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ON STOCHASTIC ELLIPTIC SYSTEM INVOLVING HIGHER ORDER OPERATOR WITH DIRICHLET CONDITIONS

AHMED MOHAMMED ABDEL AAL ABDALLAH

Abstract. In this article, we establish the generalization for the model of the stochastic elliptic systems. The existence and uniqueness of the state variable of these systems have been derived, then the set of adjoint state variable equations and inequalities that have been described the optimality conditions are given.

1. Introduction

Lions has considered a distributed control problem for elliptic operator in deterministic system [3]. Okb El Bab et al studied the modified model in case of stochastic system [4-5].

Here, we consider the following stochastic elliptic system involving higher order operator:

$$\begin{cases} (-\Delta)^m u(x) = W(x) & \text{in } G \\ \\ u(x), \dfrac{\partial u}{\partial n}, ..., \dfrac{\partial^{m-1} u}{\partial n^{m-1}} = 0 & \text{on } \partial G \end{cases} \tag{1.1}$$

where G is a bounded, compact and strictly domain in $\mathbb{R}^n$ with boundary ∂G, $u(x) = u(x,w) \in H_0^m(\Omega, \mathcal{F}, P; G)$ is a state variable process, $(x,w) \in G \times \Omega$ and $W(x)$ is a white noise. Eq.(1.1) represents the state variable process equation and $-\Delta = \sum_{i=1}^n \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

We also prove the existence and uniqueness of the optimal stochastic control of distributed and boundary types, and we discuss the necessary and sufficient conditions of the optimality.

In the following subsection, existence and uniqueness for solution of the state process equations are discussed.

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1.1 Existence and Uniqueness for the state process of the system

In this subsection, we study the existence and uniqueness of solution for Eq.(1.1). Since

$$H_0^m(\Omega,\mathcal{F},P;G)\subseteq L^2(\Omega,\mathcal{F},P;G)\subseteq H^{-m}(\Omega,\mathcal{F},P;G), \tag{1.2}$$

The model of the stochastic system (1.1) is given by:

$$Au(x)=W,\ \ A=(-\Delta)^m.$$

The elliptic operator A in the state equation (1.1) is a bounded second order self-adjoint stochastic elliptic partial differential operator.
For this operator we define the bilinear form $b(u,.)$ on $[H_0^m(\Omega,\mathcal{F},P;G)]^2$, by:

$$b(u,\Phi)=\Big(Au,\Phi\Big)_{L^2(\Omega,\mathcal{F},P;G)},\ \ u,\ \ \Phi\in H_0^m(\Omega,\mathcal{F},P;G),$$

where A maps $H_0^m(\Omega,\mathcal{F},P;G)$ onto $H^{-m}(\Omega,\mathcal{F},P;G)$, where $H^{-m}(\Omega,\mathcal{F},P;G)$ is the conjugate space of $H_0^m(\Omega,\mathcal{F},P;G)$. Then,

$$b(u,\Phi)=\mathbb{E}\bigg[\int_G(-\Delta)^m u(x)\ \Phi(x)dx\bigg], \tag{1.3}$$

and the linear form

$$L(\Phi)=\mathbb{E}\bigg[\int_G W(x)\ \Phi(x)\ dx\bigg], \tag{1.4}$$

where $W(x)$ is the white noise. Hence, the abstract variational problem associated with Eq.(1.1) can be written as

$$\begin{cases}\text{Find } u(x)\in H_0^m(\Omega,\mathcal{F},P;G)\\ \text{such that}\\ b(u,v)=L(v).\end{cases}$$

By the Lax-Milgram lemma, we prove the following theorem.

Theorem 1.1. *The bilinear form (1.3) satisfies the stochastic coerciveness condition, and then there exists a unique solution $u\in H_0^m(\Omega,\mathcal{F},P;G)$ of the system (1.1), conversely, if there exists a unique solution $u\in H_0^m(\Omega,\mathcal{F},P;G)$ such that $b(u,\Phi)=L(\Phi)$, then we get the system (1.1).*

Proof. The proof of existence and uniqueness uses the Lax- Milgram lemma. It is necessary to show that the bilinear form (1.3) is continuous and coercive. We divide the proof into several steps.

