Trade-Offs in Bi-objective Transportation Problem corresponding to all Pivotal Times

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Abstract

The present paper explores one of the versions of bi-objective transportation problem with the concept of pivotal time in a scenario when both the problems have more than one pivotal time and time pivotal for one problem may or may not be pivotal for the other. The concerned problem has been divided into two scenarios and an algorithm has been proposed to glean the efficient set of trade-off pairs of transportation and deterioration cost corresponding to each pivotal time starting from maximum pivotal time to minimum pivotal time. To reinforce the existence of the suggested procedure a Numerical illustration is also given.

Keywords: Transportation problem, cost minimization problem, Time minimization problem, Bi-objective transportation problem, Efficient solution

MSC: 90B06, 90C05, 90C08

1. Introduction

A classical transportation problem is concerned about transporting various amount of a single commodity (under some supply and demand constraints) from a given set of supply origin to a given set of destinations in such a way that the total transportation cost is minimum. Instead of single objective, there may be two objectives involved associated with the transportation problem. Our methodology relies on bi-objective optimization to approach the problem at hand. An efficient mechanism to seek one or more than one solutions at

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the same time to a mathematical optimization problem that is concerned with two objective functions, bi-objective optimization has been widely used in a number of fields over a period of time. In order to look at the application of bi-objective optimization in addressing real world search and optimization problems, it is important to locate the trade-off produced as a result of different solutions among different objectives. On this premise, in a bi-objective transportation problem, a process is required to obtain the set of all possible efficient solutions. The search for the set of efficient solutions must consider all objectives to be significant.

In 1967, Geoffrion [7] provide theoretical background for bi-objective mathematical problems. Researchers like Ecker et al. [6] , Benson [5] and Armand [2] provide theoretical methods for the search of efficient solutions for linear multi-objective problems. These methods locate the set of efficient solutions in the decision space. Glickman and Berger [9] provide an algorithm for time-cost trade- offs. In 1979, Isermann [10] studied the multi-objective nature of transportation problems and provide an algorithm which enumerates the set of all efficient solutions. Their algorithm is divided into three phases where first two phases determine all basic efficient solutions and third phase construct the set of all efficient solutions as a union of a minimal number of convex sets of efficient solutions.

Even though there are several possible ways to approach multi-objective problems that are applicable to bi-objective transportation problems also, varied methods have been proposed by researchers like Swaroop et al. [19], Malhotra [11, 12], Pandian et al. [13], Quddoos et al. [16, 17] etc. A relation of time-cost trade off pairs in diverse circumstances (solid, three-dimensional, bulk) of transportation problems has been explored by several researchers [3, 4, 14, 15, 8].

It has been observed that the set of efficient solutions in decision space is larger than the set of efficient solutions in criterion space [1]. On that note Aneja and Nair [1] proposed an algorithm which determined the set of efficient solutions of a bi-criterion transportation problem in criterion space. However the proposed technique had some drawbacks which were later corrected by Malhotra [11]. Malhotra [11] came with a new and improved iterative procedure to construct the set of efficient pairs for a bi-criterion transportation problem. Malhotra et al. [12] developed a convergent iterative procedure which provides cost-pipeline trade-off corresponding to optimal time of transportation. On the same lines Sharma et al. [18] provides an algorithm for the enumeration of cost-pipeline trade-off but with the concept of pivotal time. As asserted by Sharma et al. [18] "time lag between commissioning a project and the time when the last consignment of goods reaches the project site motivates the study of a bi-criteria transportation problem at a pivotal time T".

Many a times we come across situations when achieving optimal time may increase the cost of transportation to an extent that it does not seems practical to always aim for optimal time. For instance, in some situations optimal time may only be achieved by air travel which may be very expensive. In order to avoid such a situation one may try to opt for a middle path in which neither the time taken is very long nor the cost involved is too high. In such scenarios we may opt for pivotal time (and not just any time). Sharma et al. [18] define the pivotal time corresponding to a single objective (objective of transportation cost) and obtained the cost pipeline trade-off corresponding to the pivotal time T.

However it has been observed that a single objective problem may have more than one pivotal time or we can say pivotal time is not unique and two different problems may have different pivotal times i.e. pivotal for one problem may or may not be pivotal for the other problem. These observations led us to a situation of gleaning trade-off pairs corresponding to the pivotal time where both the problems have more than one pivotal time and these pivotal times may or may not coincide with each other.
On the basis of our observations we divide the whole problem into two scenarios and propose an algorithm that will generate the desire set of efficient pairs of a bi- objective transportation problem (here we take objectives of transportation cost and deterioration cost) corresponding to each pivotal time starting from maximum pivotal time to minimum pivotal time and also found that the minimum pivotal time is also the optimal time of transportation.

2. Mathematical Formulation

The mathematical models of the concerned bi-criterion transportation problem are as follows:

(P_1)

$$\min z_1 = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

(P_2)

$$\min z_2 = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}$$

Subject to the constraints:

$$\left.\begin{array}{l} \sum_{j \in J} x_{ij} = a_i,\ a_i > 0, i \in I, \\ \sum_{i \in I} x_{ij} = b_j,\ b_j > 0, j \in J, \\ x_{ij} \geq 0, \forall\ (i, j) \in I \times J \end{array}\right\} \tag{1}$$

where I is the index set of starting points of transportation (origins) each with availability a_i and J is the index set of landing places (destinations) each with requirement/demand b_j.
From i^{th} origin to the j^{th} destination;
c_{ij}: per unit cost of transportation
d_{ij}: per unit deterioration cost of transportation
x_{ij}: amount of homogeneous product transported
Also let t_{ij} be the time of transportation from i^{th} source to the j^{th} destination independent of x_{ij}.
Problem (P_1) and (P_2) represent transportation cost and deterioration cost minimization problem respectively.

