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ASSOCIATED CURVES OF NON-LIGHTLIKE CURVES DUE TO THE BISHOP FRAME OF TYPE-1 IN MINKOWSKI 3-SPACE

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Abstract. In this study, we define M_1, M_2-direction curves and M_1, M_2-donor curves of non-lightlike curve γ via the Bishop frame in E_1^3. We give some relations about the forementioned curves via the link of the Frenet and Bishop frames. We study the condition for associated curves to be slant helices via the Bishop frame. After defining the spherical indicatrices of associated curves, we obtain some relations between associated curves and their spherical indicatrices in terms of the frames used in the present work.

1. Introduction

There are lots of interesting and important problems in the theory of curves at differential geometry. One of the interesting problems is the problem of characterization of a regular curve in the theory of curves in the Euclidean and Minkowski spaces, see, [9]. Also there are special curves which are obtained under some definitions such as Smarandache curves, spherical indicatrices, and curves of constant breadth, and etc.

Special curves are classical differential geometric objects. These curves are obtained by assuming a special property on the original regular curve. Some of them are Smarandache curves, curves of constant breadth, Bertrand curves, and Mannheim curves, etc. Studying curves can be differed according to frame used for curve. Recently, in the studies of classical differential geometry of curves, one of the most used frames is parallel transport frame, also called Bishop frame which is an alternative frame needed for non-continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [3].

The construction of the Bishop frame is due to L. R. Bishop in [3]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [4, 5, 20]. The advantages of Bishop frame, and comparisons of Bishop frame with the Frenet frame in Euclidean 3-space were given by Bishop [3]

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Choi and Kim introduced the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve in E^3 and gave the relationship of curvature and torsion of its mates [7]. Also Choi et al. introduced the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve in E^3 and gave the relationship of curvature and torsion of its mates [7]. Körpınar et al. obtained new associated curves by using the Bishop frame in Euclidean 3-space E^3 [12]. There are also new approaches to the theory of curves due to the Bishop frame in Heisenberg groups [13, 14].

Macit and Düldül gave new associated curves by using the unit Darboux vector field of the Frenet curve in E^3 and E^4 [17]. Yılmaz investigated associated curves according to the Bishop frame of type-2 in Euclidean 3-space E^3 [21]. Kızıltuğ and Önder gave the general definition of associated curves of a Frenet curve in three dimensional compact Lie group G, and also characterized these curves in that Lie group [11].

Mak and Altınbaş studied special associated curve of a non-degenerate Frenet curve according to the Sabban frame in anti de Sitter 3-space, and then gave some characterizations of these curves in the forementioned space [18].

In this study, we define M_1, M_2-direction curves and M_1, M_2-donor curves of non-lightlike curve γ via the Bishop frame in E_1^3. We give some relations about the forementioned curves via the link of the Frenet and Bishop frames. We study the condition for associated curves to be slant helices via the Bishop frame. After defining the spherical indicatrices of associated curves, we obtain some relations between associated curves and their spherical indicatrices in terms of the frames used in the present work.

2. Preliminaries

The Minkowski three dimensional space E_1^3 is a real vector space E^3 endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is rectangular coordinate system of E_1^3. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in E_1^3, the Lorentzian cross product of u and v defined by

$$u \times v = -\begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$; timelike if $g(v, v) < 0$ and null(lightlike) if $g(v, v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vector $\delta^{\shortmid}$ are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by

$$\|a\| = \sqrt{|\langle a, a\rangle|}.$$

The curve $\delta = \delta(s)$ is called a unit speed curve if velocity vector $\delta^{\shortmid}$ is unit i.e, $\|\delta^{\shortmid}\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v, w) = 0$.

Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\delta = \delta(s)$ in the space E_1^3 [16, 19].

For a unit speed spacelike curve with first and second curvature(torsion), $\kappa(s)$ and $\tau(s)$ the following Serret-Frenet formulae in E_1^3 are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \gamma\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where $\gamma = \mp 1$ [16]. If $\gamma = 1$, then $\delta(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B and define Serret-Frenet invariants, (see, [16])

$$T(s) = \delta'(s), \quad \kappa(s) = \|T'(s)\|, \quad N(s) = \frac{T'(s)}{\kappa(s)},$$
$$B(s) = T(s) \times N(s) \text{ and } \tau(s) =< N'(s), B(s) > .$$

If $\gamma = -1$, then $\delta(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B then we write

$$T(s) = \delta'(s), \kappa(s) = \sqrt{- < T'(s), T'(s) >}, N(s) = \frac{T'(s)}{\kappa(s)},$$
$$B(s) = T(s) \times N(s) \text{ and } \tau(s) =< N'(s), B(s) > .$$

