# Optimal Versus No Arbitrage Hedge Ratio[1]: A Stochastic Control Approach

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**Abstract.** In a portfolio containing a stock and an its derivative, the optimal hedge ratio is derived using the stochastic control. Then, this ratio is compared with its value derived using the no arbitrage assumption. When the optimal ratio is used instead of ratio derived by no arbitrage, the existence of arbitrage opportunity is checked and ways to remove this zero risk benefit is studied. Numerical results of a simulated example are presented. Finally, conclusions are given.

**Keywords**: Arbitrage; Stock derivative; Optimal hedge ratio; Portfolio; Stochastic control

**1 Introduction.** In this paper, following Mudchanatongsuk *et al*. (2008), the optimal hedge ratio in a portfolio including a stock and an its derivative is determined using the stochastic control approach. To this end, let $s$ be the stock satisfying the geometric Brownian motion

$$ds = \mu s dt + \sigma s dz = A_s dt + B_s dz,$$

$t \in (0,T)$, where $z$ is standard Brownian motion, $A_s = \mu s$ and $B_s = \sigma s$. Let $g = g(s,t)$ be a derivative defined on $s$. Then, using the Ito lemma (see Oksendal, 1998), one can see that

$$dg = \left(\frac{\partial g}{\partial t} + \mu s \frac{\partial g}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 g}{\partial s^2}\right)dt + \sigma s \frac{\partial g}{\partial s} dz = A_g dt + B_g dz,$$

where $A_g = \frac{\partial g}{\partial t} + \mu s \frac{\partial g}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 g}{\partial s^2}$ and $B_g = \sigma s \frac{\partial g}{\partial s}$. Let $\pi$ be the portfolio containing the long position of one share of derivative $g$ and $\alpha$ shares of stock $s$, that is $\pi = g - \alpha s$. Assuming $\pi$ is a self-finance portfolio, see Bjork (2009), then

$$d\pi = dg - \alpha ds = A_\pi dt + B_\pi dz,$$

$A_\pi = A_g - \alpha A_s$, $B_\pi = B_g - \alpha B_s$. Here, an optimal control approach is proposed to determine $\alpha$. Indeed, it is interested to minimize $min_{\{\alpha_t\}} E(\pi_T^2)$, where $\pi_0 = 0$ with respect to

$$d\pi = A_\pi dt + B_\pi dz,$$

and $ds = \mu s dt + \sigma s dz$, at which $s|_{t=0} = s_0$ and $\pi_0 = 0$. Let $G(t,\pi,s)$ denote the value function. The Hamilton-Jacobi-Bellamn (HJB) equation, in this case, see Oksendal (1998), is given by

$$G_t' + min_\alpha\{G_\pi' A_\pi + G_s' A_s + 0.5[G_{\pi\pi}'' B_\pi^2 + G_{ss}'' B_s^2 + 2G_{\pi s}'' B_\pi B_s]\} = 0,$$

where $G(T,\pi,s) = \pi_T^2$, $G_\pi'$, $G_{\pi\pi}''$, are the first and second order partial derivatives with respect to $\pi$, respectively. Other notations are defined, analogously. To minimize the second term of HJB equation with respect to $\alpha$, notice that

$$G_\pi' A_s - B_s G_{\pi\pi}''(B_g - \alpha B_s) - B_s^2 G_{\pi s}'' = 0.$$

It is easy to see that

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$$\alpha = \frac{B_g}{B_s} + \frac{G'_\pi \gamma_s + G''_{\pi s}}{G''_{\pi\pi}} = \frac{\partial g}{\partial s} + \frac{G'_\pi \gamma_s + G''_{\pi s}}{G''_{\pi\pi}},$$

where $\gamma_s = \frac{A_s}{B_s^2}$. Denote the $\frac{\partial g}{\partial s} + \frac{G'_\pi \gamma_s + G''_{\pi s}}{G''_{\pi\pi}}$ by $\alpha_{hedge}$ and the $\frac{\partial g}{\partial s}$ by $\alpha_{arb}$, where $\alpha_{arb}$ is the $\alpha$ obtained using the no arbitrage arguments (see, Bjork, 2009). Thus,

$$\alpha_{hedge} = \alpha_{arb} + \Delta\alpha,$$

where $\Delta\alpha = \frac{G'_\pi \gamma_s + G''_{\pi s}}{G''_{\pi\pi}}$. Choosing $\alpha_{hedge}$ constructs an arbitrage portfolio. Here, it is verify the amount of arbitrage and the ways of removing this opportunity.