$S = \{X = (x_{ij}) \mid X$ satiesfies (1)$\}$

For any time T, define
$(P_1)^T$

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c^*_{ij} x_{ij}$$

where

$$c^*_{ij} = \begin{cases} c_{ij} & \text{if } t_{ij} \le T \\ \infty & \text{if } t_{ij} > T \end{cases}$$

and
$(P_2)^T$

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} d^*_{ij} x_{ij}$$

where

$$d^*_{ij} = \begin{cases} d_{ij} & \text{if } t_{ij} \le T \\ \infty & \text{if } t_{ij} > T \end{cases}$$

Let the optimal value of the problems $(P_1)^T$ and $(P_2)^T$ are denoted by z_1^T and z_2^T respectively.
Optimal solution of $(P_1)^T$ and $(P_2)^T$ provides minimum transportation cost and minimum deterioration cost respectively at time T or less than T.

Remark 2.1. *If* $\exists$ *at least one optimal basic feasible solution* $X = (x_{ij})$ *for the problem* $(P_1)^T$ *(yielding cost* z_1^T*) such that* $\forall$ (i, j) *with* $t_{ij} = T$, $x_{ij} = 0$ *and* $\max\limits_{x_{ij}>0} t_{ij} = T'(< T)$, *then* $z_1^T = z_1^{T'}$.
Similarly, if $\exists$ *at least one optimal basic feasible solution* $X = (x_{ij})$ *for the problem* $(P_2)^T$ *(yielding cost* z_2^T*) such that* $\forall$ (i, j) *with* $t_{ij} = T$, $x_{ij} = 0$ *and* $\max\limits_{x_{ij}>0} t_{ij} = T'(< T)$*,then* $z_2^T = z_2^{T'}$.

Definition of pivotal time as given by Sharma et al. [18].
"**Pivotal Time:** Time T is called Pivotal for the problem (P_1) if for any other time $T', T' > T \Rightarrow z_1^{T'} < z_1^T$ and $T' < T \Rightarrow z_1^{T'} > z_1^T$, where z_1^T and $z_1^{T'}$ are optimal costs of $(P_1)^T$ at times T and T' respectively".

Remark 2.2. *Interchanging the role of* (P_1) *with* (P_2), $(P_1)^T$ *with* $(P_2)^T$, z_1^T *with* z_2^T *and* $z_1^{T'}$ *with* $z_2^{T'}$ *the obtained time would be pivotal for the problem* (P_2).

A cost (transportation and deterioration) minimization problem may have more than one pivotal time. In the concerned bi- criterion problem, two types of costs (transportation and deterioration) are involved and therefore might

have different pivotal time. Mathematically which means that if $t_1^1, t_2^1, ..., t_r^1$ are r pivotal times for the problem (P_1) and $t_1^2, t_2^2, ..., t_l^2$ are l pivotal times for the problem (P_2),then t_i^1 for some $i \in \{1, 2, ..., r\}$ may or may not be equal to t_j^2 for some $j \in \{1, 2, ..., l\}$.
In order to obtain the efficient trade-off pairs of transportation and deterioration costs corresponding to the chosen pivotal time T^M, the concerned problem has been divided into two scenarios.

Scenario-A: If time T^M is pivotal for the problem (P_1)
There will be two cases associated with this;
Case-1: T^M is pivotal for the problem (P_1) only.
Case-2: T^M is pivotal for the problem (P_2) also.

Scenario-B: If time T^M is pivotal for the problem (P_2)
There will be two cases associated with this;
Case-1: T^M is pivotal for the problem (P_2) only.
Case-2: T^M is pivotal for the problem (P_1) also.

The theory that has been proposed here will work efficiently on both the scenarios and provide the exhaustive sets of efficient pairs corresponding to each pivotal time of the problems (P_1) and (P_2).

Remark 2.3. *Definitions, notations, symbols and theoretical developments (Section 3 and Section 4) are given in context to Scenario-A. The same can be implied for Scenario-B, by interchanging the role of* (P_1) *with* (P_2), $(P_1^{T^M})$ *with* $(P_2^{T^M})$ *and* Δ_{ij}^1 *with* Δ_{ij}^2.

3. Definitions, Notations and Symbols

$\{\mathbf{T^1}, \mathbf{T^2}, \mathbf{T^3}, ..., \mathbf{T^h}\}$ = Set of all pivotal times of the problem P_1 and P_2 such that $T^1 > T^2 > ... > T^h$, where $T^h = min\left\{t_i^1, t_j^2 \ \forall\ i = 1, 2, .., r \text{ and } \forall\ j = 1, 2, ..., l\right\}$.

$\mathbf{T^M}$: M^{th} pivotal time such that $T^M \in \{T^1, T^2, T^3, ..., T^h\}$.

For $M \in \{1, 2, ..., h\}$
$(\mathbf{T^M}; \mathbf{z_1^s}, \mathbf{z_2^s})$: s^{th} efficient pair of transportation and deterioration cost at pivotal time T^M.

Dominated pair: A pair $(T^M; z_1, z_2)$ is said to be dominated by another pair $(T^M; z_1^{'}, z_2^{'})$, if $z_k \geq z_k^{'}\ \forall\ k \in \{1, 2\}$ and $z_k > z_k^{'}$ for at least one $k \in \{1, 2\}$.

Non- Dominated Pair: A pair which is not dominated by any of the pair.
$\mathbf{\Delta_{ij}^1}$: Relative cost co-efficient for the cell (i, j) corresponding to the problem $P_1^{T^M}$.

$\mathbf{\Delta^2_{ij}}$: Relative cost co-efficient for the cell (i, j) corresponding to the problem $P_2^{T^M}$.

