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On Sufficiency for Discrete Minmax Fractional Programming Problems Involving Generalized (θ, η, ρ)-V-Univex Functions

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Abstract

In this paper, we introduce a new class of generalized (θ, η, ρ)-V-univex functions for a discrete minmax fractional programming problem and discuss numerous sets of global parametric sufficient optimality conditions under various generalized (θ, η, ρ)-V-univexity assumptions for a discrete minmax fractional programming problem involving arbitrary norms.

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1 Introduction

Minimax programming has been an interesting field of active research for a long time. These problems are of pivotal importance in many areas of modern research such as economics, engineering design, portfolio selection, game theory, rational Chebyshev approximations and financial planning, see [6, 7, 43] and the references therein. Necessary optimality conditions for finite-dimensional constrained minimax problems in terms of Lagrange multipliers have been originally investigated by Bram [10] and Danskin [15]. Schmitendorf [41] has established the necessary and sufficient optimality conditions for the following minimax programming problem:
(P^*)

$$\min_{y\in Y} sup f(x, y)$$

$$subject\ to\ h(x) \leq 0,$$

where $f(.,.) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $h(.) : \mathbb{R}^n \to \mathbb{R}^p$ are differentiable convex functions and Y is a compact subset of $\mathbb{R}^m$. Bector and Bhatia [8] and Weir [45] relaxed the convexity assumption in proving the sufficient optimality conditions for problem (P^*) and formulated several dual models. They proved such results under pseudoconvexity and quasiconvexity assumptions imposed on the functions constituting problem (P^*) and its duals. Bector et al. [9] derived duality results for minimax programming problems involving V-invex functions.

In this paper, we establish the parametric sufficient optimality results under various generalized (θ, η, ρ)-V-univexity assumptions for the following discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1\leq i\leq p} \frac{f_i(x) + \|A_i x\|_{a(i)}}{g_i(x) - \|B_i x\|_{b(i)}}$$

subject to

$$G_j(x) + \|C_j x\|_{c(j)} \leq 0, \;\; j \in \underline{q},$$

$$H_k(x) = 0, \; k \in \underline{r},$$

$$x \in X,$$

where p, q, and r are positive integers and X is an open convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, for each $i \in \underline{p} = \{1, 2, \cdots, p\}$, $j \in \underline{q} = \{1, 2, \cdots, q\}$ and $k \in \underline{r} = \{1, 2, \cdots, r\}$, f_i, g_i, G_j and H_k are real-valued functions defined on X. A_i, B_i and C_j are, respectively, $l_i \times n, m_i \times n$ and $n_j \times n$ matrices, $\|\cdot\|_{a(i)}, \|\cdot\|_{b(i)}$ and $\|\cdot\|_{c(j)}$ are arbitrary norms in $\mathbb{R}^{l_i}, \mathbb{R}^{m_i}$ and $\mathbb{R}^{n_j}$, respectively, and for each $i \in \underline{p}$, $g_i(x) - \|B_i x\|_{b(i)} > 0$ for all x satisfying the constraints of the problem (P). This types of problems in the literature of mathematical programming are known as generalized fractional programming problem. Based on the constraints for (P), we define the feasible set, $\mathbb{F}$ (assumed to be nonempty) of (P) by

$$\mathbb{F} = \{x \in X : G_j(x) + \|C_j x\|_{c(j)} \leq 0, \;\; j \in \underline{q}, \; H_k(x) = 0, \;\; k \in \underline{r}\}.$$

Many authors investigated the optimality conditions and duality results for minimax fractional programming problems using generalized convexity assumptions, see for example [4, 5, 13, 26, 27, 28, 34, 42, 44] and the references therein. Lai et al. [25] established the necessary and sufficient optimality conditions and Lai and Lee [24] obtained duality results for a class of nondifferentiable minimax programming problems with generalized convex functions. Several authors have developed interesting results in nondifferentiable minimax fractional programming problems; see for example, [1, 2, 21, 22, 27, 29, 35, 36] and the references therein.

The concept of invex functions introduced by Hanson [18] and named by Craven [14] is a significant generalization of convex functions. The theory of mathematical programming has grown remarkably when further extensions of invexity have been introduced to establish the optimality conditions and duality results. Bector et al. [11] introduced the classes of univex functions by relaxing the definition of an invex function and obtained optimality and duality results for a nonlinear multiobjective programming problem. Zalmai [46] formulate a number of parametric sufficient optimality results for (P) under various generalized (θ, η, ρ)-V-invexity assumptions.

Motivated by earlier research works by Bector et al. [11], Mishra [33], Mishra et al. [37] and Zalmai [46], the rest of the paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we begin our discussion of sufficient optimality conditions where we formulate and prove numerous sets of sufficiency criteria under a variety of generalized (θ, η, ρ)-V-univexity assumptions that are placed on the individual as well as certain combinations of the problem functions. Utilizing two partitioning schemes, in Section 4 we establish several sets of generalized parametric sufficient optimality results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results.

2 Preliminaries

This section begins by the definition of generalized univex functions which are taken from Mishra et al. [37]. Now, we mention some notations which are used in throughout the paper. Let X be a nonempty open subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\phi : \mathbb{R} \to \mathbb{R}$, and $b : X \times X \times [0,1] \to \mathbb{R}_+$, $b = b(x, u, \lambda)$. If the function f is differentiable, then b does not depend on λ.

Definition 2.1. *A differentiable function f is said to be univex at $y \in X$ with respect to b, ϕ and η if there exist functions b, ϕ and η such that for each $x \in X$,*

$$b(x, y)\, \phi\left[f(x) - f(y)\right] \geqq \langle \nabla f(y), \eta(x, y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle x, y \rangle$ denotes the inner product of the vectors x and y; f is said to be univex on X if the above inequality holds for all $x, y \in X$.

In a similar manner, one can readily define pseudounivex and quasiunivex functions as generalizations of differentiable pseudoinvex and quasiinvex functions.

Definition 2.2. *A differentiable function f is said to be pseudounivex at $y \in X$ with respect to b, ϕ and η if there exist functions b, ϕ and η such that for each $x \in X$,*

$$\langle \nabla f(y), \eta(x, y) \rangle \geqq 0 \Rightarrow b(x, y)\, \phi\left[f(x) - f(y)\right] \geqq 0,$$

or equivalently,

$$b(x, y)\, \phi\left[f(x) - f(y)\right] < 0 \Rightarrow \langle \nabla f(y), \eta(x, y) \rangle < 0.$$

Definition 2.3. *A differentiable function f is said to be quasiunivex at $y \in X$ with respect to b, ϕ and η if there exist functions b, ϕ and η such that for each $x \in X$,*

$$b(x,y)\,\phi\,[f(x)-f(y)] \leqq 0 \Rightarrow \langle \nabla f(y), \eta(x,y)\rangle \leqq 0,$$

or equivalently,

$$\langle \nabla f(y), \eta(x,y)\rangle > 0 \Rightarrow b(x,y)\,\phi\,[f(x)-f(y)] > 0.$$

Definition 2.4. *A differentiable function f is said to be strict pseudounivex at $y \in X$ with respect to b, ϕ and η if there exist functions b, ϕ and η such that for each $x \in X$,*

$$b(x,y)\,\phi\,[f(x)-f(y)] \leqq 0 \Rightarrow \langle \nabla f(y), \eta(x,y)\rangle < 0.$$

The concept of ρ-invexity has been extended in many ways, and various types of generalized ρ-invex functions have been utilized for establishing a wide range of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [12, 14, 16, 17, 19, 30, 31, 38, 40], and for recent surveys of these and related functions, the reader is referred to [23, 39].