(a): Continuity: Applying Green's formaula, we find out

$$\bigg[\int_G(-\Delta)^m u(x)\ \Phi(x)dx\bigg]=\bigg[\int_G\nabla^m u(x).\ \nabla^m\Phi(x)\ dx-\int_{\partial G}F(u,\Phi)d\partial G\bigg]$$

Obviously, we can rewrite Eq.(1.3) by stochastic Green's formaula

$$b(u,\Phi)\ \ =\ \ \mathbb{E}\bigg[\int_G\nabla^m u(x).\ \nabla^m\Phi(x)\ dx\bigg],$$

hence $F(u,\Phi) = (-\Delta)^{m-1} u \dfrac{\partial \Phi}{\partial n} = 0$ on ∂G by virtue Dirchlet condition. Then, by Cauchy Schwartz inequality:

$$|b(u,\Phi)| \leq \mathbb{E}\Big[\int_G |\nabla^m u(x)|^2 dx\Big]^{\frac{1}{2}} \mathbb{E}\Big[\int_G |\nabla^m \Phi(x)|^2 dx\Big]^{\frac{1}{2}},$$

since $\|u\|_{H_0^m(\Omega,\mathcal{F},P;G)} = \mathbb{E}\Big[\int_G |\nabla^m u(x)|^2 dx\Big]^{\frac{1}{2}}$, then

$$|b(u,\Phi)| \leq \|u\|_{H_0^m(\Omega,\mathcal{F},P;G)} \|\Phi\|_{H_0^m(\Omega,\mathcal{F},P;G)}. \tag{1.5}$$

Thus, the linear form L(.) is continuous on $H_0^m(\Omega,\mathcal{F},P;G)$, by using Cauchy Schwartz inequality, we find out

$$\begin{aligned} L(\Phi) &= \mathbb{E}\Big[\int_G W(x)\ \Phi(x)\ dx\Big] \\ &\leq \mathbb{E}\Big[\int_G (W(x))^2\ dx\Big]^{\frac{1}{2}}\ \mathbb{E}\Big[\int_G (\Phi(x))^2\ dx\Big]^{\frac{1}{2}}, \\ &= \|W\|_{L^2(\Omega,\mathcal{F},P;G)} \|\Phi\|_{L^2(\Omega,\mathcal{F},P;G)}, \end{aligned}$$

from Eq.(1.2), we have $\|u\|_{L^2(\Omega,\mathcal{F},P;G)} \leq \|u\|_{H_0^m(\Omega,\mathcal{F},P;G)}$. Then, we get

$$L(\Phi) \leq \|W\|_{H_0^m(\Omega,\mathcal{F},P;G)} \|\Phi\|_{H_0^1(\Omega,\mathcal{F},P;G)} \tag{1.6}$$

(b): Coerciveness: From definition of the norm on $H_0^m(\Omega,\mathcal{F},P;G)$

$$\begin{aligned} \|u\|^2_{H_0^m(\Omega,\mathcal{F},P;G)} &= \mathbb{E}\Big[\int_G |\nabla u^m|^2\ dx\Big] \\ &= \mathbb{E}\Big[\int_G \nabla^m u.\nabla^m u\ dx\Big] \\ &\leq \mathbb{E}\Big[\int_G \nabla^m u.\nabla^m u\ dx\Big] + \mathbb{E}\Big[\int_{\partial G} F(u,\Phi) d\partial G\Big]. \end{aligned}$$

By Green's formula, we obtain

$$\begin{aligned} \|u\|^2_{H_0^m(\Omega,\mathcal{F},P;G)} &\leq \mathbb{E}\Big[\int_G -(\Delta)^m u.u\ dx\Big] \\ &= b(u,u). \end{aligned}$$

