2008).

[14] S. Yılmaz and M. Turgut, Partially Null Curves of Constant Breadth in Semi-Riemannian Space. Modern Applied Science, 3 (3), (2009), 60-63.

[15] S. Yılmaz, Y. Ünlütürk and A. Mağden, A Study on the Characterizations of Non-Null Curves According to the Bishop Frame of Type-2 in Minkowski 3-Space. SAU J. Sci. 20, 2 (1), (2016), 325-335.

[1,2]Department of Mathematics, Kirklareli University, 39100 Kirklareli, Turkey
E-mail address: muradiye.1001@hotmail.com