The results of this paper are more general compare to results of the Zalmai [46] paper.

3 Sufficient Optimality Conditions

Let the function $F = (F_1, F_2, \ldots, F_m) : X \to \mathbb{R}^m$ be differentiable at x^*. We introduce the following generalizations of the notion of univexity as follows:

Definition 3.1. *The function F is said to be (strictly) $(\alpha, \eta, \bar{\rho})$-$V$-univex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\alpha_i : X \times X \to \mathbb{R}_+ \backslash \{0\} \equiv (0, +\infty)$, and $\bar{\rho}_i \in \mathbb{R}, i \in \underline{m}$, such that for each $x \in X (x \neq x^*)$,*

$$b(x,x^*)\,\phi\,[F_i(x) - F_i(x^*)]\,(>) \geq \langle \alpha_i(x,x^*)\nabla F_i(x^*), \eta(x,x^*)\rangle + \bar{\rho}_i \left\| x - x^* \right\|^2.$$

Definition 3.2. *The function F is said to be (strictly) $(\beta, \eta, \tilde{\rho})$-$V$-pseudounivex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\beta_i : X \times X \to \mathbb{R}_+ \backslash \{0\}, i \in \underline{m}$ and $\tilde{\rho} \in \mathbb{R}$ such that for each $x \in X (x \neq x^*)$,*

$$\left\langle \sum_{i=1}^{m} \nabla F_i(x^*), \eta(x,x^*) \right\rangle \geq -\tilde{\rho} \left\| x - x^* \right\|^2$$

$$\Longrightarrow b(x,x^*)\,\phi \left[\sum_{i=1}^{m} \beta_i(x,x^*) F_i(x) - \sum_{i=1}^{m} \beta_i(x,x^*) F_i(x^*) \right] (>) \geq 0,$$

or equivalently,

$$b(x,x^*)\,\phi \left[\sum_{i=1}^{m} \beta_i(x,x^*) F_i(x) - \sum_{i=1}^{m} \beta_i(x,x^*) F_i(x^*) \right] < 0$$

$$\Longrightarrow \left\langle \sum_{i=1}^{m} \nabla F_i(x^*), \eta(x,x^*) \right\rangle < -\tilde{\rho} \left\| x - x^* \right\|^2 .$$

Definition 3.3. *The function F is said to be (prestrictly) $(\gamma, \eta, \hat{\rho})$-$V$-quasiunivex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\gamma_i : X \times X \to \mathbb{R}_+ \backslash \{0\}, i \in \underline{m}$ and $\hat{\rho} \in \mathbb{R}$ such that for each $x \in X$,*

$$b(x,x^*)\,\phi\left[\sum_{i=1}^{m} \gamma_i(x,x^*) F_i(x) - \sum_{i=1}^{m} \gamma_i(x,x^*) F_i(x^*)\right] (<) \le 0$$

$$\Longrightarrow \left\langle \sum_{i=1}^{m} \nabla F_i(x^*), \eta(x,x^*) \right\rangle \le -\hat{\rho} \left\| x - x^* \right\|^2$$

In this section, we present several sets of sufficiency results under various generalized (θ, η, ρ)-V -univexity assumptions. In our sufficiency results, we shall use the following two auxiliary results, namely, the generalized Cauchy inequality and an alternative expression for the objective function of (P).

Lemma 3.1. *[20] For each $a, b \in \mathbb{R}^m$, $\langle a, b \rangle \le \|a\| * \|b\|$.*

Lemma 3.2. *For each $x \in X$,*

$$\varphi(x) \equiv \max_{1 \le i \le p} \frac{f_i(x) + \|A_i x\|_{a(i)}}{g_i(x) - \|B_i x\|_{b(i)}} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i \left[f_i(x) + \|A_i x\|_{a(i)}\right]}{\sum_{i=1}^{p} u_i \left[g_i(x) - \|B_i x\|_{b(i)}\right]}.$$

Throughout this paper, we use the following list of symbols:

$$\mathcal{A}_i(x,\alpha) = f_i(x) + \left\langle \alpha^i, A_i x \right\rangle, \; i \in \underline{p},$$

$$\mathcal{B}_i(x,\beta) = -g_i(x) + \left\langle \beta^i, B_i x \right\rangle, \; i \in \underline{p},$$

$$\mathcal{C}_j^{\circ}(x,\gamma) = G_j(x) + \left\langle \gamma^j, C_j x \right\rangle, \; j \in \underline{q},$$

$$\mathcal{C}_j(x,v,\gamma) = v_j \left[G_j(x) + \left\langle \gamma^j, C_j x \right\rangle\right], \; j \in \underline{q},$$

$$\mathcal{D}_k(x,w) = w_k H_k(x), \; k \in \underline{r},$$

$$\varepsilon_i(x,\lambda,u,\alpha,\beta) = u_i \left\{ f_i(x) + \left\langle \alpha^i, A_i x \right\rangle - \lambda \left[g_i(x) - \left\langle \beta^i, B_i x \right\rangle\right] \right\}, \; i \in \underline{p},$$

$$\alpha = \left(\alpha^1, \alpha^2, ..., \alpha^p\right), \; \beta = \left(\beta^1, \beta^2, ..., \beta^p\right), \; \gamma = \left(\gamma^1, \gamma^2, ..., \gamma^q\right),$$

and assume that ϕ is linear with $\phi(x) \ge 0 \Rightarrow x \ge 0$, unless otherwise stated.

Theorem 3.1. *Let* $x^* \in \mathbb{F}, \lambda^* = \varphi(x^*) \geqq 0$*, the functions* $f_i, g_i, i \in \underline{p}$*,* G_j *and* H_k *be differentiable at* x^* *for* $j \in \underline{q}, k \in \underline{r}$*, and assume that there exist* $u^* \in U, v^* \in \mathbb{R}^q_+, w^* \in \mathbb{R}^r, \alpha^{*i} \in \mathbb{R}^{l_i}, \beta^{*i} \in \mathbb{R}^{m_i}, i \in \underline{p}$*, and* $\gamma^{*j} \in \mathbb{R}^{n_j}, j \in \underline{q}$*, such that*

$$\sum_{i=1}^{p} u_i^* \left\{\nabla f_i(x^*) + A_i^T \alpha^{*i} - \lambda^* \left[\nabla g_i(x^*) - B_i^T \beta^{*i}\right]\right\} + \sum_{j=1}^{q} v_j^* \left[\nabla G_j(x^*) + C_j^T \gamma^{*j}\right] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \quad (3.1)$$

$$u_i^* \left\{f_i(x^*) + \left\langle \alpha^{*i}, A_i x^*\right\rangle - \lambda^* \left[g_i(x^*) - \left\langle \beta^{*i}, B_i x^*\right\rangle\right]\right\} = 0, \ i \in \underline{p}, \quad (3.2)$$

$$v_j^* \left[G_j(x^*) + \left\langle \gamma^{*j}, C_j x^*\right\rangle\right] = 0, \ j \in \underline{q}, \quad (3.3)$$

$$\left\|\alpha^{*i}\right\|_{a(i)}^* \leq 1, \ \left\|\beta^{*i}\right\|_{b(i)}^* \leq 1, \ i \in \underline{p}, \quad (3.4)$$

$$\left\|\gamma^{*j}\right\|_{c(j)}^* \leq 1, \ j \in \underline{q}. \quad (3.5)$$

