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On Characterizations Of Surfaces With Spacelike Induced Metric In Lorentzian Heisenberg Group

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Abstract. In this paper some characterizations for surfaces which have spacelike induced metric in three dimensional Lorentzian Heisenberg group is studied.

Key words. Heisenberg group, Levi-Civita connection.
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1 Introduction

In Euclidean Geometry there are two equivalent approaches from which the notion of mean curvature of a submanifold arises. One starts with the definition of the second fundamental form as the orthogonal component of the directional derivative of a tangent vector field to the submanifold, and the mean curvature appears as the trace of the second fundamental form. The other one considers the volume functional defined on the submanifolds of the same dimension and the mean curvature appears as the gradient of this functional.

Much of the modern global theory of complete minimal surfaces in three dimensional Euclidean space studied by Osserman during the 1960's. Recently, many of the global questions arose in this classical subject. These questions deal with analytic and conformal properties, the geometry and asymptotic behavior, and the topology and classiffication of the images of certain injective minimal immersions $\varphi : M \longrightarrow \mathbb{E}^3$ which are complete in the induced Riemannian metric.[1]

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2 Heisenberg Group $\mathbb{H}_3$

The Heisenberg group historically originates in and still has its strongest ties to quantum physics: there it is a group of unitary operators acting on the space of states induced from those observables on a linear phase space, which are given by linear or by constant functions. So any Heisenberg group is a subgroup of a group of observables in certain simple examples of quantum mechanical systems.

The Heisenberg group $\mathbb{H}_3$ is defined as $\mathbb{R}^3$ with the group operation

$$(x, y, z) * (x_1, y_1, z_1) = \left(x + x_1, y + y_1, z + z_1 + \frac{1}{2} (xy_1 - x_1 y) \right).$$

The left-invariant Riemannain metric on $\mathbb{H}_3$ is given by

$$g = ds^2 = dx^2 + dy^2 - (xdy + dz)^2 .$$

The left invariant orthonormal frame on $\mathbb{H}_3$, which is belong to Riemannian metric g

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \ \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \ \mathbf{e}_3 = \frac{\partial}{\partial x}.$$

For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g,

$$\nabla_{\mathbf{e}_i}\mathbf{e}_j = \frac{1}{2}\begin{bmatrix} 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ -\mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{bmatrix},$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis. Also, we have

$$-g (e_1, e_1) = g (e_2, e_2) = g (e_3, e_3) = 1.$$

3 Spacelike Induced Metric and Some Conclusions in H_3

Theorem 3.1. Let

$$\psi(x,y) = (f(x), g(y), h(x,y)) \tag{1}$$

be a spacelike surface in $(\mathbb{H}_3, g)$, where $f(x)$, $g(y)$ and $h(x,y)$ are continous functions. If the mean curvature of the surface is zero, then one of the following conditions holds.

i)

$$\begin{aligned} & g''(-f'^3 g'^4 + 2g'^3\vartheta' - 3f^2 g'^3\vartheta' - \vartheta'^3 \\ & + fg'^2\left(g'^2 + 3\vartheta'^2\right) + (fg' + \vartheta')(f^2 g'(fg' + \vartheta') - fg'^2) \\ & + \vartheta''(-g'^4 + (fg' + \vartheta')(g'^2 g'' + g'(fg' + \vartheta')) = 0 \end{aligned} \tag{2}$$

ii)

$$f' h_{xx}(f'^2 - h_x^2) + h_x(-2f'f'' + f''h_x + f'^2 f'') = 0 \tag{3}$$

iii)

$$\psi(x,y) = (p, qy + t, -p(qy + t) + c_2). \tag{4}$$

Proof. From the derivative of the (1), we have components of the first fundamental form

$$E = g_1(\psi_x, \psi_x) = f'^2 - h_x^2, \tag{5}$$

$$F = g_1\left(\psi_x, \psi_y\right) = -h_x\left(fg' + h_y\right), \tag{6}$$

$$G = g_1\left(\psi_y, \psi_y\right) = g'^2 - \left(fg' + h_y\right)^2 \tag{7}$$

Since $\psi(x,y)$ is a spacelike surface, we have

$$f'^2 > h_x^2 \text{ and } g'^2 > \left(fg' + h_y\right)^2 \tag{8}$$

Theni according to induced metric, following conditions holds.

