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On some optimization principles based on hesitant fuzzy entropies

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Abstract

In an extension to the notion of fuzzy set to the hesitant fuzzy set, we study some entropy measures of hesitant fuzzy element and check essential properties of these measures.Also, due to the applicability of entropy measure in optimization problems,we apply three existing as well as one newly introduced entropy measures of the hesitant fuzzy element to study the principle of maximum entropy.

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1 Introduction

Zadeh [10] introduces the concept of fuzzy set and fuzzy logic to model some realistic situations where vagueness is involved. The notion of fuzziness drastically changed the inclination of researchers in all the fields. Fuzzy entropy is a measure of ambiguity in a fuzzy set and is an important tool in fuzzy set theory. It plays vital role in many real life problems such as pattern recognition, medical diagnosis, cluster analysis, image processing and decision-making.Shannon [4] introduces the concept of entropy to quantify the amount of uncertainty

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in a random experiment. De Luca and Termini [1] proposes fuzzy entropy to quantify vagueness/ambiguity present in a fuzzy set. Mathematically, fuzzy entropy seems to have some analogy with Shannon's pobabilistic entropy. But practically both are different. One quantifies the uncertainty present in a random experiment while other gives the amount of ambiguity present in the fuzzy set. Bhandari and Pal [5] generalizes De Luca and Termini fuzzy entropy and obtain measures of discrimination between two fuzzy sets. Kaufmann [2] proposes an entropy formula for the fuzzy set by the metric distance between its membership degree function and membership function of its nearest crisp set. In order to view the fuzziness degree of the sets in terms of a lack of distinction between the fuzzy set and its complement another method was given by Yager [14].Later on, various entropies for fuzzy sets have been given from different point of views (refer [7],[3],[18],[12],[19] and references therein).

K.T. Atanassov [8] proposes a generalization of fuzzy set by characterizing a fuzzy set with a membership and non-membership function. Atanassov, De *et al.* [9],[15] defines some operations on intuitionistic fuzzy sets. Szmidt and Kacprzyk [6] formulates axioms to define entropy for intuitionistic fuzzy sets. Vlachos and Sergiadis [11] defines intuitionistic fuzzy information measures with its application in pattern recognition. Zhang *et al.* [13]defines some information measures for interval-valued intuitionistic fuzzy sets.

In order to capture some intriguing features present in a fuzzy set, Torra and Narukawa [17] introduces another generalization of a fuzzy set and this generalization is called hesitant fuzzy set. A hesitant fuzzy set is characterized by a number of membership functions irrespective to the conventional fuzzy set which is characterized by a single membership function.

Definition 1.1 *([17],[16]) Let X be a fixed set, then a HFS N on X is given as a function $g_N(x)$ that returns a subset of [0,1] when applied to X. Mathematically, it can be represented as*

$$N = \{< x, g_N(x) > | x \in X\},$$

where $g_N(x)$ is a set of some values in [0,1] denoting the possible membership degrees of the element $x \in X$ to the set N.For convenience, Xia and Xu[20] called $g_N(x)$ as an HFE and H the set of all HFEs.

For given three HFEs, g, g_1 and g_2 Torra and Narukawa[17] and Torra[16] defined some operations which are given below:

1. $g^c(x) = \bigcup_{\delta \in g(x)} \{1 - \delta\}$,
2. $(g_1 \cup g_2(x)) = \bigcup_{\delta_1 \in g_1(x), \delta_2 \in g_2(x)} \max\{\delta_1, \delta_2\}$,
3. $(f_1 \cap g_2)(x) = \bigcup_{\delta_1 \in g_1(x), \delta_2 \in g_2(x)} \min\{\delta_1, \delta_2\}$.

Also some operations are defined by Xia and Xu[16] on the HFEs f, f_1, f_2 which are given below :

1. $g^\lambda(x) = \bigcup_{\delta \in g(x)} \{\delta^\lambda\}, \lambda > 0$,
2. $(\lambda g)(x) = \bigcup_{\delta \in g(x)} \{1 - (1 - \delta)^\lambda\}, \lambda > 0$,
3. $(g_1 \oplus g_2)(x) = \bigcup_{\delta_1 \in g_1(x), \delta_2 \in g_2(x)} \{\delta_1 + \delta_2 - \delta_1 \delta_2\}$,
4. $(g_1 \otimes g_2)(x) = \bigcup_{\delta_1 \in g_1(x), \delta_2 \in g_2(x)} \{\delta_1, \delta_2\}$.