Assume, furthermore, that either one of the following three sets of conditions holds:

(a) (i) $(\mathcal{A}_1(., \alpha^*), \ldots, \mathcal{A}_1(., \alpha^*))$ *is* $(\theta, \eta, \bar{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(ii) $(\mathcal{B}_1(., \beta^*), \ldots, \mathcal{B}_p(., \beta^*))$ *is* $(\xi, \eta, \tilde{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(iii) $\left(\mathcal{C}_1^\circ(., \gamma^*), \ldots, \mathcal{C}_q^\circ(., \gamma^*)\right)$ *is* $(\pi, \eta, \hat{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(iv) $(\mathcal{D}_1(., w^*), \ldots, \mathcal{D}_r(., w^*))$ *is* $(\delta, \eta, \breve{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(v) $\theta_i = \xi_i = \pi_j = \delta_k = \sigma$ *for all* $i \in \underline{p}, j \in \underline{q}$ *and* $k \in \underline{r}$*;*
(vi) $\sum_{i=1}^p u_i^* (\bar{\rho}_i + \lambda_i^* \tilde{\rho}_i) + \sum_{j=1}^q v_j^* \hat{\rho}_j + \sum_{k=1}^r \breve{\rho}_k \geq 0$*;*

(b) (i) $(\mathcal{A}_1(., \alpha^*), \ldots, \mathcal{A}_1(., \alpha^*))$ *is* $(\theta, \eta, \bar{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(ii) $(\mathcal{B}_1(., \beta^*), \ldots, \mathcal{B}_p(., \beta^*))$ *is* $(\xi, \eta, \tilde{\rho})$*-V-univex at* x^* *with respect to* b, ϕ *and* η*;*
(iii) $(\mathcal{C}_1(., v^*, \gamma^*), \ldots, \mathcal{C}_q(., v^*, \gamma^*))$ *is* $(\pi, \eta, 0)$*-V-quasiunivex at* x^* *with respect to* b, ϕ *and* η*;*
(iv) $(\mathcal{D}_1(., w^*), \ldots, \mathcal{D}_r(., w^*))$ *is* $(\delta, \eta, 0)$*-V-quasiunivex at* x^* *with respect to* b, ϕ *and* η*;*
(v) $\theta_i = \xi_i = \sigma$ *for all* $i \in \underline{p}$*;*
(vi) $\sum_{i=1}^p u_i^* (\bar{\rho}_i + \lambda_i^* \tilde{\rho}_i) \geq 0$*;*

(c) *The function* $(L_1(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*), \ldots, L_p(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*))$ *is* $(\theta, \eta, 0)$*-V-pseudounivex at* x^* *with respect to* b, ϕ *and* η *where*

$$L_i(z, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) = u_i^* \Big\{ f_i(z) + \left\langle \alpha^{*i}, A_i z\right\rangle - \lambda^* \left[g_i(z) - \left\langle \beta^{*i}, B_i z\right\rangle\right] + \sum_{j=1}^{q} v_j^* \left[G_j(z) + \left\langle \gamma^{*j}, C_j z\right\rangle\right] + \sum_{k=1}^{r} w_k^* H_k(z) \Big\}, \ i \in \underline{p}.$$

Then, x^* *is an optimal solution of (P).*

Proof. *Let* x *be an arbitrary feasible solution of (P).*

(a) For $u^* \geq 0,\ v^* \geq 0$ and $\lambda^* \geq 0$, we have

$$\sum_{i=1}^{p} u_i^* b(x,x^*)\,\phi\left\{f_i(x) + \|A_i x\|_{a(i)} - \lambda^*\left[g_i(x) - \|B_i x\|_{b(i)}\right]\right\}$$
$$\geq \sum_{i=1}^{p} u_i^* b(x,x^*)\,\phi\left\{f_i(x) + \left\|\alpha^{*i}\right\|_{a(i)}^{*}\|A_i x\|_{a(i)} - \lambda^*\left[g_i(x) - \left\|\beta^{*i}\right\|_{b(i)}^{*}\|B_i x\|_{b(i)}\right]\right\}\ (by\ (3.4))$$
$$\geq \sum_{i=1}^{p} u_i^* b(x,x^*)\,\phi\left\{f_i(x) + \left\langle\alpha^{*i}, A_i x\right\rangle - \lambda^*\left[g_i(x) - \left\langle\beta^{*i}, B_i x\right\rangle\right]\right\}\ (by\ Lemma\ 3.1)$$
$$= \sum_{i=1}^{p} u_i^* b(x,x^*)\,\phi\Big\{f_i(x) + \left\langle\alpha^{*i}, A_i x\right\rangle - \left[f_i(x^*) + \left\langle\alpha^{*i}, A_i x^*\right\rangle\right]$$
$$- \lambda^*\left\{g_i(x) - \left\langle\beta^{*i}, B_i x\right\rangle - \left[g_i(x^*) - \left\langle\beta^{*i}, B_i x^*\right\rangle\right]\right\}\Big\}\ \ (by\ (3.2))$$
$$= \sum_{i=1}^{p} u_i^*\Big\{b(x,x^*)\,\phi\left\{f_i(x) + \left\langle\alpha^{*i}, A_i x\right\rangle - \left[f_i(x^*) + \left\langle\alpha^{*i}, A_i x^*\right\rangle\right]\right\}$$
$$-\lambda^* b(x,x^*)\,\phi\left\{g_i(x) - \left\langle\beta^{*i}, B_i x\right\rangle - \left[g_i(x^*) - \left\langle\beta^{*i}, B_i x^*\right\rangle\right]\right\}\Big\}\ \ (by\ the\ linearity\ of\ \phi)$$
$$\geq \sum_{i=1}^{p} u_i^*\Big\{\sigma(x,x^*)\left\langle\nabla f_i(x^*) + A_i^T\alpha^{*i} - \lambda^*\left[\nabla g_i(x^*) - B_i^T\beta^{*i}\right], \eta(x,x^*)\right\rangle$$
$$+ (\bar{\rho}_i + \lambda_i^*\tilde{\rho}_i)\|x - x^*\|^2\Big\}\ \ (by (i), (ii)\ and (v))$$
$$= -\Big\langle\sum_{j=1}^{q} v_j^*\sigma(x,x^*)\left[\nabla G_j(x^*) + C_j^T\gamma^{*j}\right] + \sum_{k=1}^{r} w_k^*\nabla H_k(x^*), \eta(x,x^*)\Big\rangle$$
$$+ \sum_{i=1}^{p} u_i^*(\bar{\rho}_i + \lambda^*\tilde{\rho}_i)\|x - x^*\|^2\ \ \ (by\ (3.1))$$
$$\geq \sum_{j=1}^{q} v_j^*\left\{G_j(x^*) + \langle\gamma^*, C_j x^*\rangle - \left[G_j(x) + \langle\gamma^*, C_j x\rangle\right]\right\}$$
$$+ \left[\sum_{i=1}^{p} u_i^*(\bar{\rho}_i + \lambda^*\tilde{\rho}_i) + \sum_{j=1}^{q} v_j^*\hat{\rho}_j + \sum_{k=1}^{r}\breve{\rho}_k\right]\|x - x^*\|^2$$
$$(by\ (iii), (iv), (v)\ and\ the\ primal\ feasibility\ of\ x\ and\ x^*)$$
$$\geq -\sum_{j=1}^{q} v_j^*\left[G_j(x) + \left\|\gamma^{*j}\right\|_{c(j)}^{*}\|C_j x\|_{c(j)}\right]\ (by\ (vi),\ (3.3),\ and\ Lemma\ 3.1)$$
$$\geq -\sum_{j=1}^{q} v_j^*\left[G_j(x) + \|C_j x\|_{c(j)}\right]\ (by\ (3.5))$$
$$\geq 0\ (by\ feasibility\ of\ x). \quad (3.6)$$