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p \rangle = r^2\}.$$

The pseudo-hyperbolic space H_0^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$H_0^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p \rangle = -r^2\}.$$

The Bishop derivative formula of type-1 of a spacelike curve with spacelike principal normal is given

$$\begin{bmatrix} T'(s) \\ M_1'(s) \\ M_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} . \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \end{bmatrix} \tag{2.3}$$

in E_1^3. Also, the relation between Frenet and Bishop frame of type-1 is given as

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} . \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \end{bmatrix}, \tag{2.4}$$

where the angle

$$\theta = \arctan h \frac{k_2}{k_1}. \tag{2.5}$$

And also there are the following expressions

$$\kappa(s) = \sqrt{k_2^2(s) - k_1^2(s)}, \quad \tau(s) = -\frac{d\theta(s)}{ds} \tag{2.6}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve $\alpha(s)$, see [20].

Proposition 2.1. Let $\varphi(s)$ be a spacelike curve with curvatures κ and τ. The curve φ lies on the Lorentzian sphere if and only if

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})\right]=\frac{\tau}{\kappa},$$

see [9].

Proposition 2.2. Let $\varphi(s)$ be a spacelike curve with curvatures κ and τ. The curve φ is a general helix if and only if

$$\frac{\kappa}{\tau}=\text{constant}, \tag{2.7}$$

see [9].

Proposition 2.3. Let $\varphi(s)$ be a spacelike curve with curvatures κ and τ. The curve φ is a slant helix if and only if

$$\sigma(s)=\left[\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})^{\prime}\right]=\text{constant}, \tag{2.8}$$

see [9].

Theorem 2.4. Let $\gamma : I \to E_1^3$ be a unit speed spacelike curve with a spacelike binormal curve with nonzero natural curvatures. Then γ is a slant helix if and only if $\dfrac{k_1}{k_2}$ is constant [4].

3. Main Results

In this section, we give some characterizations of associated curves of a spacelike curve via the Bishop frame of type-1 in Minkowski 3-space.

3.1. Associated Curves of Spacelike Curves Due to the Bishop Frames of Type-1.

In this section, we define some associated curves of a spacelike curve γ due to the Bishop frame of type-1 in E_1^3. For a Frenet frame $\gamma : I \to E_1^3$, consider a vector field V given by the Bishop frame o type-1 as follows:

$$V(s)=u(s)T(s)+v(s)M_1(s)+w(s)M_2(s), \tag{3.1}$$

where $u, v,$ and w are functions on I satisfying

$$u^2(s)-v^2(s)+w^2(s)=1. \tag{3.2}$$

Then, an integral curve $\overline{\gamma}(s)$ of V defined on I is a unit speed curve in E_1^3.

Definition 3.1. (M_1-direction curve) Let γ be a spacelike curve in E_1^3. An integral curve of M_1 is called M_1-direction curve of γ due to the Bishop frame of type-1.

Remark 3.2. A M_1-direction curve is an integral curve of the equation (3.1) with $u(s)=w(s)=0, v(s)=1$.

Definition 3.3. (M_2-direction curve) Let γ be a spacelike curve in E_1^3. An integral curve of M_2 is called M_2-direction curve of γ due to the Bishop frame of type-1.

Remark 3.4. A M_1-direction curve is an integral curve of the equation (3.1) with $u(s) = v(s) = 0, w(s) = 1$.

Theorem 3.5. Let γ be a spacelike curve in E_1^3 with the curvature κ and the torsion τ, and $\overline{\gamma}$ be the M_1-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have

$$\begin{aligned} &\overline{T} = M_1, \quad \overline{N} = T, \quad \overline{B} = M_2 \\ &\overline{\kappa} = k_1, \quad \overline{\tau} = -k_2. \end{aligned}$$

Proof. First, from Definition 3.1, we write that

$$\overline{\gamma}' = \overline{T} = M_1. \tag{3.3}$$

Differentiating (3.3) and then taking its norm, we find

$$\overline{\kappa} = k_1 \tag{3.4}$$

for $k_1 > 0$. Differentiation of (3.3) with using of (3.4) gives us

$$\overline{N} = T. \tag{3.5}$$

The vectorial product of $\overline{T}$ and $\overline{N}$ is as follows:

$$\overline{B} = \overline{T} \times \overline{N}. \tag{3.6}$$

Using (3.3), (3.5) in (3.6) we find that

$$\overline{B} = M_2. \tag{3.7}$$

Finally, differentiating (3.7) and using (3.5) in it, we have

$$\overline{\tau} = -k_2. \tag{3.8}$$

Corollary 3.6. Let γ be a spacelike curve in E_1^3 and $\overline{\gamma}$ be the M_1-direction curve of γ. The Frenet frame of $\overline{\gamma}$ is given in terms of the Bishop frame of type-1 as follows:

$$\begin{aligned} &\overline{T}(s) = M_1(s), \\ &\overline{N}(s) = \cosh(\int k_2(s)ds)\overline{M}_1(s) + \sinh(\int k_2(s)ds)\overline{M}_2(s), \\ &\overline{B}(s) = \sinh(\int k_2(s)ds)\overline{M}_1(s) + \cosh(\int k_2(s)ds)\overline{M}_2(s). \end{aligned} \tag{3.9}$$

Proof. It is straightforwardly seen by substituting (3.4) and (3.8) into (2.4).

Corollary 3.7. If the curve γ is a M_1-donor curve of the curve $\overline{\gamma}$ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$, then the curvature κ and the torsion τ of the curve γ are given by

$$\kappa = \sqrt{\overline{\tau}^2 - \overline{\kappa}^2}, \quad \tau = (\frac{\overline{\kappa}^2}{\overline{\tau}^2 - \overline{\kappa}^2})(-\frac{\overline{\tau}}{\overline{\kappa}})' \tag{3.10}$$

Proof. Taking the squares of (3.4) and (3.8) , then subtracting them side by side by using (2.6) gives us the equation (3.10).

Corollary 3.8. Let γ be a spacelike curve with the curvature κ and the torsion τ in E_1^3 and $\overline{\gamma}$ be the M_1-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then it satisfies

$$\frac{\overline{\tau}}{\overline{\kappa}} = -\tanh\theta, \quad \frac{\tau}{\kappa} = \frac{\overline{\kappa}^2}{(\overline{\tau}^2 - \overline{\kappa}^2)^{\frac{3}{2}}}(-\frac{\overline{\tau}}{\overline{\kappa}})' \tag{3.11}$$

Proof. It is straightforwardly seen by substituting (3.4), (3.8) and (3.10) into (2.5).

Proposition 3.9. Let γ be a spacelike curve in E_1^3 and $\overline{\gamma}$ be the M_1-direction curve of γ. Then the M_1-direction curve of $\overline{\gamma}$ equals to γ up to translation if and only if

$$u(s) = \sin(-\int k_2(s)ds), \quad v(s) = 0 \text{ and } \quad w = \sin(-\int k_2(s)ds).$$

Proof. Differentiating (3.2) with respect to s gives

$$uu' - vv' + ww' = 0. \tag{3.12}$$

Similarly differentiating (3.1) with respect to s, we obtain

$$V' = (u' + vk_1 - wk_2)T + (uk_1 + v')M_1 + (uk_2 + w')M_2, \tag{3.13}$$

since $V'(s) = \overline{\gamma}''(s) = \overline{T}' = \overline{\kappa}\overline{N}$, $\overline{\gamma}$ is the M_1-direction curve of γ, i.e., $\overline{\gamma}'(s) = \overline{T} = M_1$ if and only if

$$\begin{cases} u' + vk_1 - wk_2 = 0, \\ uk_1 + v' \neq 0, \\ uk_2 + w' = 0. \end{cases} \tag{3.14}$$

Multiplying the third equation in (3.14) with w and substituting it into (3.12), we have

$$uwk_2 = uu' - vv'. \tag{3.15}$$

Similarly multiplying the first equation with u and putting such an obtained equation into (3.15), we have $v(uk_1 + v') = 0$. Since $uk_1 + v' \neq 0$, it follows that $v = 0$. Hence the solutions of (3.14) which hold (3.12) are given by

$$u(s) = \sin(-\int k_2(s)ds), \quad v = 0, \text{ and } \quad w = \cos(-\int k_2(s)ds).$$

Definition 3.10. An integral curve of $\sin(-\int k_2(s)ds)T(s) + \cos(-\int k_2(s)ds)M_2(s)$ in (3.1) is called a M_1-donor curve of γ.