Xu and Xia [20]introduces hesitant fuzzy entropy and cross entropy and suggests their applications in multi attribute decision making.

Definition 1.2 *An entropy HFE f is a real-valued function* $E : H \to [0, 1]$, *satisfying the following axiomatic requirements:*

HE1: $E(f) = 0 \;\; iff \;\; f = 0 \;\; or \;\; f = 1$,

HE2: $E(f) = 1 \;\; iff \;\; f_{\sigma(j)} + f_{\sigma(l_f - j + 1)} = 1, \; for \;\; j = 1, 2, ..., l_\alpha$,

HE3: $E(f) \leq E(g) \; if \;\; f_{\sigma(i)} \leq g_{\sigma(i)} \;\; for \;\; g_{\sigma_i} + g_{\sigma(l-i+1)} \leq 1 \;\; or \;\; g_{\sigma(l-i+1)} \geq 1, i = 1, 2, ..., l$,

HE4: $E(f) = E(f^c)$.

The remainder of the paper is organized as follows. In section 2, we prove optimization principle for three existing measures for hesitant fuzzy entropy. In section 3, we propose a new measure of hesitant fuzzy entropy and prove the optimization principle. Finally, section 4 presents the conclusion.

2 Optimization principles of some hesitant fuzzy entropies

Let f be a hesitant fuzzy element of length l_α then we have following formulae of entropy the hesitant fuzzy element f due to Xu and Xia[22].

1. $E^1(f) = -\frac{1}{l_\alpha \log 2}\sum_{j=1}^{l_\alpha}\left[\frac{\xi_j(x)}{2}\log\frac{\xi_j(x)}{2} + \frac{2-\xi_j(x_i)}{2}\log\frac{2-\xi_j(x_i)}{2}\right]$, where $\xi_j(x) = \xi_{\sigma_j} + \xi_{(\sigma_{l_\alpha - j+1})}$.

2. $E^2(f) = -\frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{j=1}^{l_\alpha}\left[\sin\frac{\pi\xi_j(x)}{4} + \sin\frac{\pi(2-\xi_j(x))}{4} - 1\right]$, where $\xi_j(x) = \xi_{\sigma_j} + \xi_{(\sigma_{l_\alpha - j+1})}$.

3. $E^3(f) = -\frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{j=1}^{l_\alpha}\left[\cos\frac{\pi\xi_j(x)}{4} + \cos\frac{\pi(2-\xi_j(x))}{4} - 1\right]$, where $\xi_j(x) = \xi_{\sigma_j} + \xi_{(\sigma_{l_\alpha - j+1})}$.

These measures satisfies the axiomatic requirements **HE1-HE4**.

Principle of maximum fuzzy entropy

Here, we provide the applications of entropy measures of hesitant fuzzy element f for the study of maximum hesitant fuzziness. For this study, we consider the existing measures $E^1(f)$,$E^2(f)$, $E^3(f)$ and a newly proposed measure $E(f)$.

Problem 1. Maximize

$$E^1(f) = -\frac{1}{l_\alpha \log 2}\sum_{j=1}^{l_\alpha}\left[\frac{\xi_j(x)}{2}\log\frac{\xi_j(x)}{2} + \frac{2-\xi_j(x)}{2}\log\frac{2-\xi_j(x)}{2}\right]. \tag{1}$$

subject to the following hesitant fuzzy constraints

$$\sum_{j=1}^{l_\alpha}\xi_j(x) = \alpha_0 \tag{2}$$

and

$$\sum_{j=1}^{l_\alpha} \xi_j(x) g_j(x) = k. \quad (3)$$

Consider the following Lagrangian

$$L = -\frac{1}{l_\alpha \log 2} \sum_{j=1}^{l_\alpha} \left[\frac{\xi_j(x)}{2} \log \frac{\xi_j(x)}{2} + \frac{2-\xi_j(x)}{2} \log \frac{2-\xi_j(x)}{2} \right] - \lambda_1 \left\{ \sum_{j=1}^{l_\alpha} \xi_j(x) - \alpha_0 \right\}$$
$$-\lambda_2 \left\{ \sum_{j=1}^{l_\alpha} \xi_j(x) g_j(x) - k \right\} \quad (4)$$