By using $u^* > 0$, and Lemma 3.2 with assumption on ϕ the above inequality implies that

$$\begin{aligned}\varphi(x^*) = \lambda^* &\leq \frac{\sum_{i=1}^{p} u_i^* \left[f_i(x) + \|A_i x\|_{a(i)}\right]}{\sum_{i=1}^{p} u_i^* \left[g_i(x) - \|B_i x\|_{b(i)}\right]} \\ &\leq \max_{u\in U} \frac{\sum_{i=1}^{p} u_i \left[f_i(x) + \|A_i x\|_{a(i)}\right]}{\sum_{i=1}^{p} u_i \left[g_i(x) - \|B_i x\|_{b(i)}\right]} = \varphi(x).\end{aligned}$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b) Since for each $j \in \underline{q}$,

$$\begin{aligned}v_j^* \left[G_j(x) + \langle \gamma^{*j}, C_j x\rangle\right] &\leq v_j^* \left[G_j(x) + \left\|\gamma^{*j}\right\|_{c(j)}^* \|C_j x\|_{c(j)}\right] \ (by\ Lemma\ 3.1) \\ &\leq v_j^* \left[G_j(x) + \|C_j x\|_{c(j)}\right] \ (by (3.5)) \\ &\leq 0 \ (since\ x \in \mathbb{F}) \\ &= v_j^* \left[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle\right] \ (by\ (3.3)),\end{aligned}$$

it follows that

$$b(x, x^*)\,\phi\Big\{\sum_{j=1}^{q} v_j^* \pi_j(x, x^*) \left[G_j(x) + \langle \gamma^{*j}, C_j x\rangle\right] - \sum_{j=1}^{q} v_j^* \pi_j(x, x^*) \left[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle\right]\Big\} \leq 0, \quad (3.7)$$

which in view of (iii) implies that

$$\left\langle \sum_{j=1}^{q} v_j^* \left[\nabla G_j(x^*) + C_j^T \gamma^{*j}\right], \eta(x, x^*) \right\rangle \leq 0. \quad (3.8)$$

Similarly, using our assumption in (iv) and the feasibility of x and x^*, we can show that

$$\left\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*), \eta(x, x^*) \right\rangle \leq 0. \quad (3.9)$$

Now proceeding as in the proof of part (a), we obtain

$$\sum_{i=1}^{p} u_i^* b(x,x^*)\,\phi\left\{f_i(x)+\|A_i x\|_{a(i)}-\lambda^*\left[g_i(x)-\|B_i x\|_{b(i)}\right]\right\}$$
$$\geq -\left\langle \sum_{j=1}^{q} v_j^* \sigma(x,x^*)\left[\nabla G_j(x^*)+C_j^T\gamma^{*j}\right]+\sum_{k=1}^{r} w_k^* \sigma(x,x^*)\,\nabla H_k(x^*),\eta(x,x^*)\right\rangle$$
$$+\sum_{i=1}^{p} u_i^*\left(\bar{\rho}_i+\lambda^*\tilde{\rho}_i\right)\|x-x^*\|^2,$$

which in view of (3.8), (3.9) and (vi), reduces to (3.6). Hence, the rest of the proof is similar to that of part (a).

(c) By our $(\theta,\eta,0)$-V-pseudounivexity assumption, (3.1) implies that

$$b(x,x^*)\,\phi\Big[\sum_{i=1}^{p}\theta_i(x,x^*)L_i(x,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$$
$$-\sum_{i=1}^{p}\theta_i(x,x^*)L_i(x^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\Big]\geq 0.$$

Because $x,x^*\in\mathbb{F},\ v^*\geq 0$, and (3.2) and (3.3) hold, then the above inequality implies that

$$0\leq\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)b(x,x^*)\,\phi\left\{f_i(x)+\left\|\alpha^{*i}\right\|_{a(i)}^{*}\|A_i x\|_{a(i)}-\lambda^*\left[g_i(x)-\left\|\beta^{*i}\right\|_{b(i)}^{*}\|B_i x\|_{b(i)}\right]\right\}$$
$$+\sum_{j=1}^{q}v_j^* b(x,x^*)\,\phi\left[G_j(x)+\left\|\gamma^{*j}\right\|_{c(j)}^{*}\|C_j x\|_{c(j)}\right]\ (by\ Lemma\ 3.1)$$
$$\leq\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)b(x,x^*)\,\phi\left\{f_i(x)+\|A_i x\|_{a(i)}-\lambda^*\left[g_i(x)-\|B_i x\|_{b(i)}\right]\right\}$$
$$+\sum_{j=1}^{q}v_j^* b(x,x^*)\,\phi\left[G_j(x)+\|C_j x\|_{c(j)}\right]\ (by\ (3.4) and (3.5))$$
$$\leq\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)b(x,x^*)\,\phi\left\{f_i(x)+\|A_i x\|_{a(i)}-\lambda^*\left[g_i(x)-\|B_i x\|_{b(i)}\right]\right\}$$
$$(by\ the\ feasibility\ of\ x). \quad (3.10)$$

Since u^* and $\theta_i(x,x^*)>0$, and by the assumption on ϕ, with the Lemma 3.2, the above inequality implies that

$$\begin{aligned}\varphi(x) = \max_{1\le i\le p}\frac{f_i(x)+\|A_ix\|_{a(i)}}{g_i(x)-\|B_ix\|_{b(i)}} &= \max_{1\le i\le p}\frac{\theta_i(x,x^*)\left[f_i(x)+\|A_ix\|_{a(i)}\right]}{\theta_i(x,x^*)\left[g_i(x)-\|B_ix\|_{b(i)}\right]}\\ &\left(since\ \theta_i(x,x^*)>0,\ i\in \underline{p}\right)\\ &= \max_{u\in U}\frac{\sum_{i=1}^p u_i\theta_i(x,x^*)\left[f_i(x)+\|A_ix\|_{a(i)}\right]}{\sum_{i=1}^p u_i\theta_i(x,x^*)\left[g_i(x)-\|B_ix\|_{b(i)}\right]}\ (by\ Lemma\ 3.1)\\ &\geq \max_{u\in U}\frac{\sum_{i=1}^p u_i^*\theta_i(x,x^*)\left[f_i(x)+\|A_ix\|_{a(i)}\right]}{\sum_{i=1}^p u_i^*\theta_i(x,x^*)\left[g_i(x)-\|B_ix\|_{b(i)}\right]}\\ &\geq \lambda^*\ (by(3.10))\\ &= \varphi(x^*).\end{aligned}$$

Since $x\in\mathbb{F}$ was arbitrary, we conclude from the above inequality that x^* is an optimal solution of (P).