The rest of paper is designed as follows. Section 2 discusses ways of removing the arbitrage and, following Mudchanatongsuk *et al.* (2008), for some selections of $g$ and $G$, a closed form for the amount of arbitrage opportunity is derived. Numerical analysis is presented in section 3. Conclusions are given in section 4.

**2 Arbitrage strategies.** In this section, the existence of arbitrage opportunity, by selecting $\alpha_{hedge}$, instead of $\alpha_{arb}$, is checked and way for removing the arbitrage opportunity is studied. Notice that $\pi_{hedge} - \pi_{arb} = -s\Delta\alpha$. When$\Delta\alpha = 0$, then there is no arbitrage benefit.

**2.1 Arbitrage removing**. Here, following Mudchanatongsuk *et al.* (2008), let $G(t,\pi,s) = f(t,s)\pi^2$ and $f(T,s) = 1$. It is seen that $G'_\pi = 2\pi f(t,s), G''_{\pi\pi} = 2f(t,s)$ and $G''_{\pi s} = 2\pi f'_s$. Then, $\Delta\alpha = \pi(\gamma_s + \frac{f'_s}{f(t,s)})$. To remove the arbitrage opportunity, it is enough to let $\frac{f'_s}{f(t,s)} = -\gamma_s$, then one can see that

$$f(t,s) = \exp\left\{C - \int_0^s \gamma_x \, dx\right\},$$

where $C$ is a constant. Since $f(T,s) = 1$, therefore, $f(t,s) = \exp\left\{\int_{s_t}^{s_T} \gamma_x \, dx\right\}$. Also, $\gamma_s = \frac{\mu}{\sigma^2 s}$. Hence, $f(t,s) = (\frac{s_T}{s_t})^{\frac{\mu}{\sigma^2}}$. Therefore, by choosing

$$G(t,\pi,s) = (\frac{s_T}{s_t})^{\frac{\mu}{\sigma^2}}\pi^2,$$

the arbitrage opportunity is removed.

**2.2 Amount of arbitrage**. Here, assuming $f(t,s) = (\frac{s_T}{s_t})^\gamma$, one can see that

$$\Delta\alpha = \frac{\pi}{s}\left(\frac{\mu}{\sigma^2} - \gamma\right) = \left(\alpha - \frac{g}{s}\right)\left(\gamma - \frac{\mu}{\sigma^2}\right).$$

Thus, $\pi_{hedge} - \pi_{arb} = -s\left(\alpha - \frac{g}{s}\right)\left(\gamma - \frac{\mu}{\sigma^2}\right)$. The following proposition implies the arbitrage opportunities at time $T$, as follows

**Proposition 1**. (*a*)-(*c*) are correct.

(*a*) $\Delta\alpha_T = 0$ iff $\gamma = \frac{\mu}{\sigma^2}$ or $\alpha_T = \frac{g_T}{s_T}$. There is no arbitrage opportunity.

(*b*) $\Delta\alpha_T > 0$ iff $\gamma > \frac{\mu}{\sigma^2}$ and $\alpha_T > \frac{g_T}{s_T}$ or $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T < \frac{g_T}{s_T}$. Buying $\pi_{hedge}$ has zero risk benefit.

(*c*) $\Delta\alpha_T < 0$ iff $\gamma > \frac{\mu}{\sigma^2}$ and $\alpha_T < \frac{g_T}{s_T}$ or $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T > \frac{g_T}{s_T}$. Short selling $\pi_{hedge}$ has zero risk benefit.

As follows, the arbitrage opportunity, using forward and option contracts, are presented.

*(a) Forward contract.* Consider a long position in forward contract with strike price $K$ where $g_T = s_T - K$. If $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T < 1 - \frac{K}{s_T}$, therefore, buying $\pi_{hedge}$ has arbitrage opportunity. However, If $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T > 1 - \frac{K}{s_T}$, therefore, short selling $\pi_{hedge}$ has arbitrage opportunity. If $\gamma > \frac{\mu}{\sigma^2}$, then the above arbitrage strategies are conversed.

*(b) Option contract.* Again, consider a call option. Thus, if $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T > \max(0, 1 - \frac{K}{s_T})$. Hence, short selling $\pi_{hedge}$ has zero risk benefit. if $\gamma < \frac{\mu}{\sigma^2}$ and $\alpha_T < \max(0, 1 - \frac{K}{s_T})$. Hence, buying $\pi_{hedge}$ has zero risk benefit.