$$= -\frac{1}{l_\alpha \log 2} \left[\frac{\xi_1(x)}{2} \log \frac{\xi_1(x)}{2} + \frac{2-\xi_1(x)}{2} \log \frac{2-\xi_1(x)}{2} + \frac{\xi_2(x)}{2} \log \frac{\xi_2(x)}{2} + \right.$$
$$\left. \frac{2-\xi_2(x)}{2} \log \frac{2-\xi_2(x)}{2} + ... + \frac{\xi_{l_\alpha}(x)}{2} \log \frac{\xi_{l_\alpha}(x)}{2} + \frac{2-\xi_{l_\alpha}(x)}{2} \log \frac{2-\xi_{l_\alpha}(x)}{2} \right] - \lambda_1$$
$$\left\{ \left[\xi_1(x) + \xi_2(x) + ... + \xi_{l_\alpha}(x) \right] - \alpha_0 \right\} - \lambda_2 \left\{ \left[\xi_1(x) g_1(x) + \xi_2(x) g_2(x) + ... + \xi_{l_\alpha}(x) g_{l\alpha}(x) \right] \right.$$
$$\left. -k \right\}.$$

Now, $\frac{\partial L}{\partial \xi_1(x)} = 0 \Rightarrow \xi_1(x) = \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_1(x)\}2l_\alpha \log 2}}$.
Similarly, $\xi_2(x) = \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_2(x)\}2l_\alpha \log 2}}$,..., $\xi_{l_\alpha}(x) = \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_{l_\alpha}(x)\}2l_\alpha \log 2}}$.
Thus, $\frac{\partial L}{\partial \xi_j(x)} = 0$ gives $\xi_j(x) = \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha log 2}}$.
From (2) and (3), we get

$$\alpha_0 = \sum_{j=1}^{l_\alpha} \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}} \quad (5)$$

and

$$k = \sum_{j=1}^{l_\alpha} \frac{2}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}} g_j(x). \quad (6)$$

where λ_1, λ_2 can be determined from (5) and (6)
It is seen that from (6), to every value of λ_2 there is a unique value of k and vice-versa.
Further let $g_1 < g_2 < ... < g_{l_\alpha}$.

When $\lambda_2 \to -\infty$, $k = 2l_\alpha \sum_{j=1}^{l_\alpha} g_j(x) = 2l_\alpha \overline{g}$ and $\alpha_0 = 2l_\alpha$.
When $\lambda_2 \to 0$, $\alpha_0 = \sum_{j=1}^{l_\alpha} \frac{2}{1+e^{\{\lambda_1\}2l_\alpha \log 2}}$ and $k = \sum_{j=1}^{l_\alpha} \frac{2}{1+e^{\{\lambda_1\}2l_\alpha \log 2}} g_j(x)$
Thus when $\lambda_2 > 0$ then $g_1 < k < \overline{g}$ and hence

$$\begin{aligned} E^1_{\max}(f) &= -\frac{1}{l_\alpha \log 2}\sum_{j=1}^{l_\alpha}\Bigg[\frac{2}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}\log 2}} + \log\frac{2}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}} + \\ &\quad \frac{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}} + \log\frac{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}}{1+e^{\{\lambda_1+\lambda_2 g_j(x)\}2l_\alpha \log 2}}\Bigg]. \end{aligned}$$

Since $-x\log x$ is a concave function and the sum of a concave function is also a concave function, therefore $E^1_{\max}(f)$ given in (1) is a concave function.

Problem 2. In this problem, we maximize the hesitant fuzzy entropy $E^2(f)$ under the set of constraints (2) and (3).
Consider the following Lagrangian

$$\begin{aligned} L &= \frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{j=1}^{l_\alpha}\left\{\left[\sin\frac{\pi\xi_j(x)}{4} + \sin\frac{\pi(2-\xi_j(x))}{4}\right] - 1\right\} + \lambda_1\left\{\sum_{j=1}^{l_\alpha}\xi_j(x) - \alpha_0\right\} \\ &\quad + \lambda_2\left\{\sum_{j=1}^{l_\alpha}\xi_j(x)g_j(x) - k\right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{l_\alpha(\sqrt{2}-1)}\Bigg\{\left[\sin\frac{\pi\xi_1(x)}{4} + \sin\frac{\pi(2-\xi_1(x))}{4} - 1\right] + \left[\sin\frac{\pi\xi_2(x)}{4} + \sin\frac{\pi(2-\xi_2(x))}{4} - 1\right] + \ldots \\ &\quad \left[\sin\frac{\pi\xi_j(x)}{4} + \sin\frac{\pi(2-\xi_j(x))}{4} - 1\right]\Bigg\} + \lambda_1\left\{\left[\xi_1(x) + \xi_2(x) + \ldots + \xi_{l_\alpha}(x)\right] - \alpha_0\right\} + \\ &\quad \lambda_2\left\{\left[\xi_1(x)g_1(x) + \xi_2(x)g_2(x) + \ldots + \xi_{l_\alpha}(x)g_{l\alpha}(x)\right] - k\right\}. \end{aligned} \quad ($$