□

Theorem 3.2. *Let $x^*\in\mathbb{F}, \lambda^*=\varphi(x^*)\geqq 0$, the functions $f_i, g_i, i\in\underline{p}$, G_j and H_k be differentiable at x^* for $j\in\underline{q}, k\in\underline{r}$, and assume that there exist $u^*\in U, v^*\in\mathbb{R}^q_+, w^*\in\mathbb{R}^r, \alpha^{*i}\in\mathbb{R}^{l_i}, \beta^{*i}\in\mathbb{R}^{m_i}, i\in\underline{p}$, and $\gamma^{*j}\in\mathbb{R}^{n_j}, j\in\underline{q}$, such that* (3.1)-(3.5) *hold. Furthermore, assume that any one of the following four sets of assumptions is satisfied:*

(a) *(i)* $(\varepsilon_1(.,\lambda^*,u^*,\alpha^*,\beta^*),\ldots,\varepsilon_p(.,\lambda^*,u^*,\alpha^*,\beta^*))$, *is* (θ,η,ρ)*-V-pseudounivex at* x^* *with respect to* b,ϕ *and* η;

(ii) $(\mathcal{C}_1(.,v^*,\gamma^*),\ldots,\mathcal{C}_q(.,v^*,\gamma^*))$ *is* $(\pi,\eta,\tilde\rho)$*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iii) $(\mathcal{D}_1(.,w^*),\ldots,\mathcal{D}_r(.,w^*))$ *is* $(\delta,\eta,\hat\rho)$*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iv) $\rho+\tilde\rho+\hat\rho\geq 0$;

(b) *(i)* $(\varepsilon_1(.,\lambda^*,u^*,\alpha^*,\beta^*),\ldots,\varepsilon_p(.,\lambda^*,u^*,\alpha^*,\beta^*))$, *is prestrictly* (θ,η,ρ)*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(ii) $(\mathcal{C}_1(.,v^*,\gamma^*),\ldots,\mathcal{C}_q(.,v^*,\gamma^*))$ *is* $(\pi,\eta,\tilde\rho)$*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iii) $(\mathcal{D}_1(.,w^*),\ldots,\mathcal{D}_r(.,w^*))$ *is* $(\delta,\eta,\hat\rho)$*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iv) $\rho+\tilde\rho+\hat\rho> 0$;

(c) *(i)* $(\varepsilon_1(.,\lambda^*,u^*,\alpha^*,\beta^*),\ldots,\varepsilon_p(.,\lambda^*,u^*,\alpha^*,\beta^*))$, *is prestrictly* (θ,η,ρ)*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(ii) $(\mathcal{C}_1(.,v^*,\gamma^*),\ldots,\mathcal{C}_q(.,v^*,\gamma^*))$ *is strictly* $(\pi,\eta,\tilde{\rho})$*-*V*-pseudounivex at* x^* *with respect to* b,ϕ *and* η;

(iii) $(\mathcal{D}_1(.,w^*),\ldots,\mathcal{D}_r(.,w^*))$ *is* $(\delta,\eta,\hat{\rho})$*-*V*-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iv) $\rho+\tilde{\rho}+\hat{\rho}\geq 0$;

(d) *(i)* $(\varepsilon_1(.,\lambda^*,u^*,\alpha^*,\beta^*),\ldots,\varepsilon_p(.,\lambda^*,u^*,\alpha^*,\beta^*))$, *is prestrictly* (θ,η,ρ)*-*V*-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(ii) $(\mathcal{C}_1(.,v^*,\gamma^*),\ldots,\mathcal{C}_q(.,v^*,\gamma^*))$ *is* $(\pi,\eta,\tilde{\rho})$*-*V*-quasiunivex at* x^* *with respect to* b,ϕ *and* η;

(iii) $(\mathcal{D}_1(.,w^*),\ldots,\mathcal{D}_r(.,w^*))$ *is* $(\delta,\eta,\hat{\rho})$*-*V*-pseudounivex at* x^* *with respect to* b,ϕ *and* η;

(iv) $\rho+\tilde{\rho}+\hat{\rho}\geq 0$.

Then x^* *is an optimal solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).
(a) Proceeding as in the proof of part (b) of Theorem 3.1, we obtain (3.7), which in view of (ii) implies that

$$\left\langle \sum_{j=1}^{q} v_j^* \left[\nabla G_j(x^*) + C_j^T\gamma^{*j}\right], \eta(x,x^*)\right\rangle \leq -\tilde{\rho}\left\|x-x^*\right\|^2. \tag{3.11}$$

Similarly, we can show that our assumptions in (iii) combined with the feasibility of x and x^* lead to the following inequality:

$$\left\langle \sum_{k=1}^{r} w_k^*\nabla H_k(x^*), \eta(x,x^*)\right\rangle \leq -\hat{\rho}\left\|x-x^*\right\|^2. \tag{3.12}$$

Now because of (3.11), (3.12) and (iv), (3.1) reduces to

$$\left\langle \sum_{i=1}^{p} u_i^*\left\{\nabla f_i(x^*) + A_i^T\alpha^{*i} - \lambda^*\left[\nabla g_i(x^*) - B_i^T\beta^{*i}\right]\right\}, \eta(x,x^*)\right\rangle \leq -\rho\left\|x-x^*\right\|^2,$$

which in view of (i) implies that

$$\sum_{i=1}^{p} u_i^*\theta_i(x,x^*)b(x,x^*)\phi\left\{\left[f_i(x)+\left\langle\alpha^{*i},A_ix\right\rangle\right]-\lambda^*\left[g_i(x)-\left\langle\beta^{*i},B_ix\right\rangle\right]\right\}$$
$$\geq \sum_{i=1}^{p} u_i^*\theta_i(x,x^*)b(x,x^*)\phi\left\{\left[f_i(x^*)+\left\langle\alpha^{*i},A_ix^*\right\rangle\right]-\lambda^*\left[g_i(x^*)-\left\langle\beta^{*i},B_ix^*\right\rangle\right]\right\},$$

Because of (3.2), the above inequality implies that

$$0 \leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \left\{ \left[f_i(x) + \langle \alpha^{*i}, A_i x \rangle \right] - \lambda^* \left[g_i(x) - \langle \beta^{*i}, B_i x \rangle \right] \right\}$$

$$\leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \left\{ \left[f_i(x) + \left\| \alpha^{*i} \right\|_{a(i)}^* \left\| A_i x \right\|_{a(i)} \right] - \lambda^* \left[g_i(x) - \left\| \beta^{*i} \right\|_{b(i)}^* \left\| B_i x \right\|_{b(i)} \right] \right\}$$

(*by Lemma* 3.1)

$$\leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \left\{ \left[f_i(x) + \left\| A_i x \right\|_{a(i)} \right] - \lambda^* \left[g_i(x) - \left\| B_i x \right\|_{b(i)} \right] \right\} \ (by\ (3.4)),$$

which is precisely (3.10). As shown in the proof of part (c) of Theorem 3.1, this inequality leads to the conclusion that x^* is an optimal solution of (P).
(b)-(d) The proofs are similar to that of part (a).

□

In the remainder of this section, we briefly discuss certain modifications of Theorems 3.1 and 3.2 obtained by replacing (3.1) with an inequality.