For $s \geq 1, l = 0, 1, .., t_s$
$\mathbf{X^{ls}_M}$: Basic feasible solutions yielding the s^{th} efficient pair at pivotal time T^M.

$\mathbf{X^S_M} = \left\{X^{ls}_M | l = 0, 1, .., t_s\right\}$ be the set of all basic feasible solutions for the s^{th} efficient pair.

$\mathbf{B^{ls}_M}$: Set of basic cells corresponding to the basic feasible solutions X^{ls}_M.

$\mathbf{R^{ls}_M} = \{(i, j) \notin B^{ls}_M | \Delta^1_{ij} < 0, \Delta^2_{ij} > 0$, entering which into B^{ls}_M, $\exists$ at least one cell of positive allocation in the resulting basis with time $T^M\}$.

$\mathbf{B^{ls_{(rm)}}_M}$: Set of basic cells obtained by entering a cell $(r, m) \in R^{ls}_M$ into the basis B^{ls}_M.

$\mathbf{X^{ls_{(rm)}}_M}$: Basic feasible solution corresponding to the basis $B^{ls_{(rm)}}_M$.

$\mathbf{O^{ls}_M} = \{(T^M; z_1, z_2) | z_1 = \sum_{i \in I} \sum_{j \in J} c^*_{ij} x^{ij_{(rm)}}, z_2 = \sum_{i \in I} \sum_{j \in J} d^*_{ij} x^{ij_{(rm)}},$ where $(x^{ij_{(rm)}}) \in X^{ls_{(rm)}}_M\}$.

$\mathbf{O^s_M} = \cup_{l=0}^{t_s} O^{ls}_M$.

$\mathbf{U^s_M} = N^{s-1}_M - \{(T^M; z^s_1, z^s_2)\}, U^1_M = \emptyset$.

$\mathbf{N^s_M} = U^s_M \cup O^s_M - \{(T^M; z_1, z_2) | (T^M; z_1, z_2)\}$ is a dominated pair in $U^s_M \cup O^s_M\}$.

$\mathbf{E^M}$: Collection of all the efficient pairs of transportation and deterioration cost at pivotal time T^M.

4. Theoretical Development

The theoretical developments of the concerned problem are given in context to Scenario-A i.e. when time T^M is pivotal for the problem (P_1) and may or may not be pivotal for the problem (P_2).The same can be implied for Scenario-B, by interchanging the role of (P_1) with (P_2), $(P_1^{T^M})$ with $(P_2^{T^M})$, Δ^1_{ij} with Δ^2_{ij} and redefining the theorems accordingly.

Theorem 4.1. *Time T is called pivotal for the problem (P_1) or (P_2), if and only if each optimal basic feasible solution of the problem $(P_1)^T$ or $(P_2)^T$ has at least one cell (i, j) with time $t_{ij} = T$ and $x_{ij} > 0$.*

Proof. Let time T be pivotal for the problem (P_1). Suppose there exits an optimal basic feasible solution $X = \left(x_{ij}\right)$ of $(P_1)^T$ (yielding cost $(z_1)^T$) such that $\forall (i, j)$ with

$t_{ij} = T, x_{ij} = 0$. This indicates that cells containing time T has no allocation which further implies that there exists an optimal basic feasible solution $X = \left(x_{ij}\right)$ with $\max_{x_{ij}>0} t_{ij} = T'$, where $T' < T$. Therefore from Remark 2.1, $(z_1)^{T'} = (z_1)^T$. Thus there exits a time time $T'(< T)$ with $(z_1)^{T'} = (z_1)^T$, which is a contradiction to the definition of pivotal time.
Conversely, suppose that T is not pivotal for the problem (P_1). Therefore there exits a time T' (say) less that T with $(z_1)^{T'} \leq (z_1)^T$. Therefore each cell (i, j) corresponding to the optimal basic feasible solution of the problem $(P_1)^T$ with time $t_{ij} = T$ has $x_{ij} = 0$. □

Note: Theorem 4.1 can be proved for the problem (P_2) also.

Theorem 4.2. *Corresponding to pivotal time T^M, there are finite number of efficient pairs $(T^M; {z_1}^s, {z_2}^s)$ and each corresponds to a basic feasible solution.*

Proof. The proof has been divided into two parts.
Part-1:(First efficient pair)
Let time $T = T^M [M \in \{1, 2, ..., h\}]$ be the pivotal for the problem (P_1) with optimal cost ${z_1}^1$ corresponding to the optimal basic feasible solution Y_0 with basis B_0.
Define $(P_2)^T$ for $T = T^M$(pivotal time) and read the value of z_2 at Y_0.
Construct the set

$$F_0 = \{(i, j) \notin B_0 | \Delta^1_{ij} = 0, \Delta^2_{ij} > 0\}$$

If $F_0 = \emptyset$, the obtained value of z_2 is minimum corresponding to ${z_1}^1$ at T^M and therefore denoted by ${z_2}^1$. Thus the obtained pair $(T^M; {z_1}^1, {z_2}^1)$ is the first efficient pair at T^M.
If $F_0 \neq \emptyset$, choose $(i_0, j_0) \in F_0$ such that $\Delta^2_{i_0 j_0} = max\{\Delta^2_{ij} | (i, j) \in F_0\}$ and enter the same into the basis B_0. Let Y_1 be the resulting basic feasible solution with basis B_1.
Construct the set $F_1 = \{(i, j) \notin B_1 | \Delta^1_{ij} = 0, \Delta^2_{ij} > 0\}$.
If $F_1 = \emptyset$, the pair corresponding to Y_1 would be the first efficient pair, otherwise repeat the process.
The process will continue unless get $F_f = \emptyset$ for some basis B_f at some f^{th} stage.
In this case B_f will yield the first efficient pair $(T^M; {z_1}^1, {z_2}^1)$.