Now, $\frac{\partial L}{\partial \xi_1(x)} = 0 \Rightarrow \xi_1(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1 + \lambda_2 g_1(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\} + 1.$
Similarly, $\xi_2(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1 + \lambda_2 g_2(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\} + 1$,..., $\xi_{l_\alpha}(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1 + \lambda_2 g_{l_\alpha}(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\} + 1.$
Thus, $\frac{\partial L}{\partial \xi_j(x)} = 0$ gives $\xi_j(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1 + \lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\} + 1.$
From (2) and (3), we get

$$\alpha_0 \;=\; \sum_{j=1}^{l_\alpha}\left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1. \qquad (8)$$

and

$$k \;=\; \sum_{j=1}^{l_\alpha}\left(\left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1\right)g_j(x). \quad (9)$$

When $\lambda_2 \to 0$,

$$\alpha_0 \;=\; \sum_{j=1}^{l_\alpha}\left\{\frac{4}{\pi}\sin^{-1}\left[\lambda_1\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1.$$

and

$$k \;=\; \sum_{j=1}^{l_\alpha}\left(\left\{\frac{4}{\pi}\sin^{-1}\left[\lambda_1\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1\right)g_j(x).$$

Thus when $\lambda_2 > 0$, we have

$$E^2_{\max}(f) \;=\; \frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{l_\alpha}^{j=1}(\sin\theta+\cos\theta-1),$$

where $\theta = \frac{\pi}{4}\left\{\frac{4}{\pi}\sin^{-1}\left[\frac{2(2-\sqrt{2})}{\pi}l\alpha\{\lambda_1+\lambda_2 g_j(x)\}\right]+1\right\}$
Thus,$E^2_{\max}(f) = \frac{1}{l\alpha(\sqrt{2}-1)}f(\theta)$,
where, $f(\theta) = (\sin\theta+\cos\theta-1)$,
$f'(\theta) = (\cos\theta-\sin\theta)$ and
$f''(\theta) = -(\sin\theta+\cos\theta) < 0$.
Thus above equation shows that $E^2_{\max}(f)$ is concave.

Problem 3. Here we maximize another hesitant fuzzy entropy $E^3_{\max}(f)$ under the set of fuzzy constraints (2) and (3).
Consider the following Lagrangian

$$L \;=\; \frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{j=1}^{l_\alpha}\left\{\left[\cos\frac{\pi\xi_j(x)}{4}+\cos\frac{\pi(2-\xi_j(x))}{4}\right]-1\right\}+\lambda_1\left\{\sum_{j=1}^{l_\alpha}\xi_j(x)-\alpha_0\right\}$$
$$+\lambda_2\left\{\sum_{j=1}^{l_\alpha}\xi_j(x)g_j(x)-k\right\}$$

$$= \frac{1}{l_\alpha(\sqrt{2}-1)}\Bigg\{\Bigg[\cos\frac{\pi\xi_1(x)}{4}+\cos\frac{\pi(2-\xi_1(x))}{4}-1\Bigg]+\Bigg[\cos\frac{\pi\xi_2(x)}{4}+\cos\frac{\pi(2-\xi_2(x))}{4}-1\Bigg]+ ...+\Bigg[\cos\frac{\pi\xi_{l\alpha}(x)}{4}+\cos\frac{\pi(2-\xi_{l\alpha}(x))}{4}-1\Bigg]\Bigg\}+\lambda_1\Bigg\{\Bigg[\xi_1(x)+\xi_2(x)+...+\xi_{l_\alpha}(x)\Bigg]-\alpha_0\Bigg\}+ \lambda_2\Bigg\{\Bigg[\xi_1(x)g_1(x)+\xi_2(x)g_2(x)+...+\xi_{l_\alpha}(x)g_{l\alpha}(x)\Bigg]-k\Bigg\}. \quad (10)$$