Theorem 3.3. *Let $x^* \in \mathbb{F}, \lambda^* = \varphi(x^*) \geqq 0$, the functions $f_i, g_i, i \in \underline{p}$, G_j and H_k be differentiable at x^* for $j \in \underline{q}, k \in \underline{r}$, and assume that there exist $u^* \in U, v^* \in \mathbb{R}_+^q, w^* \in \mathbb{R}^r, \alpha^{*i} \in \mathbb{R}^{l_i}, \beta^{*i} \in \mathbb{R}^{m_i}, i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}, j \in \underline{q}$, such that* (3.2)-(3.5) *and the following inequality hold:*

$$\sum_{i=1}^{p} u_i^* \left\{ \nabla f_i(x^*) + A_i^T \alpha^{*i} - \lambda^* \left[\nabla g_i(x^*) - B_i^T \beta^{*i} \right] \right\} + \sum_{j=1}^{q} v_j^* \left[\nabla G_j(x^*) + C_j^T \gamma^{*j} \right]$$
$$+ \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) \geq 0, \ \ for\ all\ x \in \mathbb{F}, \quad (3.13)$$

where $\eta : X \times X \to \mathbb{R}^n$ is a given function. Furthermore, assume that any one of the three sets of conditions specified in Theorem 3.1 *is satisfied. Then x^* is an optimal solution of (P).*

Although the proofs of Theorems 3.1 and 3.3 are essentially the same, their contents are some what different. This can easily be seen by comparing (3.1) with (3.13). We observe that any octuple $(x, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ that satisfies (3.1)-(3.5) also satisfies (3.2)-(3.5) and (3.13), but the converse is not necessarily true. Moreover, (3.1) is a system of n equations, whereas (3.13) is a single inequality. Evidently, from a computational point of view, (3.1) is preferable to (3.13) because of the dependence of the latter on the feasible set of (P).

The modified version of Theorem 3.2 can be stated in a similar manner.

4 Generalized Sufficiency Conditions

In this section, we discuss several families of sufficient optimality results under various generalized (θ, η, ρ)-V-univexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [32] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0, J_1, \ldots, J_m\}$ and $\{K_0, K_1, \ldots, K_m\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$ respectively; thus,$J_\mu \subseteq \underline{q}$ for each $\mu \in \underline{m} \bigcup \{0\}$, $J_\mu \bigcap J_\nu = \phi$ for each $\mu, \nu \in \underline{m} \bigcup \{0\}$ with $\mu \neq \nu$ and $\bigcup_{\mu=0}^{m} J_\mu = \underline{q}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_m\}$. Moreover, if m_1 and m_2 are the members of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $m = \max\{m_1, m_2\}$ and $J_\mu = \phi$ or $K_\mu = \phi$ for $\mu > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i\left(., \lambda, u, v, w, \alpha, \beta, \gamma\right),\ i \in \underline{p}$, and $\Lambda_t\left(., v, w, \gamma\right),\ t \in \underline{m}$, defined for fixed $\lambda, u, v, w, \alpha, \beta$, and γ, on X as follows:

$$\Phi_i\left(x, \lambda, u, v, w, \alpha, \beta, \gamma\right) = u_i \Big\{ f_i(x) + \left\langle \alpha^i, A_i x \right\rangle - \lambda\left[g_i(x) - \left\langle \beta^i, B_i x\right\rangle\right] + \sum_{j \in J_0} v_j\left[G_j\left(x\right) + \left\langle \gamma^j, C_j x\right\rangle\right] + \sum_{k \in K_0} w_k H_k\left(x\right) \Big\},\ i \in \underline{p},$$

$$\Lambda_t\left(x, v, w, \gamma\right) = \sum_{j \in J_t} v_j\left[G_j\left(x\right) + \left\langle \gamma^j, C_j x\right\rangle\right] + \sum_{k \in K_t} w_k H_k\left(x\right),\ t \in \underline{m}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

Theorem 4.1. *Let $x^* \in \mathbb{F}, \lambda^* = \varphi(x^*) \geqq 0$, the functions $f_i, g_i, i \in \underline{p}$, G_j and H_k be differentiable at x^* for $j \in \underline{q}, k \in \underline{r}$, and assume that there exist $u^* \in U, v^* \in \mathbb{R}_+^q, w^* \in \mathbb{R}^r, \alpha^{*i} \in \mathbb{R}^{l_i}, \beta^{*i} \in \mathbb{R}^{m_i}, i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}, j \in \underline{q}$, such that* (3.1)-(3.5) *hold. Furthermore, assume that any one of the following three sets of assumptions is satisfied:*

(a) *(i)* $\left(\Phi_1\left(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*\right), \ldots, \Phi_p\left(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*\right)\right)$ *is* $(\theta, \eta, \bar{\rho})$-$V$-*pseudounivex at* x^* *with respect to* b, ϕ *and* η;

(ii) $\left(\Lambda_1\left(., v^*, w^*, \gamma^*\right), \ldots, \Lambda_m\left(., v^*, w^*, \gamma^*\right)\right)$ *is* $(\pi, \eta, \tilde{\rho})$-$V$-*quasiunivex at* x^* *with respect to* b, ϕ *and* η;

(iii) $\bar{\rho} + \tilde{\rho} \geq 0$;

(b) *(i)* $\left(\Phi_1\left(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*\right), \ldots, \Phi_p\left(., \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*\right)\right)$ *is prestrictly* $(\theta, \eta, \bar{\rho})$-$V$-*quasiunivex at* x^* *with respect to* b, ϕ *and* η;

(ii) $\left(\Lambda_1\left(., v^*, w^*, \gamma^*\right), \ldots, \Lambda_m\left(., v^*, w^*, \gamma^*\right)\right)$ *is* $(\pi, \eta, \tilde{\rho})$-$V$-*quasiunivex at* x^* *with respect to* b, ϕ *and* η;

(iii) $\bar{\rho} + \tilde{\rho} > 0$;

(c) *(i)* $(\Phi_1(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*),\ldots,\Phi_p(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*))$ *is prestrictly* $(\theta,\eta,\bar{\rho})$*-V-quasiunivex at* x^* *with respect to* b,ϕ *and* η*;*

(ii) $(\Lambda_1(.,v^*,w^*,\gamma^*),\ldots,\Lambda_m(.,v^*,w^*,\gamma^*))$ *is strictly* $(\pi,\eta,\tilde{\rho})$*-V-pseudounivex at* x^* *with respect to* b,ϕ *and* η*;*

(iii) $\bar{\rho}+\tilde{\rho}\geq 0$.

