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Analysis of batch arrival single and bulk service queue with two phases of service multiple vacation closedown and balking

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Abstract

In this paper, we analyze batch arrival queueing model with multiple vacation, closedown, balking and two phases of service (for both single and bulk service) is considered. The single server provides two consecutive phases of service. The server provides bulk service only if the queue length is atleast 'a' and the maximum bulk service capacity is 'b'. If the queue length is less than 'a' the server provides single service. After two successive phases of service, if there is no customer waiting in the queue the server starts closedown and then goes for a vacation of random length. After the vacation, still there is no customer waiting in the queue, the server goes for another vacation and so on until he finds atleast one customer waiting in the queue. Otherwise, the server resumes service to the waiting customers. The batch of customers may join the queue with probability η or may not join (i.e. balk) the queue with probability $1-\eta$. Using supplementary variable technique, the steady-state probability generating function of the system size at an arbitrary time is obtained. The performance measures and cost model are also derived. Numerical illustrations are presented to visualize the effect of system parameters.

Keywords: Batch arrival, Single and bulk service, Closedown, Multiple vacation, Balking

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1 Introduction

Queueing models where the server performs closedown work and resumes vacation when there is no sufficient batch size (less than the minimum threshold) for service, is quit common in various practical situations related to manufacturing systems, service systems, etc. Neuts [14] initiated the concept of bulk queues and analyzed a general class of such models. A literature survey on vacation queueing models can be found in Doshi [8] and Takagi [15] which include some applications. Krishna Reddy et al. [13] considered an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Arumuganathan and Jeyakumar [1] obtained the probability generating function of queue length distributions at an arbitrary time epoch for the bulk queueing model with multiple vacation and closedown times. Also they have developed a cost model with a numerical study for their queueing model.

Arumuganathan and Jeyakumar [2] obtained the probability generating function of queue size distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, setup times and N-policy. Ke [12] investigated an $M^{[X]}/G/1$ queueing model with vacation policies, breakdown and startup/closedown times where the vacation, startup, closedown and repair times are generally distributed. Jeyakumar

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and Arumuganathan [11] obtained the PGF of queue size at an arbitrary time epoch in the steady state case for the $M^{[X]}/G/1$ queueing model with two service modes and multiple vacation. Jain and Pandey [9] analyzed the $M^{[X]}/G^{(a,b)}/1$ queueing model with multiple vacation of alternate type, setup times and balking. They computed the PGF of queue size at an arbitrary time epoch.

Arumuganathan and Jeyakumar [3] derived the PGF of queue size at an arbitrary epoch for an $M^X/G/1$ queueing model with two service modes. They have considered the situation in which the server provides single service if the queue length at least 'a' until it reaches 'c' ($a < c$) and resumes bulk service if the queue size is 'c' with a maximum batch size of 'b'. Choudhury and Deka [5] derived the queue size distribution of the $M^X/G/1$ retrial queueing model with two phase of heterogeneous service and Bernoulli vacation schedule. Ayyappan and Shyamala [4] derived the PGF of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time for both steady state and transient cases. Using matrix geometric method, Jain [10] derived the queue size distribution for the priority queueing model with batch arrival, balking, threshold recovery, unreliable server and optional service. Choudhury and Deka [6] obtained the queue size distribution at various time epochs for the $M^X/G/1$ queueing model with two phases of service, multiple vacation and unreliable server.

The rest of the paper is organized as follows. In section 2, batch arrival single and bulk service queue with two phases of service, closedown, multiple vacation and balking is described and the steady-state system size equations are considered. In section 3, using supplementary variable technique, the probability generating function of the queue size are derived and some particular cases are provided. In section 4, performance measures like expected length of busy and idle periods, expected queue length and waiting time are obtained. In section 5, the cost model is provided. In section 6, numerical illustrations are presented to validate the analytical results. In section 7, this research work is concluded with the proposed future work.

2 Model Description

In this paper, we analyze batch arrival queueing model with multiple vacation, closedown, balking and two phases of service (for both single and bulk service). The single server provides two consecutive phases of service. The server provides bulk service only if the queue length is atleast 'a' and the maximum bulk service capacity is 'b'. If the queue length is less than 'a', the server provides single service. After two successive phases of service, if there is no customer waiting in the queue, the server starts closedown and then goes for a vacation of random length. After the vacation, still there is no customer waiting in the queue the server goes for another vacation and so on until he finds atleast one customer waiting in the queue. Otherwise, the server resumes service to the waiting customers. The batch of customers may join the queue with probability η or may not join (i.e. balk) the queue with probability $1 - \eta$.

2.1 Notations

The following notations are used in this paper.
λ - Arrival rate,
X - Group size random variable,
$g_k - Pr\{X = k\}$,
$X(z)$ - Probability generating function (PGF) of X.