Part-2: (s^{th} efficient pair)
Let X^{01}_M be basic feasible solution with basis B^{01}_M yielding the first efficient pair $(T^M; z^1_1, z^1_2)$ at pivotal time T^M.
For $s = 1$, $l = 1, 2, ..., t_1$, construct the sets

$$L = \{(i, j) \notin B^{01}_M | \Delta^1_{ij} = 0, \Delta^2_{ij} = 0\}$$

Enter all the elements (cells) of L into the basis B^{01}_M one by one. Let X^{l1}_M and B^{l1}_M $(l = 1, 2, .., t_1)$ be the resulting basic feasible solutions and basis respectively, each yielding the first efficient pair $(T^M; z^1_1, z^1_2)$.
Let $X^1_M = \{X^{l1}_M | l = 0, 1...t_1\}$ be the collection of all alternate solutions yielding the first efficient pair $(T^M; z^1_1, z^1_2)$.
Construct the sets

$R^{ls}_M = \{(i,j) \notin B^{ls}_M | \Delta^1_{ij} < 0, \Delta^2_{ij} > 0$, entering which into B^{ls}_M, $\exists$ at least one cell of positive allocation in the resulting basis with time $T^M\}$,
$O^{ls}_M = \{(T^M; z_1, z_2) | z_1 = \sum_{i\in I}\sum_{j\in J} c^*_{ij} x^{ij_{(rm)}}, z_2 = \sum_{i\in I}\sum_{j\in J} d^*_{ij} x^{ij_{(rm)}}$ where $(x^{ij_{(rm)}}) \in X^{ls_{(rm)}}_M\}$,
$O^s_M = \cup^{t_s}_{l=0} O^{ls}_M$,
$U^s_M = N^{s-1}_M - \{(T^M; z^s_1, z^s_2)\}, U^1_M = \emptyset$.
$N^s_M = U^s_M \cup O^s_M - \{(T^M; z_1, z_2) | (T^M; z_1, z_2)\}$ is a dominated pair in $U^s_M \cup O^s_M\}$.
If $N^s_M = \emptyset$, the process will terminate.
If $N^s_M \neq \emptyset$, choose $(T^M; z^{s+1}_1, z^{s+1}_2) \in N^s_M$ such that $z^{s+1}_1 = min\{z_1 | (T^M; z_1, z_2) \in N^s_M\}$.
Now set $s = s + 1$ and again construct the sets $R^{ls}_M, O^{ls}_M, U^s_M, N^s_M$. The process will continue until $N^s_M = \emptyset$ for some s.
From the above two parts it is clearly visible that each efficient pair corresponds to a basic feasible solution. Since basic feasible solutions are finite in numbers, the efficient pairs would also be same.

□

Note: Theorem 4.3, Remark 4.4 and Theorem 4.5 are motivated from [18].

Theorem 4.3. *In each efficient pair $(T^M; z^s_1, z^s_2)$, z^s_1 and z^s_2 are minimum corresponding to each other at Pivotal time T^M.*

Proof. Suppose there exists a pair $(T^M; z'_1, z'_2)$ such that $z'_1 = z^s_1, z'_2 < z^s_2$. This indicates that $(T^M; z^s_1, z^s_2)$ is dominated by $(T^M; z'_1, z'_2)$. Being an efficient pair, $(T^M; z^s_1, z^s_2)$ is non-dominated by the pairs corresponds to basic feasible solutions and therefore $(T^M; z'_1, z'_2)$ must be a convex combination of basic feasible solutions.
This contradicts the efficient character of $(T^M; z^s_1, z^s_2)$ as these basic feasible solutions must yield at least one pair $(T^M; z^s_1, z''_2)$ for which $z''_2 < z'_2$.
Thus z^s_2 minimum corresponding to z^s_1. The converse can also be prove on the same lines. □

Remark 4.4. *A pair $(T^M; z'_1, z'_2)$ for which $z'_1 = z^s_1$ for some pair $(T^M; z^s_1, z^s_2) \in E^M$ and $z'_2 \neq z^s_2$, is dominated pair.*

Theorem 4.5. *If $(T^M; z'_1, z'_2)$ is a non-dominated pair which is not in E^M, then*

$$z'_1 = \sum_{r=1}^{Q} \alpha^r z^r_1,\ z'_2 \leq \sum_{r=1}^{Q} \alpha^r z^r_2, \sum_{r=1}^{Q} \alpha^r = 1, \alpha^r \geq 0\ \forall\ r \in \{1, 2, ..Q\}$$

where Q is the index corresponding to the last efficient pair $(T^M; z^Q_1, z^Q_2)$ in E^M.

Proof. Since $(T^M; z'_1, z'_2) \notin E^M$, therefore must be yielded by a feasible solution. Also by Remark 4.4, $z'_1 \neq z^1_1$ and $z'_1 \neq z^Q_1$.
$\Rightarrow z^1_1 < z'_1 < z^Q_1$. This implies that $\exists$ scalars $\alpha^r \geq 0$ for $r \in \{1, 2, ..Q\}$ such that

$$z'_1 = \sum_{r=1}^{Q} \alpha^r z^r_1$$

where $\sum_{r=1}^{Q} \alpha^r = 1$. Let $z_2^{''} = \sum_{r=1}^{Q} \alpha^r z_2^r$. If $z_2^{'} > z_2^{''}$, then $(T^M; z_1^{'}, z_2^{'})$ is dominated by $(T^M; z_1^{'}, z_2^{''})$, which contradict the fact that $(T^M; z_1^{'}, z_2^{'})$ is a non- dominated pair. Therefore $z_2^{'} \leq \sum_{r=1}^{Q} \alpha^r z_2^r$. □