Then x^* *is an optimal solution of (P).*

Proof. Let be an arbitrary feasible solution of (P).
(a) It is clear that (3.1) can be expressed as follows:

$$\sum_{i=1}^{p}u_i^*\left\{\nabla f_i(x^*)+A_i^T\alpha^{*i}-\lambda^*\left[\nabla g_i(x^*)-B_i^T\beta^{*i}\right]\right\}+\sum_{j\in J_0}v_j^*\left[\nabla G_j(x^*)+C_j^T\gamma^{*j}\right]+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)+\sum_{t=1}^{m}\left\{\sum_{j\in J_t}v_j^*\left[\nabla G_j(x^*)+C_j^T\gamma^{*j}\right]+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\right\}=0. \quad (4.1)$$

Since for each $t\in\underline{m}$,

$$\begin{aligned}\Lambda_t(x,v^*,w^*,\gamma^*) &= \sum_{j\in J_t}v_j^*\left[G_j(x)+\left\langle\gamma^{*j},C_jx\right\rangle\right]+\sum_{k\in K_t}w_k^*H_k(x)\\ &\leq \sum_{j\in J_t}v_j^*\left[G_j(x)+\left\|\gamma^{*j}\right\|_{c(j)}^*\left\|C_jx\right\|_{c(j)}\right]+\sum_{k\in K_t}w_k^*H_k(x)\ \ (\textit{by Lemma } 3.1)\\ &\leq \sum_{j\in J_t}v_j^*\left[G_j(x)+\left\|C_jx\right\|_{c(j)}\right]+\sum_{k\in K_t}w_k^*H_k(x)\ \ (\textit{by } (3.5))\\ &\leq 0\ (\textit{by the feasibility of } x)\\ &\leq \sum_{j\in J_t}v_j^*\left[G_j(x^*)+\left\langle\gamma^{*j},C_jx^*\right\rangle\right]+\sum_{k\in K_t}w_k^*H_k(x^*)\\ &= \Lambda_t(x^*,v^*,w^*,\gamma^*),\end{aligned}$$

and hence

$$b(x,x^*)\,\phi\left[\sum_{t=1}^{m}\pi_t(x,x^*)\Lambda_t(x,v^*,w^*,\gamma^*)-\sum_{t=1}^{m}\pi_t(x,x^*)\Lambda_t(x^*,v^*,w^*,\gamma^*)\right]\leq 0,$$

which because of (ii) implies that

$$\left\langle\sum_{t=1}^{m}\left\{\sum_{j\in J_t}v_j^*\left[\nabla G_j(x^*)+C_j^T\gamma^{*j}\right]+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\right\},\eta(x,x^*)\right\rangle\leqq-\tilde{\rho}\left\|x-x^*\right\|^2, \quad (4.2)$$

Combining (4.1) and (4.2), and using (iii) we get

$$\Big\langle \sum_{i=1}^{p} u_i^* \left\{ \nabla f_i(x^*) + A_i^T \alpha^{*i} - \lambda^* \left[\nabla g_i(x^*) - B_i^T \beta^{*i} \right] \right\} + \sum_{j \in J_0} v_j^* \left[\nabla G_j(x^*) + C_j^T \gamma^{*j} \right] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*), \eta(x, x^*) \Big\rangle \geq \tilde{\rho} \|x - x^*\|^2 \geq \bar{\rho} \|x - x^*\|^2, \quad (4.3)$$

which by virtue of (i) implies that

$$b(x, x^*) \phi \Big[\sum_{i=1}^{p} \theta_i(x, x^*) \Phi_i(x, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) - \sum_{i=1}^{p} \theta_i(x, x^*) \Phi_i(x^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \Big] \geq 0, \quad (4.4)$$

where the inequality follows from (3.2), (3.3) and the feasibility of x^*. Therefore, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \Big\{ f_i(x) + \left\|\alpha^{*i}\right\|_{a(i)}^* \|A_i x\|_{a(i)} - \lambda^* \Big[g_i(x) - \left\|\beta^{*i}\right\|_{b(i)}^* \|B_i x\|_{b(i)} \Big] \\ &\quad + \sum_{j \in J_0} v_j^* \Big[G_j(x) + \left\|\gamma^{*j}\right\|_{c(j)}^* \|C_j x\|_{c(j)} \Big] \Big\} \ (\textit{by Lemma 3.1 and the feasibility of } x) \\ &\leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \Big\{ f_i(x) + \|A_i x\|_{a(i)} - \lambda^* \Big[g_i(x) - \|B_i x\|_{b(i)} \Big] \\ &\quad + \sum_{j \in J_0} v_j^* \Big[G_j(x) + \|C_j x\|_{c(j)} \Big] \Big\} \ (\textit{by } (3.4) \textit{ and } (3.5)) \\ &\leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*) b(x, x^*) \phi \Big\{ f_i(x) + \|A_i x\|_{a(i)} - \lambda^* \Big[g_i(x) - \|B_i x\|_{b(i)} \Big] \Big\} \ (\textit{by the feasibility of } x) \end{aligned}$$

Now using this inequality and Lemma 3.2, as in the proof of Theorem 3.1, we obtain $\varphi(x^*) \leq \varphi(x)$. Since x was arbitrary, we conclude that x^* is an optimal solution of (P).
(b) Proceeding in exactly the same manner as in the proof of part (a), we obtain (4.3) in which the second inequality is strict. By (i), this implies that (4.4) holds and, therefore, the rest of the proof is identical to that of part (a).
(c) The proof is similar to those of parts (a) and (b).

□

In the remainder of this section we present another collection of sufficiency results which are somewhat different from those stated in Theorem 4.1. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$, and by placing appropriate generalized

(θ,η,ρ)-V-univexity requirements on certain vector functions involving $\varepsilon_i(.,\lambda,u,\alpha,\beta)$, $i\in \underline{p}, G_j, j\in\underline{q}$ and $H_k, k\in\underline{r}$.

Let $\{I_0, I_1,\ldots,I_\ell\}$ be partitions of $\underline{p}$, such that $L=\{0,1,\ldots,\ell\}\subset M=\{0,1,\ldots,m\}$, and let the function $\Pi_t(.,\lambda,u,v,w,\alpha,\beta,\gamma): X\to\mathbb{R}$ be defined, for fixed $\lambda,u,v,w,\alpha,\beta$ and γ, by

$$\Pi_t(x,\lambda,u,v,w,\alpha,\beta,\gamma)=\sum_{i\in I_t}u_i\Big\{f_i(x)+\langle\alpha^i,A_ix\rangle-\lambda\left[g_i(x)-\langle\beta^i,B_ix\rangle\right] + \sum_{j\in J_t}v_j\left[G_j(x)+\langle\gamma^j,C_jx\rangle\right]+\sum_{k\in K_t}w_kH_k(x)\Big\},\ t\in\underline{m}.$$

Theorem 4.2. *Let $x^*\in\mathbb{F}, \lambda^*=\varphi(x^*)\geqq 0$, the functions $f_i,g_i,i\in\underline{p}$, G_j and H_k be differentiable at x^* for $j\in\underline{q},k\in\underline{r}$, and assume that there exist $u^*\in U, v^*\in\mathbb{R}^q_+, w^*\in\mathbb{R}^r, \alpha^{*i}\in\mathbb{R}^{l_i}, \beta^{*i}\in\mathbb{R}^{m_i}, i\in\underline{p}$, and $\gamma^{*j}\in\mathbb{R}^{n_j}, j\in\underline{q}$, such that (3.1)-(3.5) hold. Furthermore, assume that any one of the following three sets of assumptions is satisfied:*

(a) *(i)* $(\Pi_0(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*),\ldots,\Pi_\ell(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*))$ *is $(\theta,\eta,\bar\rho)$-V-pseudounivex at x^* with respect to b,ϕ and η;*

(ii) $(\Lambda_{\ell+1}(.,v^*,w^*,\gamma^*),\ldots,\Lambda_m(.,v^*,w^*,\gamma^*))$ *is $(\pi,\eta,\tilde\rho)$-V-quasiunivex at x^* with respect to b,ϕ and η;*

(iii) $\bar\rho+\tilde\rho\geq 0$;

(b) *(i)* $(\Pi_0(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*),\ldots,\Pi_\ell(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*))$ *is prestrictly $(\theta,\eta,\bar\rho)$-V-quasiunivex at x^* with respect to b,ϕ and η;*

(ii) $(\Lambda_{\ell+1}(.,v^*,w^*,\gamma^*),\ldots,\Lambda_m(.,v^*,w^*,\gamma^*))$ *is strictly $(\pi,\eta,\tilde\rho)$-V-pseudounivex at x^* with respect to b,ϕ and η;*

(iii) $\bar\rho+\tilde\rho\geq 0$;

(c) *(i)* $(\Pi_0(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*),\ldots,\Pi_\ell(.,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*))$ *is prestrictly $(\theta,\eta,\bar\rho)$-V-quasiunivex at x^* with respect to b,ϕ and η;*

(ii) $(\Lambda_{\ell+1}(.,v^*,w^*,\gamma^*),\ldots,\Lambda_m(.,v^*,w^*,\gamma^*))$ *is $(\pi,\eta,\tilde\rho)$-V-quasiunivex at x^* with respect to b,ϕ and η;*

(iii) $\bar\rho+\tilde\rho> 0$.