**3 Numerical results**. Here, the results of previous section is studied in a simulated example, numerically. To this end, consider a daily series of stock $s$ given by

$$s_{t_{k+1}} = \exp\{\mu \Delta t + \sigma \sqrt{\Delta t} Z_{t_k}\} s_{t_k},$$

where $Z_t$ is a sequence of iid random variables with common standard normal distribution, $\Delta t = t_k - t_{k-1} = \frac{1}{251}, k = 1, \ldots, 251, t_0 = 0$. Consider an European call option with strike price $K$ as the financial derivative $g$. The following figure is a realization of $f(t, s) = (\frac{s_T}{s_t})^{\frac{\mu}{\sigma^2}}$ for $\mu = 0.5, \sigma = 0.1$ and $s_0 = 20$. Results are derived using the Vose *ModelRisk* Excel adds-in.

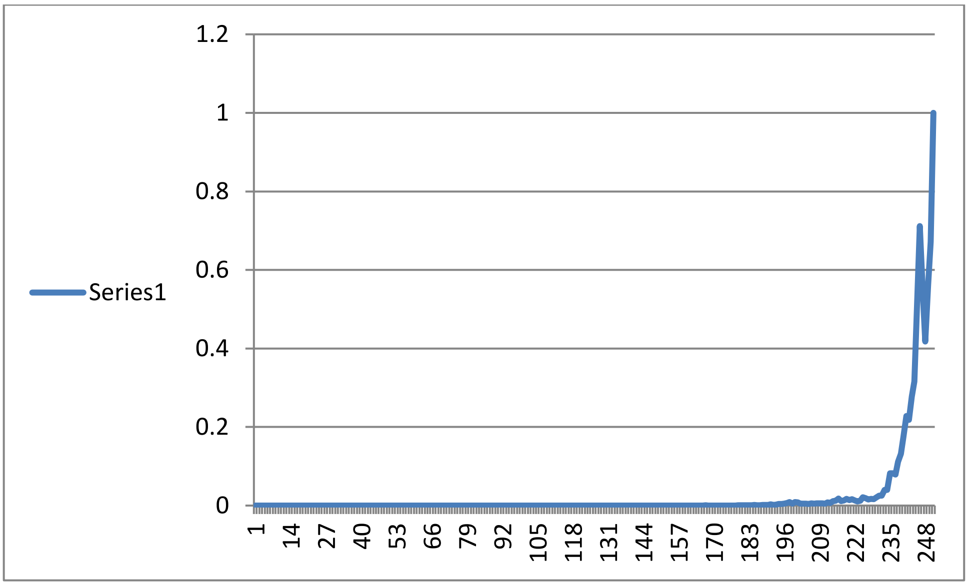


Fig. 1. Time Series Plot of $f(t, s)$

Also, the following table gives the descriptive statistics summaries of $f\left(\frac{250}{251}, s\right)$.

Table 1: Descriptive Statistics of $f\left(\frac{250}{251}, s\right)$

| Min | Max | Mean | Stdev | Skew | Kurtosis |
|---|---|---|---|---|---|
| 0.315 | 2.66 | 0.955 | 0.3034 | 0.89 | 4.13 |

The following figure gives the histogram of $f\left(\frac{250}{251}, s\right)$.