Therefore,

$$b(u,u) \geq c\|u\|^2_{H_0^m(\Omega,\mathcal{F},P;G)} \text{ (Stochastic coerciveness)} \tag{1.7}$$

It is easy to construct the following Sobolev spaces $[H_0^m(\Omega,\mathcal{F},P;G)]^2$ by the 2-times Cartesian product as follows:

$$\Big[H_0^m(\Omega,\mathcal{F},P;G)\Big]^2 = H_0^m(\Omega,\mathcal{F},P;G) \times H_0^m(\Omega,\mathcal{F},P;G).$$

Since the bilinear form $b(u,\Phi)$ is continuous and stochastic coercive on $[H_0^m(\Omega,\mathcal{F},P;G)]^2$, and the linear form is also continuous on $[H_0^m(\Omega,\mathcal{F},P;G)]^2$, by Lax Milgram lemma there exist a unique solution $u \in H_0^m(\Omega,\mathcal{F},P;G)$, such that

$$b(u,\Phi) = L(\Phi), \ \forall \Phi \in [H_0^m(\Omega,\mathcal{F},P;G)]^2. \tag{1.8}$$

Conversely, when $b(u,\Phi) = L(\Phi)$, $\forall\Phi \in [H_0^m(\Omega,\mathcal{F},P;G)]$ and $u \in [H_0^m(\Omega,\mathcal{F},P;G)]$, integrating Eqn. (1.1) on G and taking expectation, we find

$$\mathbb{E}\left[\int_G \nabla^m u(x).\ \nabla^m \Phi(x)\ dx\right] = \mathbb{E}\left[\int_G W(x)\ \Phi(x)\ dx\right]$$

By applying stochastic Green's formula:

$$\mathbb{E}\left[\int_G (-\Delta u(x))^m\ \Phi(x)\ dx + \int_{\partial G} F(u(x),\Phi(x)) d\partial G\right] = \mathbb{E}\left[\int_G W(x)\ \Phi(x)\ dx\right],$$

on ∂G and $F(u(x),\Phi(x)) = 0$ by virtue of Dirichlet condition $u(x) = 0$. By comparison of two sides, we deduce the system (1.1), which completes the proof. ■

Under the above consideration, using the theorems 1.1, 1.2 of [3], we can formulate the following Dirichlet problem, which define the state process of our control problem. Now, we formulate the control problem with adding the control in the region G and we determine the cost functional.

1.2 Formulation of the Stochastic Optimal Distributed Control Problem

In this subsection, the optimal distributed control problem corresponding to the stochastic elliptic system (1.1) with the initial boundary value Dirichlet condition is formulated.

The function y denotes the control in the space $Y = L^2(\Omega,\mathcal{F},P;G)$ and $u(y)$ is the solution (state process of the system) associated to the control y.

Let us introduce the set of admissible control by

$$Y_{ad} = \left\{ y \in L^2(\Omega,\mathcal{F},P;G) : y_a(x) \le y(x) \le y_b(x)\ \forall\ x \in G \right\}, \tag{1.9}$$

where $y_a, y_b \in L^2(\Omega,\mathcal{F},P;G)$ and $y_a(x) \le y_b(x)\ \forall\ x \in G$.

Note that $Y_{ad} \subset Y$ is non-empty, convex and bounded function in $L^2(\Omega,\mathcal{F},P;G)$.

The state process of the system u is given by the solution of the following system. We consider the following optimization problem:

$$\begin{cases} (-\Delta)^m u(y) = W + y & \text{in } G \\ \\ u(y) = 0 & \text{on } \partial G. \end{cases} \tag{1.10}$$

Eq.(1.10) represents the formulation equation for the system (1.1).
The observation equation is given by $\chi(y) = I\ u(y) \equiv u(y)$, where I is the identity operator and the cost functional is given by:

$$C(y) = \mathbb{E}\left[\int_G (u(y) - \chi_d)^2 dx\right] + \int_\Omega \left[\int_G Mzy\ dx\right]\ dp, \tag{1.11}$$

where $M > 0$ is a positive constant, χ_d is an observation function (a known element) of the space $L^2(\Omega, \mathcal{F}, P; G)$, $u(y)$ is the unique solution that satisfies the following integral equation

$$\mathbb{E}\left[\int_G \nabla^m u(x) \nabla^m \Phi(x)\ dx\right] = \mathbb{E}\left[\int_G W(x)\Phi(x)\ dx + \mathbb{E}\left[\int_{\partial G} y(x)\Phi(x) d\partial G\right]. \quad (1.12)$$