Here $S_1(.), S_2(.), G_1(.), G_2(.), V(.)$ and $C(.)$ represent the cumulative distribution function (CDF) of service time for first phase service (single service), service time for second phase service (single service),service time for first phase service (bulk service),service time for second phase service (bulk service), vacation time and closedown time and their corresponding probability density functions are $s_1(x), s_2(x), g_1(x), g_2(x), v(x)$ and $c(x)$ respectively. $S_1^0(t), S_2^0(t), G_1^0(t), G_2^0(t), V^0(t)$ and $C^0(t)$ represent the remaining single service time of first phase service, remaining single service time of second phase service, remaining bulk service time of first phase service, remaining bulk service time of second phase service , vacation time and closedown time at time t respectively. $\tilde{S_1}(\theta), \tilde{S_2}(\theta), \tilde{G_1}(\theta), \tilde{G_2}(\theta), \tilde{V}(\theta)$ and $\tilde{C}(\theta)$ represent the Laplace-Stieltjes transform of S_1, S_2, G_1, G_2, V and C respectively. The supplementary variables $S_1^0(t), S_2^0(t), G_1^0(t), G_2^0(t), V^0(t)$ and $C^0(t)$ are introduced in order to obtain the bivariate Markov process $\{N(t), Y(t)\}$, where $N(t) = \{N_q(t) \cup N_s(t)\}$ and

$$
\begin{aligned}
Y(t) &= (0)[1]\,\{2\}\,\langle 3\rangle\,(4)[5], \text{if the server is on(phase 1 single service)[phase 2 single service]} \\
&\quad \text{\{phase 1 bulk service\}}\ \langle\text{phase 2 bulk service}\rangle\ (vacation)[closedown]. \\
Z(t) &= j, \text{if the server is on } j^{th} \text{ vacation.} \\
N_s(t) &= \text{Number of customers in the service at time } t. \\
N_q(t) &= \text{Number of customers in the queue at time } t.
\end{aligned}
$$

Define the probabilities as,

$$
\begin{aligned}
B_{1,j}^{(1)}(x,t)dt &= P\left\{N_q(t) = j, x \le S_1^0(t) \le x + dx, Y(t) = 0\right\},\ j \ge 0. \\
B_{1,j}^{(2)}(x,t)dt &= P\left\{N_q(t) = j, x \le S_2^0(t) \le x + dx, Y(t) = 1\right\},\ j \ge 0. \\
P_{i,j}^{(1)}(x,t)dt &= P\left\{N_s(t) = i, N_q(t) = j, x \le G_1^0(t) \le x + dx, Y(t) = 2\right\},\ a \le i \le b,\ j \ge 0. \\
P_{i,j}^{(2)}(x,t)dt &= P\left\{N_s(t) = i, N_q(t) = j, x \le G_2^0(t) \le x + dx, Y(t) = 3\right\},\ a \le i \le b,\ j \ge 0. \\
Q_{j,n}(x,t)dt &= P\left\{N_q(t) = n, x \le V^0(t) \le x + dx, Y(t) = 4, Z(t) = j\right\},\ n \ge 0,\ j \ge 1. \\
C_n(x,t)dt &= P\left\{N_q(t) = n, x \le C^0(t) \le x + dx, Y(t) = 5\right\},\ n \ge 0.
\end{aligned}
$$

The supplementary variable technique was introduced by Cox [7]. The steady-state system size equations are obtained as follows:

$$
\begin{aligned}
-B'^{(1)}_{1,0}(x) &= -\lambda B_{1,0}^{(1)}(x) + \lambda(1-\eta)B_{1,0}^{(1)}(x) + B_{11}^{(2)}(0)s_1(x) + \sum_{m=a}^{b} P_{m,1}^{(2)}(0)s_1(x) \\
&\quad + \sum_{l=1}^{\infty} Q_{l,1}(0)s_1(x), \qquad (1) \\
-B'^{(1)}_{1,n}(x) &= -\lambda B_{1,n}^{(1)}(x) + \lambda(1-\eta)B_{1,n}^{(1)}(x) + \eta\sum_{k=1}^{n} B_{1n-k}^{(1)}(x)\lambda g_k + B_{1,n+1}^{(2)}(0)s_1(x) \\
&\quad + \sum_{m=a}^{b} P_{m,n+1}^{(2)}(0)s_1(x)\sum_{l=1}^{\infty} Q_{l,n+1}(0)s_1(x), \quad 1 \le n \le a-2, \qquad (2) \\
-B'^{(1)}_{1,n}(x) &= -\lambda B_{1,n}^{(1)}(x) + \lambda(1-\eta)B_{1,n}^{(1)}(x) + \eta\sum_{k=1}^{n} B_{1n-k}^{(1)}(x)\lambda g_k, \quad n \ge a-1, \qquad (3)
\end{aligned}
$$

$$-B'^{(2)}_{1,0}(x) = -\lambda B^{(2)}_{1,0}(x) + \lambda(1-\eta)B^{(2)}_{1,0}(x) + B^{(1)}_{10}(0)s_2(x), \quad (4)$$

$$-B'^{(2)}_{1,n}(x) = -\lambda B^{(2)}_{1,n}(x) + \lambda(1-\eta)B^{(2)}_{1,n}(x) + B^{(1)}_{1n}(0)s_2(x) + \eta\sum_{k=1}^{n} B^{(2)}_{1n-k}(x)\lambda g_k, \ n \geq 1, \quad (5)$$

