AMO - Advanced Modeling and Optimization, Volume 20, Number 1, 2018

Analysis of $M^{[X]}/G(a,b)/1$ queue with second optional service closedown multiple vacation and state dependent arrival rate

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Abstract

In this paper, an $M^{[X]}/G(a,b)/1$ queueing model with second optional service closedown, multiple vacation and state dependent arrival rate is considered. After completing the first service, the customers may opt for the second service with probability ζ or leave the system with probability $1-\zeta$. After completing a bulk service, if the queue size is less than 'a', then the server starts closedown and then goes for a vacation of random length. When he returns from the vacation, if the queue length is still less than 'a', he leaves for another vacation and so on. This process continues until he finds at least 'a' customer in the queue. After a vacation, if the server finds at least 'a' customer waiting for service, he resumes service for a batch of ξ customers ($a \leq \xi \leq b$). The arrival rate varies depends upon the state of the server. Using supplementary variable technique, the probability generating function (PGF) of the queue size, expected queue length, expected waiting time, expected busy period and expected idle period are derived. Numerical illustrations are presented to visualize the effect of system parameters.

Keywords: Bulk queue; Second optional service; Closedown; Multiple vacation; State dependent arrival rate

AMS Subject Classification (2010) 60K25

1 Introduction

Queueing models where the server performs closedown work and resumes vacation when there is no sufficient batch size (less than the minimum threshold) for service, is quit common in various practical situations related to manufacturing systems, service systems, etc. Neuts [14] initiated the concept of bulk queues and analyzed a general class of such models. A literature survey on vacation queueing models can be found in Doshi [6] and Takagi [16] which include some applications. Lee [10] developed a systematic procedure to calculate the system size probabilities for a bulk queueing model. Krishna Reddy et al. [9] considered an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Arumuganathan and Jeyakumar [1] obtained the probability generating function of queue length distributions at an arbitrary time epoch for the bulk queueing model with multiple vacation and closedown times. Also they have developed a cost model with a numerical study for their queueing model. Wang [17] considered an M/G/1 queueing model with second optional service and unreliable server. He derived the steady-state as well as the transient system size probabilities using supplementary variable method.

Footnote
AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

Madan et al. [11] considered an $M^{[X]}/(G_1, G_2)/1$ queueing model where the batch of customers can choose any one of the two service and then after the completion of service, they may opt for re-service. They derived the steady state probability generating functions for the number of customers in the queue and the system. Arumuganathan and Jeyakumar [2] obtained the probability generating function of queue size distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, setup times and N-policy. Ke [8] investigated an $M^{[X]}/G/1$ queueing model with vacation policies, breakdown and startup/closedown times where the vacation, startup, closedown and repair times are generally distributed. Parthasarathy and Sudhesh [15] derived the transient system size probabilities and the duration of busy period for a single server queueing model using continued fraction where the system alternates between arrivals and service with state dependent rates. Maraghi et al. [13] derived PGF for a number of customers in the queue under the steady-state. Also they derived the performance measures like expected queue length, expected waiting time with some special cases.

Balasubramanian and Arumuganathan [4] considered an $M^{[X]}/G(a,b)/1$ queueing model and obtained its queue size distribution for the steady state. They also derived the average length of busy and idle periods, expected queue length and waiting time. Jain et al. [7] obtained the system size distribution for the M/G/1 queueing model with unreliable server and multi-optional vacations where the arrivals received the first essential service and some of them required some other optional services. Ayyappan and Shyamala [3] derived the PGF of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time for both steady state and transient cases. Madan and Malalla [12] studied a batch arrival queue in which the server provides the second optional service on customer's request, the server may breakdown at random time and delayed repair. They also derived the queue size distribution of the system and some performance measures.

The rest of the paper is organized as follows. In section 2, an $M^{[X]}/G(a,b)/1$ queueing model with second optional service closedown, multiple vacation and state dependent arrival rate is described and the steady-state system size equations are considered. In section 3, using supplementary variable technique, the probability generating function of the queue size are derived and a particular case is provided. In section 4, performance measures like expected length of busy and idle periods, expected queue length and waiting time are obtained. In section 5, the cost model is provided. In section 6, numerical illustrations are presented to validate the analytical results. In section 7, this research work is concluded with the proposed future work.

2 Model Description

We consider an $M^{[X]}/G(a,b)/1$ queueing model with second optional service closedown, multiple vacation and state dependent arrival rate. After completing the first service, the customers may opt for the second service with probability ζ or leave the system with probability $1-\zeta$. After completing a bulk service, if the queue size is less than 'a', then the server starts closedown and then goes for a vacation of random length. When he returns from the vacation, if the queue length is still less than 'a', he leaves for another vacation and so on. This process continues until he finds at least 'a' customer in the queue. After a vacation, if the server finds at least 'a' customer waiting for service, he resumes service for a batch of ξ customers ($a \leq \xi \leq b$). The arrival rate varies depends upon the state of the server. When the server is busy, the arrival rate is λ_1. When the server is idle (closedown and multiple vacation), the arrival rate is λ_2. For the model under consideration, the PGF of the queue size at an arbitrary time and various performance measures are obtained. Cost model for the system is also derived.

2.1 Notations

The following notations are used in this paper.
λ - Arrival rate,
X - Group size random variable,
$g_k - Pr\{X = k\}$,
$X(z)$ - Probability generating function (PGF) of X.

Here $S_1(.), S_2(.), V(.)$ and $C(.)$ represent the cumulative distribution function (CDF) of first essential service time, second optional service time, vacation time and closedown time and their corresponding probability density functions are $s_1(x), s_2(x), v(x)$ and $c(x)$ respectively. $S_1^0(t), S_2^0(t), V^0(t)$ and $C^0(t)$ represent the remaining service time of first essential service of a batch, second optional service time of a batch, vacation time and closedown time at time t respectively. $\tilde{S_1}(\theta), \tilde{S_2}(\theta), \tilde{V}(\theta)$ and $\tilde{C}(\theta)$ represent the Laplace-Stieltjes transform of S_1, S_2, V and C respectively.