Now, $\frac{\partial L}{\partial \xi_1(x)} = 0 \Rightarrow \xi_1(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_1(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1.$

Similarly, $\xi_2(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_2(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1$, ,..., $\xi_{l_\alpha}(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_{l_\alpha}(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1.$

Thus, $\frac{\partial L}{\partial \xi_j(x)} = 0$ gives $\xi_j(x) = \left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1.$

From (2) and (3), we get

$$\alpha_0 = \sum_{j=1}^{l_\alpha}\left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1. \quad (11)$$

and

$$k = \sum_{j=1}^{l_\alpha}\left(\left\{\frac{4}{\pi}\sin^{-1}\left[\{\lambda_1+\lambda_2 g_j(x)\}\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1\right)g_j(x). \quad (12)$$

When $\lambda_2 \to 0$,

$$\alpha_0 = \sum_{j=1}^{l_\alpha}\left\{\frac{4}{\pi}\sin^{-1}\left[\lambda_1\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1$$

and

$$k = \sum_{j=1}^{l_\alpha}\left(\left\{\frac{4}{\pi}\sin^{-1}\left[\lambda_1\left(\frac{2(2-\sqrt{2})}{\pi}l\alpha\right)\right]\right\}+1\right)g_j(x).$$

Thus when $\lambda_2 > 0$, we have

$$E^3_{\max}(f) = \frac{1}{l_\alpha(\sqrt{2}-1)}\sum_{l_\alpha}^{j=1}(\cos\theta+\sin\theta-1),$$

where $\theta = \frac{\pi}{4}\left\{\frac{4}{\pi}\sin^{-1}\left[\frac{2(2-\sqrt{2})}{\pi} l\alpha\{\lambda_1+\lambda_2 g_j(x)\}\right]+1\right\}$

Thus,$E^3_{\max}(f) = \frac{1}{l\alpha(\sqrt{2}-1)}\sum_{j=1}^{l\alpha} f(\theta)$,
where, $f(\theta) = (\cos\theta + \sin\theta - 1)$
$f'(\theta) = (-\sin\theta + \cos\theta)$, and
$f''(\theta) = -(\cos\theta + \sin\theta) < 0$.
Thus above equation shows that $E^3_{\max}(f)$ is concave.

3 A new hesitant fuzzy entropy

We propose the following entropy measure

$$E(f) = -\frac{1}{al_\alpha \log 2}\left[\sum_{j=1}^{l\alpha}(1+a\xi_j(x))\log(1+a\xi_j(x)) + (1+a(1-\xi_j(x)))\right.$$
$$\left.\log(1+a(1-\xi_j(x))) + (1+a)\log(1+a)\right], \tag{13}$$

where $\xi_j(x) = \xi_{\sigma_j} + \xi_{(\sigma_{l_\alpha - j+1})}, a > 0$.
$E(f)$ satisfies the axiomatic requirements **HE1-HE4**.
Problem 4. In this problem, we apply another measure study maximum hesitant fuzzy entropy principle. For this, we consider the following problem: Maximize

$$E(f) = -\frac{1}{al_\alpha \log 2}\left[\sum_{j=1}^{l\alpha}(1+a\xi_j(x))\log(1+a\xi_j(x)) + (1+a(1-\xi_j(x)))\right.$$
$$\left.\log(1+a(1-\xi_j(x))) + (1+a)\log(1+a)\right]; a > 0 \tag{14}$$

subject to constraints (2) and (3).
Consider the following Lagrangian

$$L = -\frac{1}{al_\alpha \log 2}\left[\sum_{j=1}^{l\alpha}(1+a\xi_j(x))\log(1+a\xi_j(x)) + (1+a(1-\xi_j(x)))\right.$$
$$\left.\log(1+a(1-\xi_j(x))) + (1+a)\log(1+a)\right] + \lambda_1\left\{\sum_{j=1}^{l_\alpha}\xi_j(x) - \alpha_0\right\}$$
$$+\lambda_2\left\{\sum_{j=1}^{l_\alpha}\xi_j(x)g_j(x) - k\right\} \tag{15}$$