Theorem 3.11. Let γ be a spacelike curve in E_1^3 with the curvature κ and the torsion τ, and $\overline{\gamma}$ be the M_2-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then we have

$$\begin{aligned} &\overline{T} = M_2, \quad \overline{N} = T, \quad \overline{B} = M_1 \\ &\overline{\kappa} = -k_2, \quad \overline{\tau} = k_1. \end{aligned}$$

Proof. First, from Definition 3.3, we write that

$$\overline{\gamma}' = \overline{T} = M_2. \tag{3.16}$$

Differentiating (3.16) and then taking its norm, we find

$$\overline{\kappa} = -k_2 \tag{3.17}$$

for $k_1 > 0$. Differentiation of (3.16) with using of (3.17) gives us

$$\overline{N} = T. \tag{3.18}$$

The vectorial product of $\overline{T}$ and $\overline{N}$ is as follows:

$$\overline{B} = \overline{T} \times \overline{N}. \tag{3.19}$$

Using (3.16), (3.18) in (3.19) we find that

$$\overline{B} = M_1. \tag{3.20}$$

Finally, differentiating (3.20) and using (3.18) in it, we have

$$\overline{\tau} = k_1. \tag{3.21}$$

Corollary 3.12. Let γ be a spacelike curve in E_1^3 and $\overline{\gamma}$ be the M_2-direction curve of γ. The Frenet frame of $\overline{\gamma}$ is given in terms of the Bishop frame of type-1 as follows:

$$\begin{aligned}
&\overline{T}(s) = M_2(s),\\
&\overline{N}(s) = \cosh(-\int k_1(s)ds)\overline{M}_1(s) + \sinh(-\int k_1(s)ds)\overline{M}_2(s),\\
&\overline{B}(s) = \sinh(-\int k_1(s)ds)\overline{M}_1(s) + \cosh(-\int k_1(s)ds)\overline{M}_2(s).
\end{aligned} \tag{3.22}$$

Proof. It is straightforwardly seen by substituting (3.17) and (3.21) into (2.4).

Corollary 3.13. If the curve γ is a M_1-donor curve of the curve $\overline{\gamma}$ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$, then the curvature κ and the torsion τ of the curve γ are given by

$$\kappa = \sqrt{\overline{\kappa}^2 - \overline{\tau}^2}, \quad \tau = (\frac{\overline{\tau}^2}{\overline{\kappa}^2 - \overline{\tau}^2})(-\frac{\overline{\kappa}}{\overline{\tau}})' \tag{3.23}$$

Proof. Taking the squares of (3.17) and (3.21) **,** then subtracting them side by side by using (2.6) gives us the equation (3.23).

Corollary 3.14. Let γ be a spacelike curve with the curvature κ and the torsion τ in E_1^3 and $\overline{\gamma}$ be the M_2-direction curve of γ with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then it satisfies

$$\frac{\overline{\tau}}{\overline{\kappa}} = -\coth\theta, \quad \frac{\tau}{\kappa} = \frac{\overline{\tau}^2}{(\overline{\kappa}^2 - \overline{\tau}^2)^{\frac{3}{2}}}(-\frac{\overline{\kappa}}{\overline{\tau}})' \tag{3.24}$$

Proof. It is straightforwardly seen by substituting (3.17), (3.21) and (3.23) into (2.5).

Proposition 3.15. Let γ be a spacelike curve in E_1^3 and $\overline{\gamma}$ be the M_2-direction curve of γ. Then the M_2-direction curve of $\overline{\gamma}$ equals to γ up to translation if and only if

$$u(s) = \cosh(-\int k_1(s)ds), \quad v(s) = \sinh(-\int k_1(s)ds) \text{ and } \quad w = 0.$$

Proof. Differentiating (3.2) with respect to s gives

$$uu' - vv' + ww' = 0. \tag{3.25}$$

Similarly differentiating (3.1) with respect to s, we obtain

$$V' = (u' + vk_1 - wk_2)T + (uk_1 + v')M_1 + (uk_2 + w')M_2, \tag{3.26}$$

since $V'(s) = \overline{\gamma}''(s) = \overline{T}' = \overline{\kappa}\overline{N}$, $\overline{\gamma}$ is the M_2-direction curve of γ, i.e., $\overline{\gamma}'(s) = \overline{T} = M_2$ if and only if

$$\begin{cases} u' + vk_1 - wk_2 = 0, \\ uk_1 + v' = 0, \\ uk_2 + w' \neq 0. \end{cases} \tag{3.27}$$

Multiplying the second equation in (3.27) with v and substituting it into (3.25), we have

$$-uvk_1 = uu' + ww'. \tag{3.28}$$

Similarly multiplying the first equation with u and putting such an obtained equation into (3.28), we have $w(uk_2 + w') = 0$. Since $uk_2 + w' \neq 0$, it follows that $w = 0$. Hence the solutions of (3.27) which hold (3.25) are given by

$$u(s) = \cosh(-\int k_1(s)ds), \quad v = \sinh(-\int k_1(s)ds), \text{ and } \quad w = 0.$$

Definition 3.16. An integral curve of $\cosh(-\int k_1(s)ds)T(s)+\sinh(-\int k_1(s)ds)M_1(s)$ in (3.1) is called a M_2-donor curve of γ.