The supplementary variables $S_1^0(t), S_2^0(t), V^0(t)$ and $C^0(t)$ are introduced in order to obtain the bivariate Markov process $\{N(t), Y(t)\}$, where $N(t) = \{N_q(t) \cup N_s(t)\}$ and

$$\begin{aligned}
Y(t) &= (0)[1]\{2\}\langle 3\rangle, \text{if the server is on(first essential service)[second optional service]} \\
&\quad \{\text{vacation}\}\langle\text{closedown time}\rangle. \\
Z(t) &= j, \text{if the server is on } j^{th} \text{ vacation.} \\
N_s(t) &= \text{Number of customers in the service at time } t. \\
N_q(t) &= \text{Number of customers in the queue at time } t.
\end{aligned}$$

Define the probabilities as,

$$\begin{aligned}
P_{i,j}^{(1)}(x,t)dt &= P\left\{N_s(t) = i, N_q(t) = j, x \le S_1^0(t) \le x + dx, Y(t) = 0\right\},\ a \le i \le b,\ j \ge 0, \\
P_{i,j}^{(2)}(x,t)dt &= P\left\{N_s(t) = i, N_q(t) = j, x \le S_2^0(t) \le x + dx, Y(t) = 1\right\},\ a \le i \le b,\ j \ge 0, \\
Q_{j,n}(x,t)dt &= P\left\{N_q(t) = n, x \le V^0(t) \le x + dx, Y(t) = 2, Z(t) = j\right\},\ n \ge 0,\ j \ge 1, \\
C_n(x,t)dt &= P\left\{N_q(t) = n, x \le C^0(t) \le x + dx, Y(t) = 3\right\},\ n \ge 0.
\end{aligned}$$

The supplementary variable technique was introduced by Cox [5]. The steady-state system size equations are obtained as follows:

$$\begin{aligned}
-P'^{(1)}_{i,0}(x) &= -\lambda_1 P_{i,0}^{(1)}(x) + (1-\zeta)\sum_{m=a}^{b} P_{m,i}^{(1)}(0)s_1(x) + \sum_{m=a}^{b} P_{m,i}^{(2)}(0)s_1(x) \\
&\quad + \sum_{l=1}^{\infty} Q_{l,i}(0)s_1(x), \quad a \le i \le b, \qquad (1) \\
-P'^{(1)}_{i,j}(x) &= -\lambda_1 P_{i,j}^{(1)}(x) + \sum_{k=1}^{j} P_{i,j-k}^{(1)}(x)\lambda_1 g_k,\ j \ge 1,\ a \le i \le b-1, \qquad (2)
\end{aligned}$$

$$
\begin{aligned}
-P'^{(1)}_{b,j}(x) &= -\lambda_1 P^{(1)}_{b,j}(x) + (1-\zeta)\sum_{m=a}^{b} P^{(1)}_{m,b+j}(0)s_1(x) + \sum_{k=1}^{j} P^{(1)}_{b,j-k}(x)\lambda_1 g_k \\
&\quad + \sum_{m=a}^{b} P^{(2)}_{m,b+j}(0)s_1(x) + \sum_{l=1}^{\infty} Q_{l,b+j}(0)s_1(x), \quad j \geq 1, \qquad (3)\\
-P'^{(2)}_{i,0}(x) &= -\lambda_1 P^{(2)}_{i,0}(x) + \zeta P^{(1)}_{i0}(0)s_2(x), \quad a \leq i \leq b, \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
-P'^{(2)}_{i,j}(x) &= -\lambda_1 P^{(2)}_{i,j}(x) + \zeta P^{(1)}_{ij}(0)s_2(x) + \sum_{k=1}^{j} P^{(2)}_{i,j-k}(x)\lambda_1 g_k, \ j \geq 1, \ a \leq i \leq b, \qquad (5)\\
-C'_n(x) &= -\lambda_2 C_n(x) + (1-\zeta)\sum_{m=a}^{b} P^{(1)}_{m,n}(0)c(x) + \sum_{m=a}^{b} P^{(2)}_{m,n}(0)c(x) \\
&\quad + \sum_{k=1}^{n} C_{n-k}(x)\lambda_2 g_k, \quad n \leq a-1, \qquad (6)\\
-C'_n(x) &= -\lambda_2 C_n(x) + \sum_{k=1}^{n} C_{n-k}(x)\lambda_2 g_k, \quad n \geq a, \qquad (7)\\
-Q'_{1,0}(x) &= -\lambda_2 Q_{1,0}(x) + C_0(0)v(x), \qquad (8)\\
-Q'_{1,n}(x) &= -\lambda_2 Q_{1,n}(x) + C_n(0)v(x) + \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda_2 g_k, \quad n \geq 1, \qquad (9)\\
-Q'_{j,0}(x) &= -\lambda_2 Q_{j,0}(x) + Q_{j-1,0}(0)v(x), \quad j \geq 2, \qquad (10)\\
-Q'_{j,n}(x) &= -\lambda_2 Q_{j,n}(x) + Q_{j-1,n}(0)v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_2 g_k, \ j \geq 2, \ 1 \leq n < a, \qquad (11)\\
-Q'_{j,n}(x) &= -\lambda_2 Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_2 g_k, \ j \geq 2, \ n \geq a. \qquad (12)
\end{aligned}
$$

The Laplace-Stieltjes transform of $P^{(1)}_{i,j}(x), P^{(2)}_{i,j}(x), C_n(x), Q_{j,n}(x)$, are defined as follows:

$$
\begin{aligned}
\tilde{P}^{(1)}_{i,j}(\theta) &= \int_0^{\infty} e^{-\theta x} P^{(1)}_{i,j}(x)dx, \quad \tilde{P}^{(2)}_{i,j}(\theta) = \int_0^{\infty} e^{-\theta x} P^{(2)}_{i,j}(x)dx,\\
\tilde{C}_n(\theta) &= \int_0^{\infty} e^{-\theta x} C_n(x)dx, \quad \tilde{Q}_{j,n}(\theta) = \int_0^{\infty} e^{-\theta x} Q_{j,n}(x)dx.
\end{aligned}
$$