Then, the control problem is defined by: $y \in Y_{ad}$ such that $C(y) \leq C(z)\ \forall z \in Y_{ad}$. The cost (performance) functional (1.11) can be rewritten as:

$$C(y) = \mathbb{E}\left[\int_G \Big(u(y) - u(0) + u(0) - \chi_d\Big)^2 dx + \int_G Mzy\ dx\right],$$

$$C(y) = \Pi(y, z) - 2L(z),$$

where

$$\Pi(y, z) = \mathbb{E}\left[\int_G [(u(y) - u(0))^2 + (u(z) - u(0))^2 + Mzy]\ dx\right] \quad (1.13)$$

$$L(z) = \mathbb{E}\left[\int_{\partial G} ((\chi_{d_1} - u_1(0))(u_1(z) - u_1(0))) + ((\chi_{d_2} - u_2(0))(u_2(z) - u_2(0)))dx\right] \quad (1.14)$$

from continuity of bilinear and linear forms, then there exists a unique optimal control from the general theory in [3].
Moreover, we have the following theorem for distributed control which gives the characterization of the optimal control. Under the given consideration, we may apply the Theorem 1.4 in [3] to obtain the following result.

Theorem 1.2. *If the state $u(y)$ is given by Eq.(1.1) and if the cost functional is given by Eq.(1.11), then there exists a unique optimal control $y \in Y_{ad}$ such that $C(y) \leq C(z)\ \forall z \in Y_{ad}$; Moreover, it is characterized by the following equation and inequality*

$$\begin{cases} (-\Delta)^m h(y) = u(y) - \chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G, \end{cases} \quad (1.15)$$

where $h(y)$ is the adjoint state variable process. Eq.(1.15) represents the adjoint state process equation and the inequaity

$$\mathbb{E}\left[\int_G (h + My)(z - y)dx\right] \geq 0 \quad (1.16)$$

represents the necessary and sufficient condition for optimality equation.

Proof. Since $C(y)$ is differentiable and Y_{ad} is bounded, then the optimal control $y \in L^2(\Omega, \mathcal{F}, P; G)$ is characterized by [3]

$$J'(y)(z - y) \geq 0\ \forall y \in Y_{ad}$$

which is equivalent to

$$\Pi(y, z - y) - L(z - y)\ \forall z \in Y_{ad}.$$

Using Eqs. (1.13), (1.14), we get

$$\Pi(y,z-y)-L(z-y)=\mathbb{E}\left[\int_G (u(y)-\chi_d)(u(z-y)-u(0))dx+\int_G My(z-y)\ dx\right]\geq 0,$$

thus

$$\mathbb{E}\left[\int_G ((u(y)-\chi_d)(u(z)-u(y)))dx+\int_\Omega\int_G My(z-y)\ dx\right]\geq 0 \tag{1.17}$$

Now, since $(A^*h(y),u(y))=(h(y),Au(y))$, then

$$\begin{aligned}\Pi(y,z-y)-L(z-y) &= \mathbb{E}\Big[\int_G (u(y)-\chi_d)(u(z-y)-u(0))dx \\ &+ \int_\Omega\int_G My(z-y)\ dx\Big]\geq 0,\end{aligned}$$

$$(h(y),Au(y))_{[L^2(\Omega,\mathcal{F},P;G)]}=(h(y),(-\Delta)^m u(y))_{[L^2(\Omega,\mathcal{F},P;G)]},$$