3.2. Associated Curves as Slant Helices Due to the Bishop Frame in E_1^3.

Theorem 3.17. Let γ be a spacelike curve in E_1^3,

(i) $\overline{\gamma}$ be the M_1-direction curve of γ, Then γ is a slant helix due to the Bishop frame in E_1^3 if and only if $\overline{\gamma}$ is a general helix whose Bishop curvatures of type-1 satisfy

$$\frac{(\overline{k}_2^2(s) - \overline{k}_1^2(s))^{\frac{3}{2}}}{\overline{k}_1^2(s)}(\frac{\overline{k}_1(s)}{\overline{k}_2(s)})' = const. \tag{3.29}$$

(i) $\overline{\gamma}$ be the M_2-direction curve of γ, Then γ is a slant helix due to the Bishop frame of type-1 in E_1^3 if and only if $\overline{\gamma}$ is a general helix whose Bishop curvatures of type-1 satisfy

$$\frac{\overline{k}_1^2(s)}{(\overline{k}_2^2(s) - \overline{k}_1^2(s))^{\frac{3}{2}}}(\frac{\overline{k}_2(s)}{\overline{k}_1(s)})' = const. \tag{3.30}$$

Proof. **(i)** Since γ is a slant helix due to the Bishop frame of type-1 in E_1^3, it satisfies

$$\frac{k_1(s)}{k_2(s)} = const., \tag{3.31}$$

using (3.4) and (3.8) in (3.31), we have

$$\frac{k_1(s)}{k_2(s)} = -\frac{\overline{\kappa}}{\overline{\tau}}. \tag{3.32}$$

The expressions (3.31) and (3.32) together mean that $\overline{\gamma}$ is a general helix. Using the Bishop invariants of type-1 of the M_1-direction curve $\overline{\gamma}$ and making straightforward calculations in (3.32) gives the relation (3.29).

(ii) Since γ is a slant helix due to the Bishop frame of type-1 in E_1^3, it satisfies

$$\frac{k_1(s)}{k_2(s)} = const., \tag{3.33}$$

using (3.17) and (3.21) in (3.33), we have

$$\frac{k_1(s)}{k_2(s)} = -\frac{\overline{\tau}}{\overline{\kappa}}. \tag{3.34}$$

The expressions (3.33) and (3.34) together mean that the M_2-direction curve $\overline{\gamma}$ is a general helix. Using the Bishop invariants of type-1 of the M_2-direction curve $\overline{\gamma}$ and making straightforward calculations in (3.34) gives the relation (3.30).

Corollary 3.18. Let γ be a spacelike curve in E_1^3, **(i)** the M_1-direction curve $\overline{\gamma}$ is a timelike curve. **(ii)** the M_2-direction curve $\overline{\gamma}$ is a spacelike curve.

Proof. It is straightforwardly seen from definitions of M_1 and M_2direction curves that **(i)** the M_1-direction curve $\overline{\gamma}$ is a timelike curve since

$$\langle \overline{\gamma}', \overline{\gamma}' \rangle = \langle \overline{T}, \overline{T} \rangle = \langle M_1, M_1 \rangle = -1,$$

(ii) the M_2-direction curve $\overline{\gamma}$ is a spacelike curve since

$$\langle \overline{\gamma}', \overline{\gamma}' \rangle = \langle \overline{T}, \overline{T} \rangle = \langle M_2, M_2 \rangle = 1.$$

3.3. Bishop Spherical Images of M_1-Direction Curves in E_1^3.

3.3.1. ***Tangent Bishop Spherical Images of a Regular M_1-Direction Curve $\overline{\gamma}$ in E_1^3.*** **Definition 3.19.** Let $\overline{\gamma} = \overline{\gamma}(s)$ be a M_1-direction curve of γ in E_1^3. If

we translate the tangent vector field of the Bishop frame of type-1 to the center O of the unit Lorentzian sphere S^2, we obtain a spherical image $\alpha = \alpha(s_\alpha)$. This curve is called the tangent Bishop spherical image of the curve $\overline{\gamma} = \overline{\gamma}(s)$.