Taking Laplace-Stieltjes transform from (1) to (12), we get

$$
\theta \tilde{P}^{(1)}_{i,0}(\theta) - P^{(1)}_{i,0}(0) = \lambda_1 \tilde{P}^{(1)}_{i,0}(\theta) - \tilde{S}_1(\theta)\left[(1-\zeta)\sum_{m=a}^{b} P^{(1)}_{m,i}(0) + \sum_{m=a}^{b} P^{(2)}_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\right],
$$
$$
a \leq i \leq b, \qquad (13)
$$

$$\theta\tilde{P}^{(1)}_{i,j}(\theta) - P^{(1)}_{i,j}(0) = \lambda_1\tilde{P}^{(1)}_{i,j}(\theta) - \sum_{k=1}^{j}\tilde{P}^{(1)}_{i,j-k}(\theta)\lambda_1 g_k, \tag{14}$$

$$\begin{aligned}\theta\tilde{P}^{(1)}_{b,j}(\theta) - P^{(1)}_{b,j}(0) =& \lambda_1\tilde{P}^{(1)}_{b,j}(\theta) - \sum_{k=1}^{j}\tilde{P}^{(1)}_{b,j-k}(\theta)\lambda_1 g_k - \tilde{S}_1(\theta)\Bigg[(1-\zeta)\sum_{m=a}^{b}P^{(1)}_{m,b+j}(0)\\ &+\sum_{m=a}^{b}P^{(2)}_{m,b+j}(0)+\sum_{l=1}^{\infty}Q_{l,b+j}(0)\Bigg],\ j\geq 1,\end{aligned} \tag{15}$$

$$\theta\tilde{P}^{(2)}_{i,0}(\theta) - P^{(2)}_{i,0}(0) = \lambda_1\tilde{P}^{(2)}_{i,0}(\theta) - \zeta P^{(1)}_{i0}(0)\tilde{S}_2(\theta), a\leq i\leq b, \tag{16}$$

$$\theta\tilde{P}^{(2)}_{i,j}(\theta) - P^{(2)}_{i,j}(0) = \lambda_1\tilde{P}^{(2)}_{i,j}(\theta) - \sum_{k=1}^{j}\tilde{P}^{(2)}_{i,j-k}(\theta)\lambda_1 g_k - \zeta P^{(1)}_{i0}(0)\tilde{S}_2(\theta),\ a\leq i\leq b,\ j\geq 1, \tag{17}$$

$$\begin{aligned}\theta\tilde{C}_n(\theta) - C_n(0) =& \lambda_2\tilde{C}_n(\theta) - \sum_{k=1}^{n}\tilde{C}_{n-k}\lambda_2 g_k - \tilde{C}(\theta)\Bigg[(1-\zeta)\sum_{m=a}^{b}P^{(1)}_{m,n}(0)+\sum_{m=a}^{b}P^{(2)}_{m,n}(0)\Bigg],\\ & n\leq a-1,\end{aligned} \tag{18}$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda_2\tilde{C}_n(\theta) - \sum_{k=1}^{n}\tilde{C}_{n-k}(\theta)\lambda_2 g_k, \tag{19}$$

$$\theta\tilde{Q}_{1,0}(\theta) - Q_{1,0}(0) = \lambda_2\tilde{Q}_{1,0}(\theta) - \tilde{V}(\theta)C_0(0), \tag{20}$$

$$\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) = \lambda_2\tilde{Q}_{1,n}(\theta) - \tilde{V}(\theta)C_n(0) - \sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)\lambda_2 g_k,\ n\geq 1, \tag{21}$$

$$\theta\tilde{Q}_{j,0}(\theta) - Q_{j,0}(0) = \lambda_2\tilde{Q}_{j,0}(\theta) - \tilde{V}(\theta)Q_{j-1,0}(0), \tag{22}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda_2\tilde{Q}_{j,n}(\theta) - \tilde{V}(\theta)Q_{j-1,n}(0) - \sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda_2 g_k, \tag{23}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda_2\tilde{Q}_{j,n}(\theta) - \sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda_2 g_k. \tag{24}$$

To find the probability generating function (PGF) of queue size, we define the following PGFs:

$$\begin{aligned}\tilde{P}^{(1)}_i(z,\theta) &= \sum_{j=0}^{\infty}\tilde{P}^{(1)}_{i,j}(\theta)z^j, \quad P^{(1)}_i(z,0)=\sum_{j=0}^{\infty}P^{(1)}_{i,j}(0)z^j, \quad a\leq i\leq b,\\ \tilde{P}^{(2)}_i(z,\theta) &= \sum_{j=0}^{\infty}\tilde{P}^{(2)}_{i,j}(\theta)z^j, \quad P^{(2)}_i(z,0)=\sum_{j=0}^{\infty}P^{(2)}_{i,j}(0)z^j, \quad a\leq i\leq b,\\ \tilde{C}(z,\theta) &= \sum_{n=0}^{\infty}\tilde{C}_n(\theta)z^n, \quad C(z,0)=\sum_{n=0}^{\infty}C_n(0)z^n,\\ \tilde{Q}_j(z,\theta) &= \sum_{n=0}^{\infty}\tilde{Q}_{j,n}(\theta)z^n, \quad Q_j(z,0)=\sum_{n=0}^{\infty}Q_{j,n}(0)z^n, \quad j\geq 1.\end{aligned} \tag{25}$$