Then x^* is an optimal solution of (P).

Proof. (a) Suppose to the contrary that x^* is not an optimal solution of (P). Then there is $\bar x\in\mathbb{F}$ such that $\varphi(\bar x)<\varphi(x^*)=\lambda^*$, and so it follows that

$$f_i(\bar x)+\|A_i\bar x\|_{a(i)}-\lambda^*\left[g_i(\bar x)-\|B_i\bar x\|_{b(i)}\right]<0,\ i\in\underline{p}.$$

Since $u^*>0$, we see that for each $t\in L$,

$$\sum_{i\in I_t}u_i^*\left\{f_i(\bar x)+\|A_i\bar x\|_{a(i)}-\lambda^*\left[g_i(\bar x)-\|B_i\bar x\|_{b(i)}\right]\right\}<0. \tag{4.5}$$

Now using this inequality, we see that

$$\Pi_t(\bar{x},\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*) \leq \sum_{i\in I_t} u_i^* \Big\{ f_i(\bar{x}) + \left\|\alpha^{*i}\right\|_{a(i)}^* \left\|A_i\bar{x}\right\|_{a(i)}$$

$$- \lambda^* \Big[g_i(\bar{x}) - \left\|\beta^{*i}\right\|_{b(i)}^* \left\|B_i\bar{x}\right\|_{b(i)} \Big] \Big\} + \sum_{j\in J_t} v_j^* \Big[G_j(\bar{x}) + \left\|\gamma^{*j}\right\|_{c(j)}^* \left\|C_j\bar{x}\right\|_{c(j)} \Big]$$

$$(by\ Lemma\ 3.1\ and\ feasibility\ of\ \bar{x})$$

$$\leq \sum_{i\in I_t} u_i^* \Big\{ f_i(\bar{x}) + \left\|A_i\bar{x}\right\|_{a(i)} - \lambda^* \Big[g_i(\bar{x}) - \left\|B_i\bar{x}\right\|_{b(i)} \Big] \Big\}$$

$$+ \sum_{j\in J_t} v_j^* \Big[G_j(\bar{x}) + \left\|C_j\bar{x}\right\|_{c(j)} \Big] \ (by\ (3.4)\ and\ (3.5))$$

$$\leq \sum_{i\in I_t} u_i^* \Big\{ f_i(\bar{x}) + \left\|A_i\bar{x}\right\|_{a(i)} - \lambda^* \Big[g_i(\bar{x}) - \left\|B_i\bar{x}\right\|_{b(i)} \Big] \Big\} \ (by\ the\ feasibility\ of\ \bar{x})$$

$$< 0\ (by\ (4.5))$$

$$= \sum_{i\in I_t} u_i^* \left\{ f_i(x^*) + \left\langle \alpha^{*i}, A_i x^* \right\rangle - \lambda^* \left[g_i(x^*) - \left\langle \beta^{*i}, B_i x^* \right\rangle \right] \right\}$$

$$+ \sum_{j\in J_t} v_j^* \left[G_j(x^*) + \left\langle \gamma^{*j}, C_j x^* \right\rangle \right] + \sum_{k\in K_t} w_k^* H_k(x^*) \ (by\ (3.2),\ (3.3)\ and\ the\ feasibility\ of\ x^*)$$

$$= \Pi_t(x^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$$

and hence

$$b(x,x^*)\,\phi\Big[\sum_{t\in L} \theta_t(x,x^*)\Pi_t(\bar{x},\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$$

$$- \sum_{t\in L} \theta_t(x,x^*)\Pi_t(x^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*) \Big] < 0,$$

which in view of (i) implies that

$$\Big\langle \sum_{i=1}^{p} u_i^* \left\{ \nabla f_i(x^*) + A_i^T\alpha^{*i} - \lambda^* \left[\nabla g_i(x^*) - B_i^T\beta^{*i} \right] \right\}$$

$$+ \sum_{t\in L} \left\{ \sum_{j\in J_t} v_j^* \left[\nabla G_j(x^*) + C_j^T\gamma^{*j} \right] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*) \right\}, \eta(x,x^*) \Big\rangle < -\bar{\rho}\left\|\bar{x}-x^*\right\|^2. \tag{4.6}$$

As shown in the proof of Theorem (4.1), for each $t\in M\backslash L$, $\Lambda_t(\bar{x},v^*,w^*,\gamma^*) \leq \Lambda_t(x^*,v^*,w^*,\gamma^*)$, and hence

$$b(x,x^*)\,\phi\left[\sum_{t\in M\backslash L} \pi_t(\bar{x},x^*)\Lambda_t(\bar{x},v^*,w^*,\gamma^*) - \sum_{t\in M\backslash L} \pi_t(\bar{x},x^*)\Lambda_t(x^*,v^*,w^*,\gamma^*) \right] \leq 0,$$

which in view of (ii) implies that

$$\left\langle \sum_{t\in M\setminus L} \left\{ \sum_{j\in J_t} v_j^* \left[\nabla G_j\left(x^*\right) + C_j^T \gamma^{*j}\right] + \sum_{k\in K_t} w_k^* \nabla H_k\left(x^*\right) \right\}, \eta\left(\bar{x}, x^*\right) \right\rangle \leq -\tilde{\rho} \left\|\bar{x} - x^*\right\|^2 . \tag{4.7}$$

Now combining (4.6) and (4.7) and using (iii), we see that

$$\Big\langle \sum_{i=1}^{p} u_i^* \left\{\nabla f_i\left(x^*\right) + A_i^T \alpha^{*i} - \lambda^* \left[\nabla g_i\left(x^*\right) - B_i^T \beta^{*i}\right]\right\} + \sum_{j=1}^{q} v_j^* \left[\nabla G_j\left(x^*\right) + C_j^T \gamma^{*j}\right]$$
$$+ \sum_{k=1}^{r} w_k^* \nabla H_k\left(x^*\right), \eta\left(x, x^*\right) \Big\rangle < -\left(\bar{\rho} + \tilde{\rho}\right) \left\|\bar{x} - x^*\right\|^2 \leq 0,$$

which contradicts (3.1). Therefore, x^* is an optimal solution of (P).
(b) and (c): The proofs are similar to that of part (a).

□

5 Conclusion

In this paper, we have established a number of sets of global sufficient optimality conditions under various generalized (θ, η, ρ)-V-univexity hypotheses for a discrete minmax fractional programming problem. It indicates that all these results are new in the area of minmax programming.

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