by using the stochastic Green's formula, we obtain,

$$\begin{aligned}(h,(-\Delta)^m u(y))_{[L^2(\Omega,\mathcal{F},P;G)]} &= ((-\Delta)^m h(y),u(y))_{[L^2(\Omega,\mathcal{F},P;G)]} \\ &- (h,u(y))_{[L^2(\Omega,\mathcal{F},P;\partial G)]},\end{aligned}$$

by virtue of Dirichlet problem $(h,u(y))_{L^2(\Omega,\mathcal{F},P;\partial G)}=0$,

$$(h,(-\Delta)^m u(y))_{L^2(\Omega,\mathcal{F},P;G)}=((-\Delta)^m h(y),u(y))_{L^2(\Omega,\mathcal{F},P;G)}, \tag{1.18}$$

then $A^*h(y)=(-\Delta)^m h(y)$. Since the adjoint stochastic elliptic system takes the form analogous to the form in [6] then the adjoint system is proved. Where A^* is the adjoint operator for A and h is the adjoint state (co-state process). Then

$$A^*h(y)=(-\Delta)^m h(y)=u(y)-\chi_d. \tag{1.19}$$

Obviously, inequality (1.17) becomes

$$\mathbb{E}\left[\int_G ((-\Delta)^m h(y))(u(z)-u(y)))dx+\int_\Omega\int_G My(z-y)\ dx\right]\geq 0. \tag{1.20}$$

From (1.18) and (1.20), we obtain

$$\mathbb{E}\left[\int_G (h(y))((-\Delta)^m u(z)-(-\Delta)^m u(y)))dx+\int_\Omega\int_G My(z-y)\ dx\right]\geq 0. \tag{1.21}$$

Now $(-\Delta)^m u(z)-((-\Delta)^m u(y))=z-y$, hence, we get

$$\mathbb{E}\left[\int_G ((h+My)(z-y)dx\right]\geq 0.$$

Which completes the proof of the theorem . ■

If there is no restrictions on the space of control, the inequality (1.16) becomes

$$\mathbb{E}\left[\int_G (h+My)(z-y)dx\right]=0,$$

in the following proposition, we study this case.

Proposition .1.2. *If the constraints are absent, i.e., when $Y_{ad}=Y$, then the* *equality $h + My = 0$, $z \neq y$, where $z \in Y_{ad}$ the differential problem of finding the vector-function satisfies the the following relations:*

$$\begin{cases} Au + \dfrac{h}{M} = W & in\ G \\ \\ u = 0 & on\ \partial G. \end{cases} \tag{1.22}$$

this equation represents the state process equation for the system (1.1) without constraints. The equation

$$\begin{cases} Ah(y) - u(y) = -\chi_d & in\ G \\ \\ h(y) = 0 & on\ \partial G. \end{cases} \tag{1.23}$$

represents the adjoint state process equation for the system (1.1) without constraints.

Example .1.1. *For m = 1 , it is proved in [5] that the state of the system is given by*

$$\begin{cases} -\Delta u(x) = W(x) & in\ G \\ \\ u(x) = 0 & on\ \partial G \end{cases}$$

$$b(u, \Phi) = \mathbb{E}\Big[\int_G \nabla u(x)\ \nabla \Phi(x) dx\Big],$$

and the linear form

$$L(\Phi) = \mathbb{E}\Big[\int_G W(x)\ \Phi(x)\ dx\Big],$$

Example .1.2. *For m = 2, the state of the system is given by*

$$\begin{cases} \Delta^2 u(x) = W(x) & in\ G \\ \\ u(x) = \dfrac{\partial u}{\partial n} = 0 & on\ \partial G \end{cases}$$

$$b(u, \Phi) = \mathbb{E}\Big[\int_G \Delta u(x)\ \Delta \Phi(x) dx\Big],$$

and the linear form

$$L(\Phi) = \mathbb{E}\Big[\int_G W(x)\ \Phi(x)\ dx\Big],$$

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(Ahmed Mohammed Abdallah A. M. Abdallah) Higher Technological Institute, Tenth of Ramadan City, Egypt

E-mail address: ahmedabdallah6236@yahoo.com