By multiplying the equations from (13) to (24) by suitable power of z^n and summing over n, ($n = 0$ to ∞) and using (25),

$$(\theta-\lambda_2+\lambda_2X(z))\tilde{Q}_1(z,\theta)=Q_1(z,0)-C(z,0)\tilde{V}(\theta), \tag{26}$$

$$(\theta-\lambda_2+\lambda_2X(z))\tilde{Q}_j(z,\theta)=Q_j(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\geq 2, \tag{27}$$

$$(\theta-\lambda_2+\lambda_2X(z))\tilde{C}(z,\theta)=C(z,0)-\tilde{C}(\theta)\Big[(1-\zeta)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P^{(1)}_{m,n}(0)z^n+\sum_{n=0}^{a-1}\sum_{m=a}^{b}P^{(2)}_{m,n}(0)z^n\Big], \tag{28}$$

$$(\theta-\lambda_1+\lambda_1X(z))\tilde{P}^{(1)}_i(z,\theta)=P^{(1)}_i(z,0)-\tilde{S}_1(\theta)\Big[(1-\zeta)\sum_{m=a}^{b}P^{(1)}_{m,i}(0)+\sum_{m=a}^{b}P^{(2)}_{m,i}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big],a\leq i\leq b-1, \tag{29}$$

$$\begin{aligned}(\theta-\lambda_1+\lambda_1X(z))\tilde{P}^{(1)}_b(z,\theta)=P^{(1)}_b(z,0)-\frac{\tilde{S}_1(\theta)}{z^b}\Big[&(1-\zeta)\sum_{m=a}^{b}P^{(1)}_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P^{(1)}_{m,j}(0)z^j\\&+\sum_{m=a}^{b}P^{(2)}_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P^{(2)}_{m,j}(0)z^j\\&+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j\Big],\end{aligned} \tag{30}$$

$$(\theta-\lambda_1+\lambda_1X(z))\tilde{P}^{(2)}_i(z,\theta)=P^{(2)}_i(z,0)-\zeta\tilde{S}_2(\theta)P^{(1)}_i(z,0). \tag{31}$$

By Substituting $\theta=\lambda_2-\lambda_2X(z)$ in (26) to (28), we get

$$Q_1(z,0)=\tilde{V}(\lambda_2-\lambda_2X(z))C(z,0), \tag{32}$$

$$Q_j(z,0)=\tilde{V}(\lambda_2-\lambda_2X(z))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\geq 2, \tag{33}$$

$$C(z,0)=\tilde{C}(\lambda_2-\lambda_2X(z))\Big[(1-\zeta)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P^{(1)}_{m,n}(0)z^n+\sum_{n=0}^{a-1}\sum_{m=a}^{b}P^{(2)}_{m,n}(0)z^n\Big], \tag{34}$$

By Substituting $\theta=\lambda_1-\lambda_1X(z)$ in (29) to (31), we get

$$P^{(1)}_i(z,0)=\tilde{S}_1(\lambda_1-\lambda_1X(z))\Big[(1-\zeta)\sum_{m=a}^{b}P^{(1)}_{m,i}(0)+\sum_{m=a}^{b}P^{(2)}_{m,i}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big],\quad a\leq i\leq b-1, \tag{35}$$

$$z^b P_b^{(1)}(z,0) = \tilde{S}_1(\lambda_1 - \lambda_1 X(z))\Bigg[(1-\zeta)(\sum_{m=a}^{b} P_m^{(1)}(z,0) - \sum_{j=0}^{b-1}\sum_{m=a}^{b} P_{m,j}^{(1)}(0)z^j) + \sum_{m=a}^{b} P_m^{(2)}(z,0) - \sum_{j=0}^{b-1}\sum_{m=a}^{b} P_{m,j}^{(2)}(0)z^j + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1}\sum_{l=1}^{\infty} Q_{l,j}(0)z^j\Bigg], \tag{36}$$

$$\Bigg[z^b - (1-\zeta)\tilde{S}_1(\lambda_1 - \lambda_1 X(z)) - \zeta\tilde{S}_1(\lambda_1 - \lambda_1 X(z))\tilde{S}_2(\lambda_1 - \lambda_1 X(z))\Bigg] P_b^{(1)}(z,0) = \tilde{S}_1(\lambda_1 - \lambda_1 X(z))\Bigg[[(1-\zeta) + \zeta\tilde{S}_2(\lambda_1 - \lambda_1 X(z))]\sum_{m=a}^{b-1} P_m^{(1)}(z,0) + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1}\Bigg[(1-\zeta)\sum_{m=a}^{b} P_{m,j}^{(1)}(0)z^j + \sum_{m=a}^{b} P_{m,j}^{(2)}(0)z^j + \sum_{l=1}^{\infty} Q_{l,j}(0)z^j\Bigg]\Bigg], \tag{37}$$

$$P_i^{(2)}(z,0) = \tilde{S}_2(\lambda_1 - \lambda_1 X(z))\zeta P_i^{(1)}(z,0). a \le i \le b. \tag{38}$$

Here

$$\begin{aligned} p_i^{(1)} &= \sum_{m=a}^{b} P_{m,i}^{(1)}(0), \quad p_i^{(2)} = \sum_{m=a}^{b} P_{m,i}^{(2)}(0), \\ q_i &= \sum_{l=1}^{\infty} Q_{l,i}(0), \quad k_i = (1-\zeta)p_i^{(1)} + p_i^{(2)}, \quad g_i = k_i + q_i \end{aligned} \tag{39}$$

Substitute (32) to (34), (35), (37), (38) in (26) to (31), we get

$$\tilde{Q}_1(z,\theta) = \frac{\left[\tilde{V}(\lambda_2 - \lambda_2 X(z)) - \tilde{V}(\theta)\right] C(z,0)}{(\theta - \lambda_2 + \lambda_2 X(z))}, \tag{40}$$

$$\tilde{Q}_j(z,\theta) = \frac{\left[\tilde{V}(\lambda_2 - \lambda_2 X(z)) - \tilde{V}(\theta)\right] \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n}{(\theta - \lambda_2 + \lambda_2 X(z))}, \ j \ge 2, \tag{41}$$

$$\tilde{C}(z,\theta) = \frac{\left(\tilde{C}(\lambda_2 - \lambda_2 X(z)) - \tilde{C}(\theta)\right) \sum_{n=0}^{a-1} k_n z^n}{(\theta - \lambda_0 + \lambda_0 X(z))}, \tag{42}$$