Theorem 4.6. *$N_M^Q = \emptyset$ if and only if $(T^M; z_1^Q, z_2^Q)$ is the last efficient pair in E^M.*

Proof. Let us suppose $N_M^Q = \emptyset$ and $(T^M; z_1^Q, z_2^Q)$ is not the last efficient pair in E^M. This indicates that there exists an efficient pair $(T^M; z_1^*, z_2^*)$ in E^M such that $z_1^* > z_1^Q$ and $z_2^* < z_2^Q$. Therefore either $(T^M; z_1^*, z_2^*) \in O_M^Q$ or U_M^Q. This implies $(T^M; z_1^*, z_2^*) \in N_M^Q$ which is a contradiction as $N_M^Q = \emptyset$.
Conversely suppose that $N_M^Q \neq \emptyset$. Thus either $U_M^Q \neq \emptyset$ or $O_M^Q \neq \emptyset$ or both U_M^Q and O_M^Q are non-empty. This implies that there exists at least one pair $(T^M; z_1^*, z_2^*) \in N_M^Q$ such that $z_1^* > z_1^Q$ and $z_2^* < z_2^Q$. Which further implies $(T^M; z_1^*, z_2^*) \in E^M$. Therefore $(T^M; z_1^*, z_2^*)$ is not the last efficient pair in E^M.
□

Theorem 4.7. *The last efficient pair in E^M contains the minimum deterioration cost corresponding to the Pivotal time T^M.*

Proof. Let $(T^M; z_1^Q, z_2^Q)$ be the last efficient pair in E^M, therefore from Theorem 4.6 it follows that $N_M^Q = \emptyset$. This implies emptiness of U_M^Q and O_M^Q.
(a) Emptiness of U_M^Q implies that $\nexists$ any non-dominated pair $(T^M; z_1^-, z_2^-)$ for which $z_1^- > z_1^Q$ and $z_2^- < z_2^Q$.
(b) Emptiness of O_M^Q indicates the emptiness of the set R_M^{lQ} $\forall$ $l = 1, 2, ..., t_Q$. Thus $\nexists$ any $(i, j) \notin B_M^{lQ}$ $\forall$ $l = 1, 2, .., t_Q$ such that $\Delta_{ij}^1 < 0$ and $\Delta_{ij}^2 > 0$ and by the entry of which into the current basis there exits at least one cell in the resulting basis with time T^M. There are three possibilities for $(i, j) \notin B_M^{lQ}$ with $\Delta_{ij}^2 > 0 : -$
(1) $\Delta_{ij}^1 > 0$
(2) $\Delta_{ij}^1 = 0$
(3) $\Delta_{ij}^1 < 0$ but by the entry of which into the current basis there does not exits any cell in the resulting basis with time T^M.
The above three possibilities either challenge the efficient nature of the pair $(T^M; z_1^Q, z_2^Q)$ or disturb the pivotal time T^M.
Therefore (a) and (b)simultaneously exhibits that $(T^M; z_1^Q, z_2^Q)$ contain the minimum deterioration cost at pivotal time T^M. □

Theorem 4.8. *Set E^M, $\forall$ $M \in \{1, 2, ...h\}$ records each efficient pair corresponding to pivotal time T^M.*

Proof. Let time T^M be pivotal for the problem (P_1) and an efficient pair $(T^M; z_1, z_2) \notin E^M$. Suppose $(T^M; z_1^Q, z_2^Q)$ is the last efficient pair recorded in E^M and $z_1 = z_1^k$ for some $k \in \{1, 2, .., Q\}$. Since $(T^M; z_1, z_2) \notin E^M$ therefore $z_2 \neq z_2^k$. Therefore Remark 4.4 implies $(T^M; z_1, z_2)$ is a dominated pair which is not true.

$$\text{Therefore } z_1 \neq z_1^k \ \forall k \in \{1, 2, .., Q\}. \tag{2}$$

Inevitably $z_1 > z_1^1$. If $z_1 > z_1^Q$ then $z_2 < z_2^Q$ but this will contradict the fact that z_2^Q is minimum deterioration cost at pivotal time T^M.

$$\text{Therefore } z_1^1 < z_1 < z_1^Q. \tag{3}$$

(2) and (3) together implies that there exists some $w \in \{1, 2, ..., Q\}$ such that $z_1^w < z_1 < z_1^{w+1}$. This indicates the existence of some $\xi \in (0, 1)$ such that $z_1 = \xi z_1^w + (1 - \xi) z_1^{w+1}$.
If $z_2^{''} = \xi z_2^w + (1 - \xi) z_2^{w+1}$, then pair $(T^M; z_1, z_2^{''})$ is non-dominated (being convex combination of adjacent efficient pairs). This betoken that $z_2 \nless z_2^{''}$. Also if $z_2 \geqslant z_2^{''}$, then it will disturb the efficient nature of $(T^M; z_1, z_2)$. Hence there is no such pair exists. □

Theorem 4.9. *Minimum pivotal time T^h is equal to the optimal time T^* of transportation. In other words, optimal time of transportation is the minimum pivotal time.*

Proof. Let us suppose that $T^h \neq T^*$. This implies that T^* is not pivotal for any of the problem. Therefore there exists at least one optimal basic feasible solution of $(P_1)^{T^*}$ for which there is no cell (i, j) with time $t_{ij} = T^*, x_{ij} > 0$. Therefore there exists an optimal basic feasible solution $X = \{x_{ij}\}$ with $max\{t_{ij} | x_{ij} > 0\} = T^{'} (< T^*)$, which is a contradiction to the fact that T^* is optimal time of transportation. □