Let $\alpha = \alpha(s_\alpha)$ be the tangent Bishop spherical image of a regular curve $\overline{\gamma} = \overline{\gamma}(s)$. Differentiating of α with respect to s gives

$$\alpha' = \frac{d\alpha}{ds_\alpha}\frac{ds_\alpha}{ds} = \overline{k}_1 \overline{M}_1 + \overline{k}_2 \overline{M}_2.$$

Here, we denote derivative according to s by a dash, and to s_α by a dot. In terms of the Bishop frame of type-1 in (2.3), we obtain the tangent vector of the spherical image as follows:

$$\overline{T}_\alpha = \frac{\overline{k}_1 \overline{M}_1 + \overline{k}_2 \overline{M}_2}{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}} = \frac{\overline{k}_1}{k_1}\overline{M}_1 + \frac{\overline{k}_2}{k_1}\overline{M}_2, \tag{3.35}$$

where

$$\frac{ds_\alpha}{ds} = \sqrt{\overline{k}_1^2 + \overline{k}_2^2} = \overline{\kappa}(s) = k_1(s).$$

In order to determine the first curvature of α, we write

$$\dot{\overline{T}}_\alpha = -\overline{T} + (\frac{\overline{k}_1}{k_1})'\overline{M}_1 + (\frac{\overline{k}_2}{k_1})'\overline{M}_2.$$

Thus we find

$$\overline{\kappa}_\alpha = \left\|\dot{\overline{T}}_\alpha\right\| = \sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}. \tag{3.36}$$

Therefore, we have the principal normal

$$\overline{N}_\alpha = \frac{-1}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{T} + \frac{(\frac{\overline{k}_1}{k_1})'}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{M}_1$$
$$+ \frac{(\frac{\overline{k}_2}{k_1})'}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{M}_2.$$

By the cross product of $\overline{T}_\alpha \times \overline{N}_\alpha$, we obtain the binormal vector field

$$\overline{B}_\alpha = \frac{\frac{\overline{k}_1}{k_1}(\frac{\overline{k}_2}{k_1})' - \frac{\overline{k}_2}{k_1}(\frac{\overline{k}_1}{k_1})'}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{T} + \frac{\frac{\overline{k}_2}{k_1}}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{M}_1$$
$$- \frac{\frac{\overline{k}_1}{k_1}}{\sqrt{1 + [(\frac{\overline{k}_1}{k_1})']^2 + [(\frac{\overline{k}_2}{k_1})']^2}}\overline{M}_2.$$

By means of the obtained equations, we express the torsion of the tangent Bishop spherical image as follows:

$$
\begin{aligned}
\tau = {} & \Big(\frac{-1}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}}\Big)' \frac{\frac{\overline{k}_1}{k_1}(\frac{\overline{k}_2}{k_1})' - \frac{\overline{k}_2}{k_1}(\frac{\overline{k}_1}{k_1})'}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}} \\
& -\Big(\frac{(\frac{\overline{k}_1}{k_1})'}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}}\Big)' \frac{\frac{\overline{k}_2}{k_1}}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}} \\
& +\Big(\frac{(\frac{\overline{k}_2}{k_1})'}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}}\Big)' \frac{\frac{\overline{k}_1}{k_1}}{\sqrt{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}} \\
& +\frac{\overline{k}_1(\frac{\overline{k}_1}{k_1})'[\frac{\overline{k}_1}{k_1}(\frac{\overline{k}_2}{k_1})' - \frac{\overline{k}_2}{k_1}(\frac{\overline{k}_1}{k_1})']}{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2} - \frac{\overline{k}_2(\frac{\overline{k}_2}{k_1})'[\frac{\overline{k}_1}{k_1}(\frac{\overline{k}_2}{k_1})' - \frac{\overline{k}_2}{k_1}(\frac{\overline{k}_1}{k_1})']}{1+[(\frac{\overline{k}_1}{k_1})']^2+[(\frac{\overline{k}_2}{k_1})']^2}.
\end{aligned}
\tag{3.37}
$$

Consequently, we determined the Frenet-Serret invariants of the tangent Bishop spherical image of M_1-direction curve $\overline{\gamma}$ in terms of the Bishop invariants of type-1.

Corollary 3.20. Let $\alpha = \alpha(s_\alpha)$ be the tangent Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame, then the tangent spherical image α is a circle in the osculating plane.

3.3.2. $\overline{M}_1$ ***Bishop Spherical Images of a Regular*** M_1-***Direction Curve*** $\overline{\gamma}$ ***in*** E_1^3. **Definition 3.21.** Let $\overline{\gamma} = \overline{\gamma}(s)$ be a M_1-direction curve of γ in E_1^3. If we

translate the $\overline{M}_1$ vector field of the Bishop frame of type-1 to the center O of the unit Lorentzian sphere S^2, we obtain a spherical image $\beta = \beta(s_\beta)$. This curve is called the $\overline{M}_1$ Bishop spherical image of the curve $\overline{\gamma} = \overline{\gamma}(s)$.