$$\tilde{P}_i^{(1)}(z,\theta) = \frac{\left(\tilde{S}_1(\lambda_1 - \lambda_1 X(z)) - \tilde{S}_1(\theta)\right) g_i}{(\theta - \lambda_1 + \lambda_1 X(z))}, \ a \le i \le b-1, \tag{43}$$

$$\tilde{P}_b^{(1)}(z,\theta) = \frac{\left[\tilde{S}_1(\lambda_1 - \lambda_1 X(z)) - \tilde{S}_1(\theta)\right] f(z)}{\left[(\theta - \lambda_1 + \lambda_1 X(z))\Big[z^b - (1-\zeta)\tilde{S}_1(\lambda_1 - \lambda_1 X(z)) - \zeta\tilde{S}_1(\lambda_1 - \lambda_1 X(z))\tilde{S}_2(\lambda_1 - \lambda_1 X(z))\Big]\right]}, \tag{44}$$

where

$$f(z) = [(1-\zeta) + \zeta\tilde{S}_2(\lambda_1 - \lambda_1 X(z))]\sum_{m=a}^{b-1} P_m^{(1)}(z,0) + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{n=0}^{b-1} g_n z^n.$$

$$\tilde{P}_i^{(2)}(z,\theta) = \frac{\left(\tilde{S}_2(\lambda_1-\lambda_1X(z)) - \tilde{S}_2(\theta)\right)\zeta P_i^{(1)}(z,0)}{(\theta-\lambda_1+\lambda_1X(z))}, \ a \le i \le b. \tag{45}$$

3 Probability generating function of queue size

In this section, the PGF, $P(z)$ of the queue size at an arbitrary time epoch is derived.

3.1 PGF of queue size at an arbitrary time epoch

If P(z) be the PGF of the queue size at an arbitrary time epoch, then

$$P(z) = \sum_{m=a}^{b-1}\tilde{P}_m^{(1)}(z,0) + \tilde{P}_b^{(1)}(z,0) + \sum_{m=a}^{b}\tilde{P}_m^{(2)}(z,0) + \tilde{C}(z,0) + \sum_{l=1}^{\infty}\tilde{Q}_l(z,0). \tag{46}$$

By substituting $\theta = 0$ into the equations (40) to (45), then the equation (46) becomes

$$P(z) = \frac{\left[\begin{array}{l} (1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z)) + \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))(\tilde{S}_2(\lambda_1-\lambda_1X(z))-1) \\ \quad (-\lambda_2+\lambda_2X(z))\sum_{n=a}^{b-1}(z^b-z^n)g_n + (\tilde{V}(\lambda_2-\lambda_2X(z))\tilde{C}(\lambda_2-\lambda_2X(z))-1) \\ \quad\left[\left[(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z)) + \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\tilde{S}_2(\lambda_1-\lambda_1X(z))-1\right]\right. \\ \quad (-\lambda_2+\lambda_2X(z)) + \left[z^b-(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z)) - \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\right. \\ \quad \left.\tilde{S}_2(\lambda_1-\lambda_1X(z)))(-\lambda_1+\lambda_1X(z))\right]\sum_{n=0}^{a-1}k_nz^n + (\tilde{V}(\lambda_2-\lambda_2X(z))-1) \\ \quad\left[\left[(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z)) + \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\tilde{S}_2(\lambda_1-\lambda_1X(z))-1\right]\right. \\ \quad (-\lambda_2+\lambda_2X(z)) + \left[z^b-(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z))\right. \\ \quad \left. - \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\tilde{S}_2(\lambda_1-\lambda_1X(z))(-\lambda_1+\lambda_1X(z))\right]\sum_{n=0}^{a-1}q_nz^n \end{array}\right]}{\left[\begin{array}{l} (-\lambda_1+\lambda_1X(z))(-\lambda_2+\lambda_2X(z))\left[z^b-(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z))\right. \\ \quad \left. - \zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\tilde{S}_2(\lambda_1-\lambda_1X(z))\right] \end{array}\right]}. \tag{47}$$

Equation (47) has $a+b$ unknowns $g_a, g_{a+1}, \ldots, g_{b-1}, k_0, k_1, \ldots, k_{a-1}, q_0, q_1, \ldots, q_{a-1}$. Using the following theorem, we express q_i in terms of k_i in such a way that numerator has only b constants. Now equation (47) gives the PGF of the number of customers involving only 'b' unknowns.

By Rouche's theorem of complex variables, it can be proved that $z^b-(1-\zeta)\tilde{S}_1(\lambda_1-\lambda_1X(z))-\zeta\tilde{S}_1(\lambda_1-\lambda_1X(z))\tilde{S}_2(\lambda_1-\lambda_1X(z))$ has $b-1$ zeros inside and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations in b unknowns. These equations can be solved by any suitable numerical technique.

3.2 Steady-state condition

Using $P(1)=1$, the steady state condition is derived as $\rho=\lambda_1E(X)\left[E(S_1)+\zeta E(S_2)\right]/b$.

Theorem 1 *Let q_i can be expressed in terms of g_i as*

$$q_n=\sum_{i=0}^{n}L_ik_{n-i},\quad n=0,1,2,\ldots,a-1, \tag{48}$$

where

$$L_n=\frac{h_n+\sum\limits_{i=1}^{n}\gamma_iL_{n-i}}{1-\gamma_0},\quad n=1,2,3,\ldots,a-1, \tag{49}$$

with

$$h_n=\sum_{i=0}^{n}\gamma_i\alpha_{n-i},\quad L_0=\frac{\gamma_0\alpha_0}{1-\gamma_0}, \tag{50}$$

γ_i's and α_i's are the probabilities of the i customers arrive during vacation time and closedown time respectively.