i). If $h_x = 0$, $(fg' + h_y) \neq 0$, then $h(x,y) = \vartheta(y)$. So, the induced metric is

$$\tilde{g}_2 = f'^2 dx^2 + \left(g'^2 - \left(fg' + \vartheta'\right)^2\right) dy^2. \tag{9}$$

The unit normal vector field of the surface is

$$N = \frac{1}{\sqrt{\left| \left(fg' + \vartheta'\right)^2 - g'^2 \right|}} \left(-g' e_1 + \left(fg' + \vartheta'\right) e_2\right). \tag{10}$$

Then, the local orthonormal basis of the surface $\psi(x, y)$

$$E_1 = \frac{1}{f'} e_3, \tag{11}$$

$$E_2 = \frac{fg' + h_y}{g'\sqrt{g'^2 - \left(fg' + \vartheta'\right)^2}} e_1 + \frac{1}{\sqrt{g'^2 - \left(fg' + \vartheta'\right)^2}} e_2. \tag{12}$$

Components of the second fundamental form are

$$h_{11} = 0, \tag{13}$$

$$\begin{aligned} h_{12} &= \frac{1}{2f'^2 g'^2 \left(g'^2 - \left(fg' + \vartheta'\right)^2\right)} \{2f' g'^4 - g'^2 (g'^2 + \left(fg' + \vartheta'\right)^2) \\ &\quad - \left(fg' + \vartheta'\right)^2 \left(g'^2 + \left(fg' + \vartheta'\right)^2 - 2f' g'^2\right)\}, \end{aligned} \tag{14}$$

$$h_{21} = \frac{1}{2f'g'}, \tag{15}$$

$$\begin{aligned} h_{22} &= \frac{1}{g' \left(g'^2 - \left(fg' + \vartheta'\right)^2\right)^{5/2}} \{g''(-f'^3 g'^4 + 2g'^3 \vartheta' - 3f^2 g'^3 \vartheta' - \vartheta'^3 \\ &\quad + fg'^2 \left(g'^2 - 3\vartheta'^2\right) + (fg' + \vartheta')(f^2 g'(fg' + \vartheta') - fg'^2) \\ &\quad + \vartheta''(-g'^4 + (fg' + \vartheta')(g'^2 g'' + g'(fg' + \vartheta')). \end{aligned} \tag{16}$$

Then, the mean curvature of the surface is

$$\begin{aligned} H \;=\; & -\frac{1}{f'^2 g' \left(g'^2 - \left(fg' + \vartheta'\right)^2\right)^{5/2}} \{g''(-f'^3 g'^4 + 2g'^3\vartheta' - 3f^2 g'^3 \vartheta' - \vartheta'^3 \\ & + fg'^2 \left(g'^2 + 3\vartheta'^2\right) + (fg' + \vartheta')(f^2 g'(fg' + \vartheta') - fg'^2) \\ & + \vartheta''(-g'^4 + (fg' + \vartheta')(g'^2 g'' + g'(fg' + \vartheta')) \end{aligned} \tag{17}$$