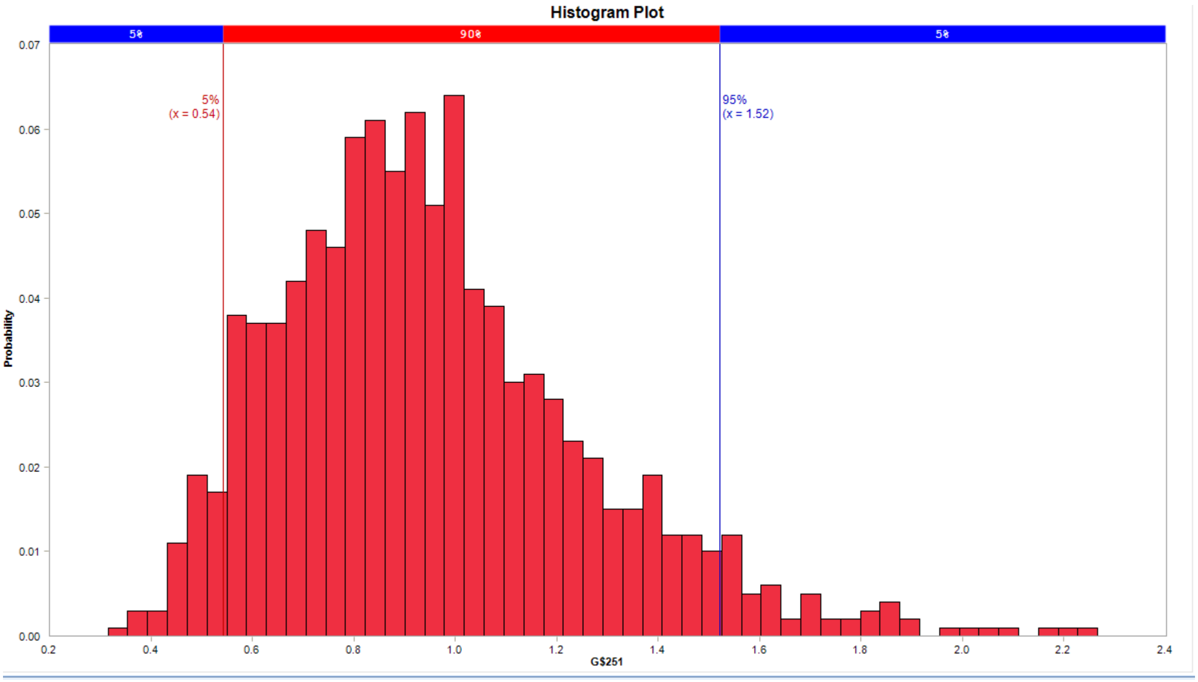


Fig.2. Histogram of $f\left(\frac{250}{251}, s\right)$

The no arbitrage hedge ratio for a call option is given by $N(d_1)$ where $d_1 = \frac{\log\left(\frac{s}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $N$ is the distribution function of standard normal distribution. Let the risk free rate be 0.05 and $K = 22$. The following figure also shows the time series of $\Delta\alpha_T, T = \frac{1}{251}, \ldots, \frac{251}{251}$. Notice that $\Delta\alpha_T \propto (\gamma_s + \frac{f_s'}{f(T,s)})$ where $\propto$ means the proportional to. Here, $f_s'$ is computed numerically.

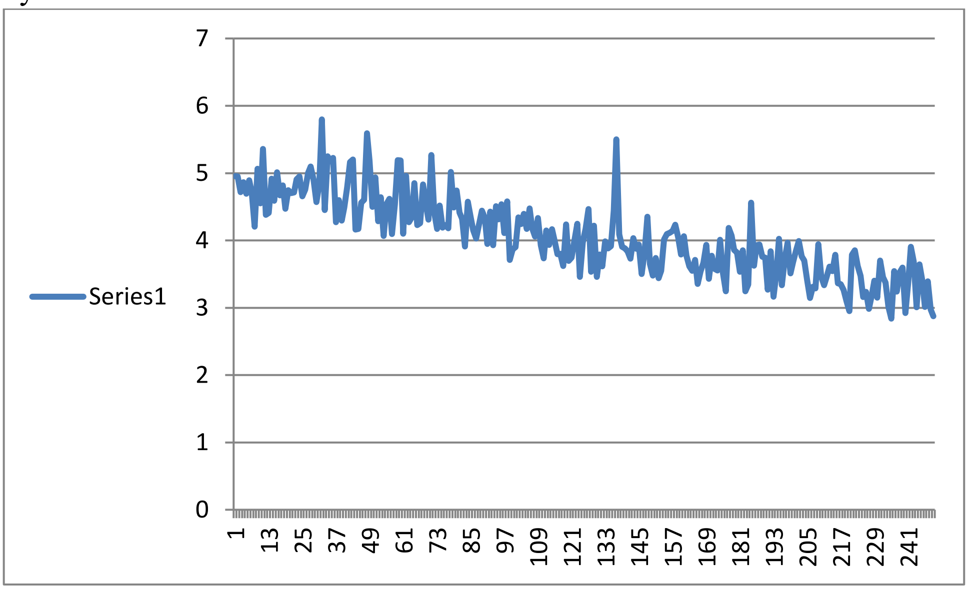


Fig. 3. Time series plot of $\Delta\alpha_T$

As follows, time series of $\alpha_T, T = \frac{1}{251}, \ldots, \frac{251}{251}$ is plotted.