Proof: From equations (32) and (33) , we have

$$\begin{aligned}\sum_{n=0}^{\infty}q_nz^n=&\tilde{V}(\lambda_2-\lambda_2X(z))\tilde{C}(\lambda_2-\lambda_2X(z))\left[\sum_{n=0}^{a-1}k_nz^n\right]+\tilde{V}(\lambda_2-\lambda_2X(z))\sum_{n=0}^{a-1}q_nz^n\\=&\sum_{n=0}^{\infty}\gamma_nz^n\left[\sum_{i=0}^{\infty}\alpha_iz^i\sum_{n=0}^{a-1}k_nz^n+\sum_{n=0}^{a-1}q_nz^n\right].\end{aligned} \tag{51}$$

Equating the coefficient of z^n, $n=0,1,2,\ldots,a-1$ on both sides of (51), we get

$$\begin{aligned}q_n&=\sum_{j=0}^{n}\sum_{i=0}^{n-j}\gamma_i\alpha_{n-i-j}k_j+\sum_{i=0}^{n-1}\gamma_{n-i}q_i+\gamma_0q_n,\\q_n&=\frac{\sum\limits_{j=0}^{n}\sum\limits_{i=0}^{n-j}\gamma_i\alpha_{n-i-j}k_j+\sum\limits_{i=0}^{n-1}\gamma_{n-i}q_i}{1-\gamma_0}.\end{aligned}$$

Co-efficient of k_n in q_n is

$$\frac{\gamma_0\alpha_0}{1-\gamma_0}=L_0\ (say).$$

Co-efficient of k_{n-1} in q_n is

$$\frac{(\gamma_0\alpha_1+\gamma_1\alpha_0)+\gamma_1\left(\frac{\gamma_0\alpha_0}{1-\gamma_0}\right)}{1-\gamma_0}$$

$$\frac{h_1 + \gamma_1 L_0}{1 - \gamma_0} = L_1 \ (say),$$

where

$$h_1 = \gamma_0 \alpha_1 + \gamma_1 \alpha_0.$$

By induction

$$L_n = \frac{h_n + \sum_{i=1}^{n} \gamma_i L_{n-i}}{1 - \gamma_0}, \ n = 0, 1, 2, \ldots, a-1,$$

$$L_0 = \frac{\gamma_0 \alpha_0}{1 - \gamma_0}, \ h_n = \sum_{i=0}^{n} \gamma_i \alpha_{n-i}.$$

3.3 Particular case

When there is no closedown and second optional service

$$P(z) = \frac{\left[\begin{array}{l} \left(\tilde{S}(\lambda_1 - \lambda_1 X(z)) - 1\right)(-\lambda_2 + \lambda_2 X(z)) \sum_{n=a}^{b-1} \left(z^b - z^n\right) g_n \\ \quad + \left(\tilde{V}(\lambda_2 - \lambda_2 X(z)) - 1\right) \left[\left(\tilde{S}(\lambda_1 - \lambda_1 X(z)) - 1\right)(-\lambda_2 + \lambda_2 X(z)) \right. \\ \qquad \left. + (z^b - \tilde{S}(\lambda_1 - \lambda_1 X(z)))(-\lambda_1 + \lambda_1 X(z))\right] \sum_{n=0}^{a-1} (k_n + q_n) z^n \end{array}\right]}{(-\lambda_1 + \lambda_1 X(z))(-\lambda_2 + \lambda_2 X(z))(z^b - \tilde{S}(\lambda_1 - \lambda_1 X(z)))}, \qquad (52)$$

which coincides with the PGF of Balasubramanian and Arumuganathan [4] if the number of vacations are infinite, (i.e) $M = \infty$.

4 Performance measures

4.1 Expected queue length

The expected queue length E(Q) at an arbitrary epoch is obtained by differentiating P(z) at z=1 and is given by

$$E(Q) = \frac{\left[\begin{array}{l} f_1(X, S_1, S_2) \left[\sum_{n=a}^{b-1} [b(b-1) - n(n-1)]\, g_n\right] \\ \quad + f_2(X, S_1, S_2) \sum_{n=a}^{b-1} (b-n) g_n + f_3(X, S_1, S_2, V) \sum_{n=0}^{a-1} (k_n + q_n) \\ \quad + f_4(X, S_1, S_2, V, C) \sum_{n=0}^{a-1} k_n + f_5(X, S_1, S_2, V) \sum_{n=0}^{a-1} (n k_n + n q_n) \\ \qquad + f_6(X, S_1, S_2, V, C) \sum_{n=0}^{a-1} n k_n \end{array}\right]}{2. \left[(\lambda_1 . X_1).(\lambda_2 . X_1).(b - S_1^{(1)} - \zeta . S_2^{(1)}\right]^2}, \qquad (53)$$

$$
\begin{aligned}
f_1(X,S_1,S_2) =& H_3.H_1,\\
f_2(X,S_1,S_2) =& H_1.H_4 - H_2.H_3,\\
f_3(X,S_1,S_2,V) =& V^{(2)}\left[\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)(\lambda_2.X_1)+\left(b-S_1^{(1)}-\zeta S_2^{(1)}\right)(\lambda_1.X_1)\right].H_1\\
&+V^{(1)}\Bigg[(\lambda_2.X_1)\left(S_1^{(2)}+\zeta.(S_2^{(2)}+2.S_1^{(1)}..S_2^{(1)})\right)+(\lambda_2.X_2)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)\\
&+\left(b(b-1)-S_1^{(2)}-\zeta.(S_2^{(2)}+2.S_1^{(1)}.S_2^{(1)})\right)(\lambda_1.X_1)\\
&+\left(b-S_1^{(1)}-\zeta.S_2^{(1)})(\lambda_1.X_2\right)\Bigg].H_1\\
&-V^{(1)}\Bigg[(\lambda_2.X_1)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)\\
&+(\lambda_1.X_1)\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)\Bigg].H_2,
\end{aligned}
$$