Let $\beta = \beta(s_\beta)$ be the $\overline{M}_2$ Bishop spherical image of a regular curve $\overline{\gamma} = \overline{\gamma}(s)$. Differentiating of β with respect to s gives

$$\beta' = \frac{d\beta}{ds_\beta}\frac{ds_\beta}{ds} = \overline{k}_1\overline{T}.$$

Thus we have

$$\overline{T}_\beta = \overline{T} = M_1 \text{ and } \quad \frac{ds_\beta}{ds} = \overline{k}_1. \tag{3.38}$$

Since M_1 is a timelike vector, β is also a timelike curve from (3.38). One calculate

$$\overline{T}'_\beta = \dot{\overline{T}}_\beta \frac{ds_\beta}{ds} = \overline{k}_1 \overline{M}_1 + \overline{k}_2 \overline{M}_2$$

or

$$\dot{\overline{T}}_\beta = \overline{M}_1 + \frac{\overline{k}_2}{\overline{k}_1} \overline{M}_2.$$

Then we find the curvature of β as

$$\overline{\kappa}_\beta = \left\| \dot{\overline{T}}_\beta \right\| = \sqrt{1 + [\frac{\overline{k}_2}{\overline{k}_1}]^2} = \frac{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}}{\overline{k}_1}. \tag{3.39}$$

Therefore, we have the principal normal

$$\overline{N}_\beta = \frac{\overline{k}_1}{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}} \overline{M}_1 + \frac{\overline{k}_2}{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}} \overline{M}_2.$$

By the cross product of $\overline{T}_\beta \times \overline{N}_\beta$, we obtain the binormal vector field

$$\overline{B}_\beta = \frac{\overline{k}_2}{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}} \overline{M}_1 - \frac{\overline{k}_1}{\sqrt{\overline{k}_1^2 + \overline{k}_2^2}} \overline{M}_2.$$

By means of the obtained equations, we express the torsion of the $\overline{M}_1$ Bishop spherical image of a regular curve $\overline{\gamma} = \overline{\gamma}(s)$ as follows:

$$\overline{\tau}_\beta = \frac{\overline{k}_1 \overline{k}'_2 - \overline{k}'_1 \overline{k}_2}{\overline{k}_1^2 + \overline{k}_2^2}. \tag{3.40}$$

Consequently, we determined the Frenet-Serret invariants of the $\overline{M}_1$ Bishop spherical image of M_1-direction curve $\overline{\gamma}$ in terms of the Bishop invariants of type-1.

Considering the equations (3.39) and (3.40) by Theorem 2.4, we have

Corollary 3.22. Let $\beta = \beta(s_\beta)$ be the $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame, then the $\overline{M}_1$ Bishop spherical image β is a circle in the osculating plane.

In the light of Propositions 2.2, and 2.3, we give the following results without proofs:

Corollary 3.23. Let $\beta = \beta(s_\beta)$ be the $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a general helix due to the Bishop frame, then the Bishop curvatures of β satisy

$$\frac{(\overline{k}_1^2 + \overline{k}_2^2)^{\frac{3}{2}}}{\overline{k}_1^2 \overline{k}'_2 - \overline{k}'_1 \overline{k}_1 \overline{k}_2} = constant.$$

Corollary 3.24. Let $\beta = \beta(s_\beta)$ be the $\overline{M}_1$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the

Bishop frame, then the Bishop curvatures of β satisy

$$\frac{\frac{\overline{k}_1^2+\overline{k}_2^2}{\overline{k}_1^2}(\frac{\overline{k}_1^3\overline{k}_2'-\overline{k}_1'\overline{k}_1^2\overline{k}_2}{(\overline{k}_1^2+\overline{k}_2^2)^{\frac{3}{2}}})'}{(\frac{\overline{k}_1^2+\overline{k}_2^2}{\overline{k}_1^2}+(\frac{\overline{k}_1\overline{k}_2'-\overline{k}_1'\overline{k}_2}{\overline{k}_1^2+\overline{k}_2^2})^2)}=constant.$$

3.3.3. $\overline{M}_2$ ***Bishop Spherical Images of a Regular*** M_1-***Direction Curve*** $\overline{\gamma}$ ***in*** E_1^3. **Definition 3.25.** Let $\overline{\gamma}=\overline{\gamma}(s)$ be a M_1-direction curve of γ in E_1^3. If

we translate the $\overline{M}_2$ vector field of the Bishop frame of type-1 to the center O of the unit Lorentzian sphere S^2, we obtain a spherical image $\delta=\delta(s_\delta)$. This curve is called the $\overline{M}_2$ Bishop spherical image of the curve $\overline{\gamma}=\overline{\gamma}(s)$.