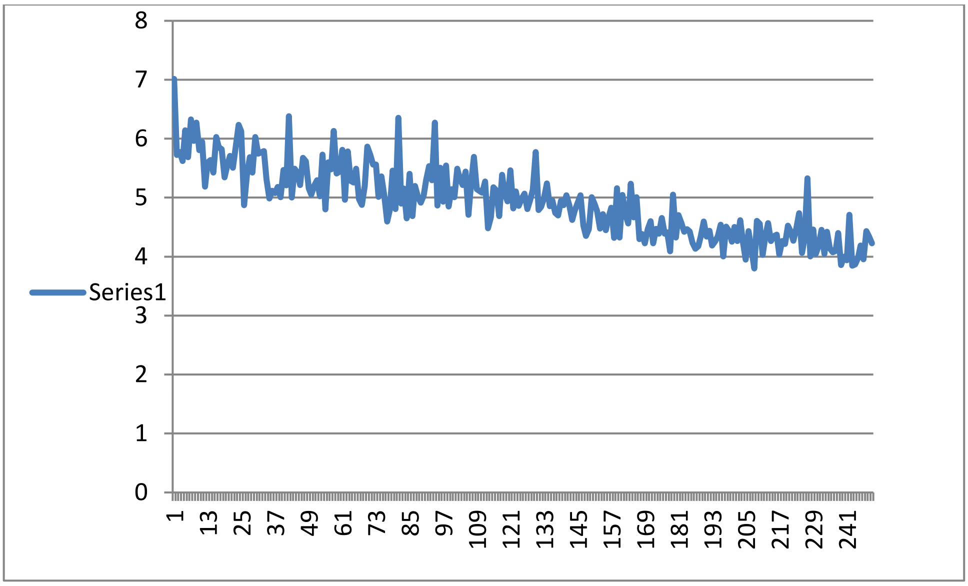


Fig. 4. Time series plot of $\alpha_T$

As follows, a distribution is fitted for $\alpha_T, T = \frac{250}{251}$. The histogram of $\Delta\alpha_T$ is plotted which shows the most probable arbitrage.

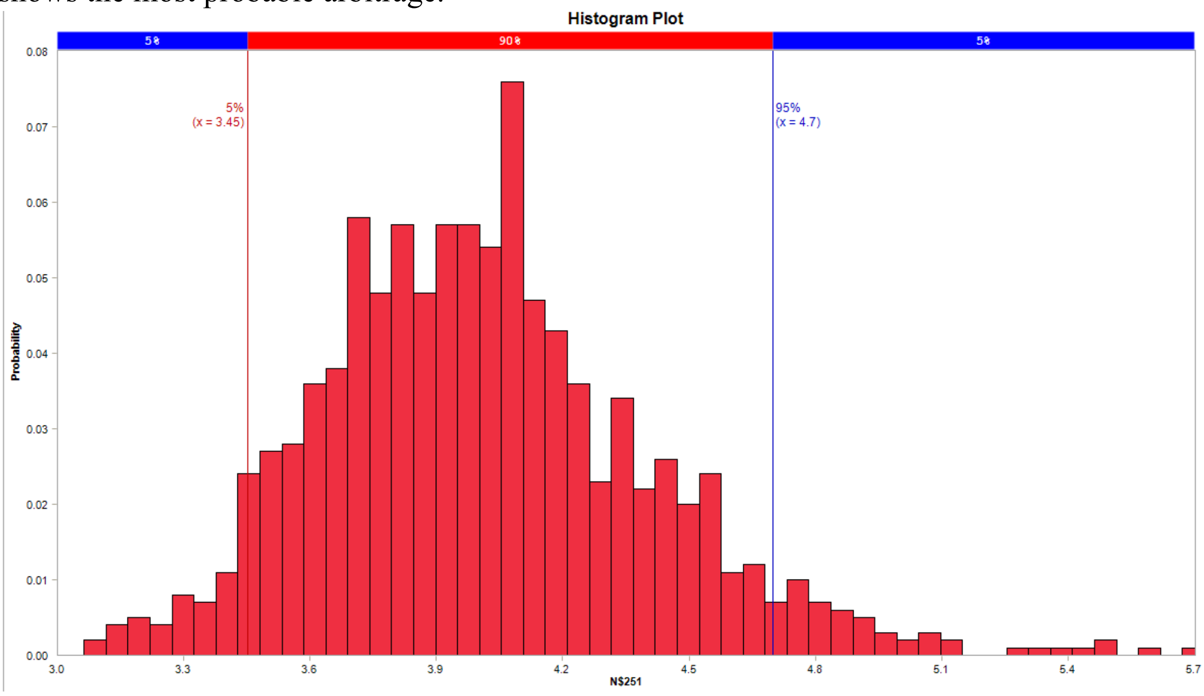


Fig.5. Histogram of $\alpha_T$

**4 Conclusions.** In this paper, the problem of finding the optimal hedge ratio is studied using the stochastic control approach. The optimal ratio is compared with the ratio derived using the no arbitrage assumption. The existence of arbitrage, its amount, and ways to remove it are surveyed. Conclusions are listed as follows:

($a$) As $T \to \infty$, then the arbitrage opportunity goes to constant, as it is expected.
($b$) Using the $\alpha_{hedge}$ instead of $\alpha_{Arb}$, almost an arbitrage with amount of 3.5 dollars always exists.

(*c*) The probable value for arbitrage opportunity is 4.12 dollars.

**References**

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