$$
\begin{aligned}
f_4(X,S_1,S_2,V,C) =& \left[C^{(2)}+2.C^{(1)}.V^{(1)}\right].\Bigg[\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)(\lambda_2.X_1)\\
&+\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)(\lambda_1.X_1)\Bigg].H_1+(\lambda_2.X_2)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)\\
&+C^{(1)}\Bigg[(\lambda_2.X_1)\left(S_1^{(2)}+\zeta.(S_2^{(2)}+2.S_1^{(1)}.S_2^{(1)})+\right)\\
&+\left(b(b-1)-S_1^{(2)}-\zeta(S_2^{(2)}+2.S_1^{(1)}.S_2^{(1)})\right)(\lambda_1.X_1)\\
&+\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)(\lambda_1.X_2)\Bigg].H_1\\
&-C^{(1)}\left[(\lambda_2.X_1)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)+(\lambda_1.X_1)\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)\right].H_2,\\
f_5(X,S_1,S_2,V,C) =& 2.V^{(1)}\left[(\lambda_2.X_1)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)+(\lambda_1.X_1)\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)\right].H_1,\\
f_6(X,S_1,S_2,V,C) =& 2.C^{(1)}\left[(\lambda_2.X_1)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)+(\lambda_1.X_1)\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)\right].H_1,
\end{aligned}
$$

where

$$
\begin{aligned}
H_1 =& (\lambda_1.X_1).(\lambda_2.X_1).\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right),\\
H_2 =& (\lambda_1.X_2).(\lambda_2.X_1).\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)+(\lambda_1.X_1).(\lambda_2.X_2).\left(b-S_1^{(1)}-\zeta.S_2^{(1)}\right)\\
&+(\lambda_1.X_1).(\lambda_2.X_1).\left(b.(b-1)-S_1^{(2)}-\zeta.(S_2^{(2)}+2.S_1^{(1)}.S_2^{(1)})\right),\\
H_3 =& (\lambda_2.X_1)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right),\\
H_4 =& (\lambda_2.X_1)\left(S_1^{(2)}+\zeta.(S_2^{(2)}+2.S_1^{(1)}.S_2^{(1)})\right)+(\lambda_2.X_2)\left(S_1^{(1)}+\zeta.S_2^{(1)}\right)
\end{aligned}
$$

and

$$\begin{aligned}
X_1 &= E(X),\ S_1^{(1)} = \lambda_1.X_1.E(S_1),\ S_2^{(1)} = \lambda_1.X_1.E(S_2),\\
V^{(1)} &= \lambda_2.X_1.E(V),\ C^{(1)} = \lambda_2.X_1.E(C),\\
S_1^{(2)} &= \lambda_1.X_2.E(S_1) + \lambda_1^2.(E(X))^2.E(S_1^2),\\
S_2^{(2)} &= \lambda_1.X_2.E(S_2) + \lambda_1^2.(E(X))^2.E(S_2^2),\\
V^{(2)} &= \lambda_2.X_2.E(V) + \lambda_2^2.(E(X))^2.E(V^2),\\
C^{(2)} &= \lambda_2.X_2.E(C) + \lambda_2^2.(E(X))^2.E(C^2).
\end{aligned}$$

4.2 Expected waiting time

The expected waiting time is obtained by using Little's formula as:

$$E(W) = \frac{E(Q)}{\lambda E(X)},$$

where $E(Q)$ is given in (53).

4.3 Expected length of busy period

Theorem 2 *Let B be the busy period random variable. Then the expected length of busy period is*

$$E(B) = \frac{E(T)}{\sum_{n=0}^{a-1} k_n}, \tag{54}$$

where

$$E(T) = E(S_1) + \zeta E(S_2).$$

Proof: Let T be the residence time that the server is rendering first essential service or second optional service.

$$E(T) = E(S_1) + \zeta E(S_2).$$

Define a random variable J_1 as

$$J_1 = \begin{cases} 0, & \text{if the server finds less than 'a' customers after the residence time,} \\ 1, & \text{if the server finds atleast 'a' customers after the residence time.} \end{cases}$$

Now the expected length of the busy period is given by

$$\begin{aligned}
E(B) &= E(B/J_1 = 0)P(J_1 = 0) + E(B/J_1 = 1)P(J_1 = 1)\\
&= E(T)P(J_1 = 0) + [E(T) + E(B)]\, P(J_1 = 1),
\end{aligned}$$

Solving for $E(B)$, we get

$$E(B) = \frac{E(T)}{P(J_1 = 0)} = \frac{E(T)}{\sum_{n=0}^{a-1} k_n}.$$

4.4 Expected length of idle period

Theorem 3 *Let I be the idle period random variable. Then the expected length of idle period is given by*

$$E(I) = E(C) + E(I_1), \tag{55}$$

where

$$E(I_1) = \frac{E(V)}{1 - \sum\limits_{n=0}^{a-1}\sum\limits_{i=0}^{n}\left[\sum\limits_{j=0}^{n-i}\gamma_j\alpha_{n-i-j}\right]g_i}, \tag{56}$$

I_1 is the idle period due to multiple vacation process, $E(C)$ is the expected closedown time.

Proof: Define a random variable J_2 as

$$J_2 = \begin{cases} 0, & \text{if the server finds atleast 'a' customers after the first vacation,} \\ 1, & \text{if the server finds less than 'a' customers after the first vacation.} \end{cases}$$

The expected length of idle period due to multiple vacations $E(I_1)$ is given by

$$\begin{aligned} E(I_1) &= E(I_1/J_2=0)P(J_2=0) + E(I_1/J_2=1)P(J_2=1) \\ &= E(V)P(J_2=0) + \left[E(V)+E(I_1)\right]P(J_2=1). \end{aligned}$$

On solving, we get

$$E(I_1) = \frac{E(V)}{P(J_2=0)} = \frac{E(V)}{1-P(J_2=1)} = \frac{E(V)}{1-\sum\limits_{n=0}^{a-1}Q_{1n}(0)}. \tag{57}$$