Theorem 4.10. *In case-2 (Scenario-A), for the last efficient pair $(T^M; z_1^Q, z_2^Q)$, $\Delta_{ij}^2 \leq 0 \ \forall (i, j) \notin B_M^{lQ} \ \forall \ l = 1, 2, ..., t_Q$.*

Proof. Let T^M be the pivotal time for the problem (P_1) and (P_2). Since $(T^M; z_1^Q, z_2^Q)$ is the last efficient pair, z_2^Q will be minimum deterioration cost at T^M(Theorem 4.7). Let us suppose that $\exists$ a cell $(i_0, j_0) \notin B_M^{lQ}$ such that $\Delta_{i_0 j_0}^2 > 0$ for some $l \in \left\{1, 2, ..., t_Q\right\}$. This implies $z_2^Q > z_2^{T^M}$,where $z_2^{T^M}$ is the optimal deterioration cost for the problem $(P_2)^{T^M}$.
Since z_2^Q is minimum deterioration cost at T^M,therefore each set of optimal basic feasible solutions of the problem $(P_2)^{T^M}$ yielding $z_2^{T^M}$ (optimal deterioration cost) doesn't have a cell with time T^M. This implies that T^M is not Pivotal for (P_2) which is clearly a contradiction. □

Theorem 4.11. *In case-2 (Scenario-A), $z_2^Q = z_2^{T^M}$, where $(T^M; z_1^Q, z_2^Q)$ is the last efficient pair at T^M and $z_2^{T^M}$ is the optimal deterioration cost for the problem $(P_2)^{T^M}$.*

OR

In case-2 (Scenario-A), minimum deterioration cost at pivotal time T^M is same as the optimal deterioration cost for the problem $(P_2)^{T^M}$.

Proof. Let us suppose that T^M is pivotal time for the problem (P_1) and (P_2). Therefore from Theorem 4.10, there doesn't exists any $(i, j) \notin B_M^{lQ}$ $(\forall \ l = 1, 2, ...t_Q)$ for which $\Delta_{ij}^2 \geq 0$ for the problem $(P_2)^{T^M}$. This implies $z_2^Q = z_2^{T^M}$. □

5. Algorithm

STEP-1 : Procedure to find the pivotal Time. To find the pivotal time for the problem (P_1), arrange the given time routes in descending order as

$$t_{ij}^1 > t_{ij}^2 > .... > t_{ij}^u$$

Where $t_{ij}^u = min\{t_{ij}|i \in I, j \in J\}$.
Take $T = t_{ij}^1$, and solve problem $(P_1)^T$. If each optimal basic feasible solution of $(P_1)^T$ has at least one cell(route) with time T and $x_{ij} > 0$, declare $T = t_{ij}^1$ as pivotal for the problem (P_1) (Theorem 4.1) otherwise take $T = t_{ij}^2$ and repeat the above procedure until get the pivotal.

Note: The time obtained by interchanging the role of (P_1) with (P_2) and $(P_1)^T$ with $(P_2)^T$ in the above procedure, would be the pivotal for the problem (P_2).

STEP-2: Choose the pivotal time.
Initially set $M = 1, L = 1$

(2.a). Choose $t_{ij}^L = T^M$ and check whether it is pivotal for the problem $(P_1)^T$ or $(P_2)^T$ (use STEP-1). If it is pivotal for at least one problem, then move to STEP-3 otherwise proceed for (2.b).

(2.b). Repeat (2.a) for $T^M = t_{ij}^{L+1}$

STEP-3 : To find the comprehensive set E^M of efficient pairs for the pivotal time T^M.

(3.a). If $T^M = t_{ij}^F$, for some $F \in \{1, 2, ...u\}$ is pivotal for the problem $(P_1)^T$, then move to (3.b).

(3.b). Obtained the first efficient pair (T^M, z_1^1, z_2^1) (as suggested in part 1 of Theorem 4.2) and set/ update $E^M = \left\{(T^M, z_1^1, z_2^1)\right\}$. Now set $s = 1$ move to (3.c).

(3.c). Construct $R_M^{ls}, O_M^{ls}, O_M^s, U_M^s, N_M^s$ for $l = 1, 2, ..., t_s$. (Part-2 of Theorem 4.2). If $N_M^s = \emptyset$, proceed to (3.e), otherwise (3.d).

(3.d). Choose $(T^M; z_1^{s+1}, z_2^{s+1}) \in N^s$ such that $z_1^{s+1} = min\{z_1|(T^M; z_1, z_2) \in N_M^S\}$. Now set/update $E^M = E^M \cup \{(T^M; z_1^{s+1}, z_2^{s+1})\}$ and move to (3.c) for $s = s + 1$ otherwise move to (3.e)

(3.e). Declare $(T^M; z_1^s, z_2^s)$ as the last efficient pair at pivotal time T^M (see Theorem 4.6) and E^M as the comprehensive set of efficient pairs for the pivotal time T^M (as Theorem 4.8). Now move to step 4.

STEP-4: Move to next pivotal time.

(4.a). Set $M = M + 1$ and $L = L + 1$ and move to STEP-2.

6. Numerical

Consider a transportation table (Table 1) containing transportation cost (upper left corner), deterioration cost (lower right corner) and time of transportation(center). Here transportation problem is denoted by (P_1) deterioration problem is denoted by (P_2).