If the mean curvature of the surface is zero,

$$\begin{aligned} & g''(-f'^3 g'^4 + 2g'^3\vartheta' - 3f^2 g'^3 \vartheta' - \vartheta'^3 \\ & + fg'^2 \left(g'^2 + 3\vartheta'^2\right) + (fg' + \vartheta')(f^2 g'(fg' + \vartheta') - fg'^2) \\ & + \vartheta''(-g'^4 + (fg' + \vartheta')(g'^2 g'' + g'(fg' + \vartheta')) \;=\; 0. \end{aligned} \tag{18}$$

ii). If $h_x \neq 0$, $(fg' + h_y) = 0$, then $h(x,y) = -f(x)\,g(y) + c(x)$. So the induced metric is

$$\tilde{g}_2 = \left(f'^2 - h_x^2\right) dx^2 + g'^2 dy^2. \tag{19}$$

Components of the first fundamental are

$$E = g_1(\psi_x, \psi_x) = f'^2 - h_x^2, \tag{20}$$

$$F = 0, \tag{21}$$

$$G = g'^2. \tag{22}$$

and the unit normal vector field is

$$N = -\frac{1}{\sqrt{|h_x^2 - f'^2|}} \left(f' e_1 + h_x e_3\right). \tag{23}$$

Local orthonormal basis system of $\psi(x,y)$ is

$$E_1 = \frac{h_x}{f'\sqrt{|h_x^2 - f'^2|}} e_1 + \frac{1}{\sqrt{|h_x^2 - f'^2|}} e_3, \tag{24}$$

$$E_2 = \frac{1}{g'} e_2. \tag{25}$$

Coefficients of the second fundamental form are

$$h_{11} = -\frac{1}{f'\left(f'^2 - h_x^2\right)^{3/2}}\left(f'h_{xx}(f'^2 - h_x^2) + h_x(-2f'f'' + f''h_x + f'^2 f'')\right), \tag{26}$$

$$h_{12} = -\frac{\left(-f'^2 + h_x^2\right)}{2g'\left(f'^2 - h_x^2\right)}, \tag{27}$$

$$h_{21} = \frac{f'^2 + 2h_{xy}f' - h_x^2}{2f'g'\left(h_x^2 - f'^2\right)}, \tag{28}$$

$$h_{22} = 0. \tag{29}$$

The mean curvature of the surface

$$H = \frac{1}{2}\frac{g'^2}{f'\left(f'^2 - h_x^2\right)^{3/2}}\left(f'h_{xx}(f'^2 - h_x^2) + h_x(-2f'f'' + f''h_x + f'^2 f''))\right). \tag{30}$$

So if $H = 0$, then

$$f'h_{xx}(f'^2 - h_x^2) + h_x(-2f'f'' + f''h_x + f'^2 f'') = 0. \tag{31}$$

iii) If

$$fg' + h_y = 0,$$

we have

$$h(x,y) = -f(x)\,g(y) + c_2(x). \tag{32}$$

Then induced metric

$$\tilde{g}_2 = p^2 dx^2 + g'^2 dy^2 \tag{33}$$

where p is a constant. The unit normal vector field of the surface is

$$N = -e_1. \tag{34}$$

Then the mean curvature of the surface

$$H = \frac{g''}{f'^2 g'^3}. \tag{35}$$

If $H = 0$,

$$g(y) = qy + t. \tag{36}$$

So, we have

$$\psi(x, y) = (p, qy + t, -p(qy + t) + c_2). \tag{37}$$

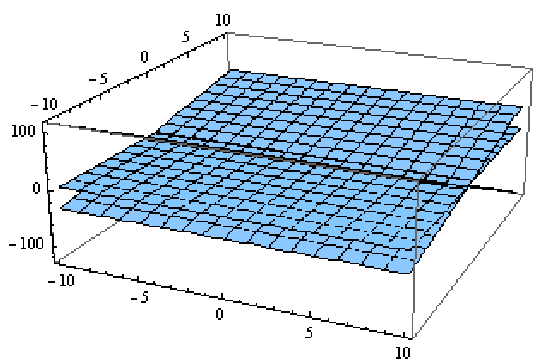
Example 3.1. Let

$$\psi(x, y) = (3, 4y + 5, -3(4y + 5) + 8) \tag{38}$$

be a spacelike surface in $(\mathbb{H}_3, g)$. The unit normal vector field of the surface is

$$N = -e_1. \tag{39}$$

Then the mean curvature surface is zero.



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