Let $\delta=\delta(s_\delta)$ be the $\overline{M}_2$ Bishop spherical image of a regular curve $\overline{\gamma}=\overline{\gamma}(s)$. Differentiating of δ with respect to s gives

$$\delta'=\frac{d\delta}{ds_\delta}\frac{ds_\delta}{ds}=\overline{k}_2\overline{T}.$$

Thus we have

$$\overline{T}_\delta=\overline{T}=M_1 \text{ and } \frac{ds_\delta}{ds}=\overline{k}_2. \tag{3.41}$$

Since M_1 is a timelike vector, δ is also a timelike curve from (3.36). One calculate

$$\overline{T}_\delta'=\dot{\overline{T}}_\delta\frac{ds_\delta}{ds}=\overline{k}_1\overline{M}_1+\overline{k}_2\overline{M}_2$$

or

$$\dot{\overline{T}}_\delta=\frac{\overline{k}_1}{\overline{k}_2}\overline{M}_1+\overline{M}_2.$$

Then we find the curvature of δ as

$$\overline{\kappa}_\delta=\left\|\dot{\overline{T}}_\delta\right\|=\sqrt{1+[\frac{\overline{k}_1}{\overline{k}_2}]^2}=\frac{\sqrt{\overline{k}_1^2+\overline{k}_2^2}}{\overline{k}_2} \tag{3.42}$$

Therefore, we have the principal normal

$$\overline{N}_\delta=\frac{\overline{k}_1}{\sqrt{\overline{k}_1^2+\overline{k}_2^2}}\overline{M}_1+\frac{\overline{k}_2}{\sqrt{\overline{k}_1^2+\overline{k}_2^2}}\overline{M}_2.$$

By the cross product of $\overline{T}_\delta\times\overline{N}_\delta$, we obtain the binormal vector field

$$\overline{B}_\delta=\frac{\overline{k}_2}{\sqrt{\overline{k}_1^2+\overline{k}_2^2}}\overline{M}_1-\frac{\overline{k}_1}{\sqrt{\overline{k}_1^2+\overline{k}_2^2}}\overline{M}_2.$$

By means of the obtained equations, we express the torsion of the $\overline{M}_2$ Bishop spherical image of a regular curve $\overline{\gamma}=\overline{\gamma}(s)$ as follows:

$$\overline{\tau}_\delta=\frac{\overline{k}_1\overline{k}_2'-\overline{k}_1'\overline{k}_2}{\overline{k}_1^2+\overline{k}_2^2}. \tag{3.43}$$

Considering the equations (3.42) and (3.43) by Theorem 2.4, we have

Corollary 3.26. Let $\delta = \delta(s_\delta)$ be the $\overline{M}_2$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame, then the $\overline{M}_2$ Bishop spherical image δ is a circle in the osculating plane.

In the light of Propositions 2.2, and 2.3, we give the following results without proofs:

Corollary 3.27. Let $\beta = \beta(s_\beta)$ be the $\overline{M}_2$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a general helix due to the Bishop frame, then the Bishop curvatures of δ satisy

$$\frac{(\overline{k}_1^2 + \overline{k}_2^2)^{\frac{3}{2}}}{\overline{k}_1\overline{k}_2\overline{k}_2' - \overline{k}_1'\overline{k}_2^2} = constant.$$

Corollary 3.28. Let $\beta = \beta(s_\beta)$ be the $\overline{M}_2$ Bishop spherical image of a regular M_1-direction curve $\overline{\gamma} = \overline{\gamma}(s)$. If the curve $\overline{\gamma} = \overline{\gamma}(s)$ is a slant helix due to the Bishop frame, then the Bishop curvatures of δ satisy

$$\frac{\dfrac{\overline{k}_1^2 + \overline{k}_2^2}{\overline{k}_2^2}\left(\dfrac{\overline{k}_1\overline{k}_2\overline{k}_2' - \overline{k}_1'\overline{k}_2^2}{(\overline{k}_1^2 + \overline{k}_2^2)^{\frac{3}{2}}}\right)'}{\left(\dfrac{\overline{k}_1^2 + \overline{k}_2^2}{\overline{k}_2^2} + \left(\dfrac{\overline{k}_1\overline{k}_2' - \overline{k}_1'\overline{k}_2}{\overline{k}_1^2 + \overline{k}_2^2}\right)^2\right)} = constant.$$

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