Table 1

					a_i
	20 / 38 / 12	13 / 60 / 37	17 / 30 / 24	14 / 58 / 10	80
	26 / 44 / 40	38 / 50 / 15	24 / 45 / 10	10 / 47 / 45	50
	21 / 56 / 40	24 / 48 / 38	33 / 35 / 25	30 / 37 / 30	30
	30 / 40 / 18	40 / 49 / 16	15 / 46 / 9	12 / 38 / 21	60
b_j	56	54	74	36	

Arrange the given time routes in descending order as;
$t_{ij}^1 = 60 > t_{ij}^2 = 58 > t_{ij}^3 = 56 > t_{ij}^4 = 50 > t_{ij}^5 = 49 > t_{ij}^6 = 48 > t_{ij}^7 = 47 > t_{ij}^8 = 46 > t_{ij}^9 = 45 > t_{ij}^{10} = 44 > t_{ij}^{11} = 40 > t_{ij}^{12} = 38 > t_{ij}^{13} = 37 > t_{ij}^{14} = 35 > t_{ij}^{15} = 30.$

Choose $t_{ij}^1 = T^1 = 60$ and solve $(P_1)^T$ and $(P_2)^T$ for $T = 60$. Table 2 and Table 3 given below exhibits the optimal basic feasible solutions of $(P_1)^{T^1}$ and $(P_2)^{T^1}$ respectively.

Table 2

12 20	**54** 13	**14** 17	14
14 26	38	24	**36** 10
30 21	24	33	30
30	40	**60** 15	12

Here $z_1^{T^1} = 3434$.

Table 3

56 12	37	24	**24** 10
17	**50** 15	10	45
40	38	**18** 25	**12** 30
18	**4** 16	**56** 9	21

Here $z_2^{T^1} = 3040$. Clearly $T^1 = 60$ is pivotal for problem (P_1) but not for (P_2) **(Case-1 of Scenario-A)**. Now move to Step 3.
Table 4 provides the basic feasible solution for the first efficient pair (60; 3434, 6076) at pivotal time $T^1 = 60$.

Table 4

20 **12** 12	13 **54** 37	17 **14** 24	14 10
26 **14** 17	38 15	24 10	10 **36** 45
21 **30** 40	24 38	33 25	30 30
30 18	40 16	15 **60** 9	12 21

Update $E^1 = \{(60; 3434, 6076)\}$.

Here $R_1^{01} = \{(1,4),(2,2),(2,3),(3,2),(3,3),(3,4),(4,4)\}$;

$$O_1^{01} = \{(60;3554,5716),(60;3700,5698),(60;3448,58100),(60;3734,5266),(60;3644,5698)\ (60;4184,4936),(60;3554,6028)\}$$

$O_1^1 = O_1^{01}$, $U_1^1 = \emptyset$ and
$N_1^1 = \{(60;3554,5716),(60;3448,58100),(60;3734,5266),(60;3644,5698),(60;4184,4936)\}$
Therefore $(T^1, z_1^2, z_2^2) = (60;3448,58100)$ and $E^1 = \{(60;3434,6076),(60;3448,58100)\}$

Continuing likewise the collection E^1 of all the efficient pairs at pivotal time $T^1 = 60$ is obtained as;

$$E^1 = \{(60;3434,6076),(60;3448,58100),(60;3554,5716),(60;3644,5698),(60;3708,5030)\ ,(60;3748,5000),(60;3818,4800),(60;3908,4460),(60;4108,3920)\}$$

Table 5 depicts the basic feasible solution providing the last efficient pair $(60;4108,3920)$ at $T^1 = 60$

Table 5

20 **[20]** 12	13 **[24]** 37	17 24	14 **[36]** 10
26 **[36]** 17	38 15	24 **[14]** 10	10 45
21 40	24 **[30]** 38	33 25	30 30
30 18	40 16	15 **[60]** 9	12 21

Here $\Delta_{22}^2 > 0, \Delta_{42}^2 > 0$. Also optimal deterioration cost is less than the minimum deterioration cost time T^1 i.e. $z_2^{T^1} = 3040 < 3920 = z_2^9$.

Next pivotal Time T^2

For the next pivotal time choose $t_{ij}^2 = T^2 = 58$ and solve $(P_1)^{T^2}$ and $(P_2)^{T^2}$ The two alternate optimal basic feasible solutions yielding $z_1^{T^2} = 4290$ (optimal cost of $(P_1)^{T^2}$) are exhibits in Table 6 and Table 7 below and Table 8 depicts the optimal basic feasible solutions yielding $z_1^{T^2} = 3040$ (optimal cost of $(P_2)^{T^2}$).

Table 6

56 20	M	**24** 17	14
26	**14** 38	24	**36** 10
21	**30** 24	33	30
30	**10** 40	**50** 15	12

Table 7

56 20	M	**14** 17	**10** 14
26	**24** 38	24	**26** 10
21	**30** 24	33	30
30	40	**60** 15	12

Table 8

56 12	M	24	**24** 10
17	**50** 15	10	45
40	38	**18** 25	**12** 30
18	**4** 16	**56** 9	21

Above tables (Table 6, Table 7, Table 8) clearly show that time $T^2 = 58$ is pivotal for the problem (P_2) only. Therefore initiating with the optimal solution of $(P_2)^{T^2}$ given in Table 8, the comprehensive set E^2 of efficient pairs at pivotal time T^2 is obtained as;

$$E^2 = \{(58; 5310, 3040), (58; 5174, 3064), (58; 5130, 3136), (58; 4886, 3160), (58; 4852, 3176), (58; 4564, 3272), (58; 4444, 3632), (58; 4290, 4318)\}.$$

Table 9 provides the last efficient pair $(58; 4290, 4318)$ at pivotal time $T^2 = 58$.

Table 9

20 **56** 12	M M	17 **14** 24	14 **10** 10
26 17	38 **24** 15	24 10	10 **26** 45
21 40	24 **30** 38	33 25	30 30
30 18	40 16	15 **60** 9	12 21

Here it could be easily seen that optimal transportation cost is equal to the minimum transportation cost at time $T^2 = 58$ i.e. $z_1^{T^2} = 4290 = z_1^8$.