$$= -\frac{1}{al_\alpha \log 2}\Big[\{(1+a\xi_1(x))\log(1+a\xi_1(x_i)) + (1+a(1-\xi_1(x)))$$
$$\log(1+a(1-\xi_1(x))) + (1+a)\log(1+a)\} +$$
$$\{(1+a\xi_2(x))\log(1+a\xi_2(x)) + (1+a(1-\xi_2(x)))$$
$$\log(1+a(1-\xi_2(x))) + (1+a)\log(1+a)\} + ... +$$
$$\{(1+a\xi_{l_\alpha}(x))\log(1+a\xi_{l_\alpha}(x)) + (1+a(1-\xi_\alpha(x)))$$
$$\log(1+a(1-\xi_1(x))) + (1+a)\log(1+a)\}\Big] +$$
$$\lambda_1\Big\{\Big[\xi_1(x)+\xi_2(x)+...+\xi_{l_\alpha}(x)\Big]-\alpha_0\Big\} +$$
$$\lambda_2\Big\{\Big[\xi_1(x)g_1(x)+\xi_2(x)g_2(x)+...+\xi_{l_\alpha}(x)g_{l_\alpha}(x)\Big]-k\Big\}.$$

Now,$\frac{\partial L}{\partial \xi_1(x)} = 0 \Rightarrow \xi_1(x) = \frac{1+a-e^{\{\lambda_1-\lambda_2 g_1(x)\}l_\alpha \log 2}}{a(1+e^{\{-\lambda_1-\lambda_2 g_1(x)\}l_\alpha \log 2})}$.
Similarly, $\xi_2(x) = \frac{1+a-e^{\{\lambda_1-\lambda_2 g_2(x)\}l_\alpha \log 2}}{a(1+e^{\{-\lambda_1-\lambda_2 g_2(x)\}l_\alpha \log 2})}$,..., $\xi_{l_\alpha}(x_i) = \frac{1+a-e^{\{\lambda_1-\lambda_2 g_2(x)\}l_\alpha \log 2}}{a(1+e^{\{-\lambda_1-\lambda_2 g_{l_\alpha}(x)\}l_\alpha \log 2})}$.
Thus,$\frac{\partial L}{\partial \xi_j(x)} = 0$ gives $\xi_j(x_i) = \frac{1+a-e^{\{\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2}}{a(1+e^{-\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2}})$.
From (2) and (3), we get

$$\alpha_0 = \sum_{j=1}^{l_\alpha} \frac{1+a-e^{\{-\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2}}{a(1+e^{\{-\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2})} \tag{16}$$

and

$$k = \sum_{j=1}^{l_\alpha} \frac{1+a-e^{\{-\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2}}{a(1+e^{\{-\lambda_1-\lambda_2 g_j(x)\}l_\alpha \log 2})} g_j(x). \tag{17}$$

When $\lambda_2 \to \infty$, $k = \sum_{j=1}^{l\alpha} \frac{-g_j(x)}{a}$ and $\alpha_0 = -\frac{l\alpha}{a}$.
When $\lambda_2 \to 0$, $\alpha_0 = \frac{1+a-e^{-\lambda_1 l\alpha \log 2}}{a(1+e^{-\lambda_1 l\alpha \log 2})}$ and $k = \frac{1}{2}\sum_{i=1}^{n} \frac{1+a-e^{-\lambda_1 l\alpha \log 2}}{a(1+e^{-\lambda_1 l\alpha \log 2)}} g_j(x)$.
Thus when $\lambda_2 > 0$, we have

$$E(f) = -\frac{1}{al_\alpha \log 2}\Big[\sum_{l_\alpha}^{j=1} \frac{2+a}{1+e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}} \log \frac{2+a}{1+e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}} +$$
$$\frac{(2+a)e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}}{1+e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}} + \log \frac{(2+a)e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}}{1+e^{-\lambda_1-\lambda_2 g_j(x)\}l\alpha \log 2}}\Big]. \tag{18}$$

Since $-x\log x$ is a concave function and the sum of a concave function is also a concave function, therefore $E_{\max}(f)$ given in (18) is a concave function.

4 Conclusion

In many problems in science and engineering redundancy and overlapping in similar situations occurs. If by some mathematical model we are able to remove this redundancy then it can increase the efficiency and robustness of the system. The development of new hesitant fuzzy entropy measures is expected to reduce uncertainty, which in turn may help to increase the efficiency of the system. Therefore we concluded that despite of development of many fuzzy entropy measures, still there is scope for the development of better measures which will find applications in a number of fields. Keeping this in mind, we have investigated some measures of hesitant fuzzy entropy and applied the results towards optimization principles. In future, we shall obtain optimization principles using discrimination measures of hesitant fuzzy sets.

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