From equation (32), we get
$Q_{1n}(0)$ = coefficient of z^n in $Q_1(z,0)$

$$\begin{aligned} Q_1(z,0) &= \tilde{V}(\lambda_2-\lambda_2X(z))\tilde{C}(\lambda_2-\lambda_2X(z))\left[\sum_{n=0}^{a-1}k_nz^n\right] \\ &= \sum_{n=0}^{\infty}\gamma_nz^n\sum_{i=0}^{\infty}\alpha_iz^i\sum_{n=0}^{a-1}k_nz^n \\ &= \left[\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\gamma_i\alpha_{n-i}\right)z^i\right]\sum_{n=0}^{a-1}k_nz^n \\ &= \sum_{n=0}^{\infty}h_nz^n\sum_{n=0}^{a-1}k_nz^n \\ &= \sum_{n=0}^{a-1}\left[\sum_{i=0}^{n}h_{n-i}k_i\right]z^n + \sum_{n=a}^{\infty}\left[\sum_{i=0}^{a-1}h_{n-i}k_i\right]z^n, \end{aligned}$$

where

$$h_n = \sum_{i=0}^{n}\gamma_i\alpha_{n-i}.$$

Equating the coefficient of $z^n, n = 0, 1, 2, \ldots, a-1$,

$$\begin{aligned} Q_{1n}(0) &= \sum_{i=0}^{n} h_{n-i} k_i, \\ &= \sum_{i=0}^{n} \left[\sum_{j=0}^{n-i} \gamma_i \alpha_{n-i-j} \right] k_i. \end{aligned}$$

Substitute in (57),we get (56).

5 Cost Model

We derive the expression for finding the total average cost with the following assumptions.

C_s - Start up cost ,
C_v - Reward per unit time due to vacation,
C_h - Holding cost per customer,
C_o - Operating cost per unit time,
C_a - Optional service cost,
C_u - Closedown cost per unit time.

The length of cycle is the sum of the idle period and busy period. Now the expected length of the cycle $E(T_c)$ is obtained as

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(J_2 = 0)} + E(C) + \frac{E(T)}{\sum_{n=0}^{a-1} k_n}.$$

Total Average Cost = Start - up cost
+ closedown cost per unit time
+ Optional service cost per unit time
+ holding cost of number of customers in the queue per unit time
+ operating cost per unit time * ρ
- reward due to vacation per unit time.

$$= \left[C_s + \zeta.C_a.E(S_2) + C_u.E(c) - C_v.\frac{E(V)}{P(J_2 = 0)} \right] . \frac{1}{E(T_c)} \\ + C_h.E(Q) + C_o.\rho,$$

where

$$\rho = \lambda_1 E(X) \left[E(S_1) + \zeta.E(S_2) \right] / b.$$

6 Numerical illustration

In this section, various performance measures which are computed in earlier sections are verified numerically. Numerical example is analyzed using MATLAB, the zeros of the function $z^b - (1-\zeta)\tilde{S}_1(\lambda_1 - \lambda_1 X(z)) - \zeta \tilde{S}_1(\lambda_1 - \lambda_1 X(z))\tilde{S}_2(\lambda_1 - \lambda_1 X(z))$ are obtained and simultaneous equations are solved.

a	$E(Q)$	$E(W)$	$E(I)$	$E(B)$	TAC
1	1.2628	0.2105	0.1226	0.1184	15.8685
2	2.5098	0.4183	0.1432	0.1123	12.0371
3	2.8315	0.4719	0.1651	0.1111	11.7277
4	3.8516	0.6419	0.1693	0.1102	11.1345
5	4.3251	0.7209	0.1715	0.0093	10.5122
6	5.0371	0.8395	0.1728	0.1102	10.0216
7	5.8675	0.9779	0.1712	0.1126	11.8271
8	6.2522	1.0420	0.1708	0.1135	12.3586
9	7.1789	1.1965	0.1699	0.1143	12.7211
10	7.7665	1.2944	0.1622	0.1155	14.1287

Table 1: Minimum threshold value vs Total average cost and performance measures $\mu_1 = 7, \mu_2 = 5, b = 11$

ζ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$	TAC
0.1	2.8653	0.4776	0.8588	0.4312	12.0316
0.2	3.8188	0.6365	0.8603	0.4303	12.7922
0.3	4.0015	0.6667	0.8623	0.4285	12.9498
0.4	4.6475	0.7746	0.8659	0.4266	13.2621
0.5	5.2912	0.8819	0.8674	0.4242	13.8188
0.6	5.7875	0.9646	0.8715	0.4216	14.0077
0.7	6.2479	1.0413	0.8764	0.4178	14.3856
0.8	7.9766	1.3294	0.8793	0.4154	14.7185
0.9	9.5411	1.5902	0.8861	0.4128	15.6914
1.0	11.6984	1.9497	0.8887	0.4111	16.2110

Table 2: Optional service probability vs Total average cost and performance measures $\mu_1 = 8, \mu_2 = 9$

A numerical example is analyzed with the following assumptions:

1. Batch size distribution of the arrival is Geometric with mean two.
2. Service time distribution is Erlang - k for both type of services with $k = 2$.
3. Vacation time and closedown time are exponential with parameter $\gamma = 9$ and $\alpha = 7$ respectively.
4. The arrival rate $\lambda_1 = 3$, when the server is busy.
5. The arrival rate $\lambda_2 = 2$, when the server is idle.
6. Start-up cost : Rs.3
7. Holding cost per customer: Rs. 0.50
8. Operating cost per unit time: Rs.2
9. Optional service cost:Rs.1
10. Reward per unit time due to vacation: Rs.3
11. Closedown cost per unit time: Rs. 0.25

In Table 1, for $\mu_1 = 7, \mu_2 = 5, b = 12$, we obtained the values of expected queue length, expected busy period, expected idle period and total average cost. From Table 1, it is clear that the total average cost is minimum when the minimum threshold value is 6. In Table 2, if we increase the value of the probability for the second optional service, the values of queue length, expected busy period, expected waiting time and total average cost increase.

7 Conclusion and future work

In this paper, the $M^{[X]}/G(a,b)/1$ queueing model with second optional service closedown, multiple vacation and state dependent arrival rate is investigated for the steady-state case. Also we have obtained various performance measures and verified numerically. In future this work may be extended into a queueing model with multiple types of service.

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