Next pivotal Time T^3

Next $t_{ij}^3 = T^3 = 56$ is not pivotal for any of the problem, therefore choose $t_{ij}^4 = T^3 = 50$. Tables (Table 10 and Table 11) given below exhibits the optimal basic feasible solutions of $(P_1)^T$ and $(P_2)^T$ respectively at time $T = 50$.

Table 10

56 20	M	**24** 17	M
26	**14** 38	24	**36** 10
M	**30** 24	33	30
30	**10** 40	**50** 15	12

Table 11

56 12	M	**24** 24	M
17	**50** 15	10	45
M	38	25	**30** 30
18	**4** 16	**50** 9	**6** 21

Above tables indicates that $T^3 = 50$ is pivotal for both the problems. Here optimal values of $(P_1)^{T^3}$ and $(P_2)^{T^3}$ are $z_1{}^{T^3} = 4290$ and $z_2{}^{T^3} = 3538$ respectively. Consider the optimal solution of $(P_1)^{T^3}$ as initial solution and obtain the first efficient pair $(T^3; z_1^1, z_2^1)$. Table given below provides the first efficient pair $(50; 4290, 4578)$.

Table 12

20 **[56]** 12	M M	17 **[24]** 24	M M
26 17	38 **[14]** 15	24 10	10 **[36]** 45
M M	24 **[30]** 38	33 25	30 30
30 18	40 **[10]** 16	15 **[50]** 9	12 21

Now proceeding as given in Step 2, the efficient set

$$E^3 = \{(50; 4290, 4578), (50; 4550, 4474), (50; 4576, 3980), (50; 5174, 3590)\}.$$

Next pivotal Time T^4
Next $t_{ij}^5 = T^4 = 49$ is pivotal for both the problems and

$$E^4 = \{(49; 4430, 4978), (49; 4444, 4712), (49; 4840, 3884), (49; 5794, 3452), (49; 5860, 3314)\}$$

Each t_{ij}^p for $p \in \{6, 7, ..., 13\}$ is not pivotal for any of the problem and $T^4 = 49$ (minimum pivotal time) is also optimal time of transportation.

7. Concluding Remarks

1. The algorithm deals with the situation of gleaning efficient pairs of transportation and deterioration costs corresponding to each pivotal time for the problem (P_1) and (P_2) in a scenario when both the problems have more than one pivotal time and time which is pivotal for one problem may or may not be pivotal for another.

In case when both the problems have same pivotal time, minimum deterioration (or transportation) cost at pivotal time T^M is same as the optimal deterioration cost for the problem $(P_2)^{T^M}$ (or $(P_1)^{T^M}$)whereas in case of different pivotal time the result may or may not hold.

2. Table 13 (given below) exhibit the findings of Numerical (Section 6) in tabular form.

Table 13

Time	Pivotal		Notation	Efficient Pair	Optimal Value		Minimum Value at T	
	(P_1)	(P_2)			$(P_1)^T$	$(P_2)^T$	(P_1)	(P_2)
$t^1_{ij} = 60$	Yes	No	T^1	E^1={(60;3434,6076),(60;3448,58100), (60;3554,5716),(60;3644,5698), (60;3708,5030),(60;3748,5000), (60;3818,4800),(60;3908,4460), (60;4108,3920)}	3434	3040	3434	3920
$t^2_{ij} = 58$	No	Yes	T^2	E^2={(58;5310,3040),(58;5174,3064), (58;5130,3136),(58;4886,3160), (58;4852,3176),(58;4564,3272), (58;4444,3632),(58;4290,4318)}	4290	3040	4290	3040
$t^3_{ij} = 56$	No	No	NA	NA	NA	NA	NA	NA
$t^4_{ij} = 50$	Yes	Yes	T^3	E^3={(50;4290,4578),(50;4550,4474) (50;4576,3980),(50;5174,3590)}	4290	3590	4290	3590
$t^5_{ij} = 49$	Yes	Yes	T^4	E^4={(49;4430,4978),(49;4444,4712), (49;4840,3884),(49;5794,3452), (49;5860,3314)}	4430	3314	4430	3314

3. Following observations has been done from Numerical (Section 6).

(a) Theorem 4.10 and Theorem 4.11 could be easily seen from the Numerical (Section 6) for pivotal time $T^3 = 50$ and $T^4 = 49$.

(b) Pivotal time $T^1 = 60$ and time $T^2 = 58$ (Numerical) exhibits that Theorem 4.10 may or may not hold for Case-1 of Scenario-A (Scenario-B). For the last efficient pair $(T^1; z^1_9, z^2_9) = (60; 4108, 3920)$ at pivotal time $T^1 = 60, \Delta^2_{22} > 0, \Delta^2_{42} > 0$ for $(2,2), (4,2) \notin B^{19}_1$ (See Table 5), whereas for last efficient pair $(T^2; z^1_8, z^2_8) = (58; 4390, 4318)$ corresponding to the pivotal time $T^2 = 58$, $\Delta^1_{ij} > 0 \ \forall \ (i, j) \notin B^{18}_2$.

(c) Theorem 4.11 may or may not hold for Case-1 of Scenario-A (Scenario-B) i.e. optimal cost of transportation (deterioration) of the problem $(P_1)^T$ may or may not be equal to the minimum cost of transportation (deterioration) at pivotal time T. Table 13 accurately exhibit this from pivotal times $T^1 = 60$ and $T^2 = 58$. On one hand for $T^1 = 60, {z_2}^{T^1}$(optimal cost of transportation) $= 3920 \neq 3040 = z^1_9$ (minimum cost of transportation at time T^1) while on the other side for $T^2 = 58, {z_1}^{T^2}$ (optimal cost of deterioration) = 4290 = z_8^1 (minimum cost of deterioration at time T^2).

(d) Minimum pivotal time $T^4 = 49$ is the optimal time of transportation.

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