$$-P'^{(1)}_{i,0}(x) = -\lambda P^{(1)}_{i,0}(x) + \lambda(1-\eta)P^{(1)}_{i,0}(x) + B^{(2)}_{1i}(0)g_1(x) + \sum_{m=a}^{b} P^{(2)}_{m,i}(0)g_1(x) + \sum_{l=1}^{\infty} Q_{l,i}(0)g_1(x), \quad a \leq i \leq b, \quad (6)$$

$$-P'^{(1)}_{i,j}(x) = -\lambda P^{(1)}_{i,j}(x) + \lambda(1-\eta)P^{(1)}_{i,j}(x) + \eta\sum_{k=1}^{j} P^{(1)}_{i,j-k}(x)\lambda g_k, \quad j \geq 1, \ a \leq i \leq b-1, (7)$$

$$-P'^{(1)}_{b,j}(x) = -\lambda P^{(1)}_{b,j}(x) + \lambda(1-\eta)P^{(1)}_{b,j}(x) + B^{(2)}_{1,b+j}(0)g_1(x) + \sum_{m=a}^{b} P^{(2)}_{m,b+j}(0)g_1(x) + \eta\sum_{k=1}^{j} P^{(1)}_{b,j-k}(x)\lambda g_k + \sum_{l=1}^{\infty} Q_{l,b+j}(0)g_1(x), \quad j \geq 1, \quad (8)$$

$$-P'^{(2)}_{i,0}(x) = -\lambda P^{(2)}_{i,0}(x) + \lambda(1-\eta)P^{(2)}_{i,0}(x) + P^{(1)}_{i0}(0)g_2(x), \quad a \leq i \leq b, \quad (9)$$

$$-P'^{(2)}_{i,j}(x) = -\lambda P^{(2)}_{i,j}(x) + \lambda(1-\eta)P^{(2)}_{i,j}(x) + P^{(1)}_{ij}(0)g_2(x) + \eta\sum_{k=1}^{j} P^{(2)}_{i,j-k}(x)\lambda g_k, \quad j \geq 1, \ a \leq i \leq b, \quad (10)$$

$$-C'_0(x) = -\lambda C_0(x) + \lambda(1-\eta)C_0(x) + \sum_{m=a}^{b} P^{(2)}_{m,0}(0)c(x) + B^{(2)}_{10}(0)c(x), \quad (11)$$

$$-C'_n(x) = -\lambda C_n(x) + \lambda(1-\eta)C_n(x) + \eta\sum_{k=1}^{n} C_{n-k}(x)\lambda g_k, \quad n \geq 1, \quad (12)$$

$$-Q'_{1,0}(x) = -\lambda Q_{1,0}(x) + \lambda(1-\eta)Q_{1,0}(x) + C_0(0)v(x), \quad (13)$$

$$-Q'_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda(1-\eta)Q_{1,n}(x) + C_n(0)v(x) + \eta\sum_{k=1}^{n} Q_{1,n-k}(x)\lambda g_k, \ n \geq 1, \quad (14)$$

$$-Q'_{j,0}(x) = -\lambda Q_{j,0}(x) + \lambda(1-\eta)Q_{j,0}(x) + Q_{j-1,0}(0)v(x), \quad j \geq 2, \quad (15)$$

$$-Q'_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda(1-\eta)Q_{j,n}(x) + \eta\sum_{k=1}^{n} Q_{j,n-k}(x)\lambda g_k, \ j \geq 2, \quad n \geq 1. \quad (16)$$

The Laplace-Stieltjes transform of $P^{(1)}_{i,j}(x), P^{(2)}_{i,j}(x), B^{(1)}_{1,j}(x), B^{(2)}_{1,j}(x), C_n(x), Q_{j,n}(x)$, are defined as follows:

$$\tilde{P}^{(1)}_{i,j}(\theta) = \int_0^\infty e^{-\theta x}P^{(1)}_{i,j}(x)dx, \quad \tilde{P}^{(2)}_{i,j}(\theta) = \int_0^\infty e^{-\theta x}P^{(2)}_{i,j}(x)dx,$$

$$\tilde{B}^{(1)}_{1,j}(\theta) = \int_0^\infty e^{-\theta x}B^{(1)}_{1,j}(x)dx, \quad \tilde{B}^{(2)}_{1,j}(\theta) = \int_0^\infty e^{-\theta x}B^{(2)}_{1,j}(x)dx,$$

$$\tilde{C}_n(\theta) = \int_0^\infty e^{-\theta x} C_n(x)dx, \quad \tilde{Q}_{j,n}(\theta) = \int_0^\infty e^{-\theta x} Q_{j,n}(x)dx.$$

Taking Laplace-Stieltjes transform from (1) to (16), we get

$$\theta\tilde{B}^{(1)}_{1,0}(\theta) - B^{(1)}_{1,0}(0) = \lambda\tilde{B}^{(1)}_{1,0}(\theta) - \lambda(1-\eta)\tilde{B}^{(1)}_{1,0}(\theta) - \tilde{S}_1(\theta)\Big[B^{(2)}_{11}(0) + \sum_{m=a}^{b} P^{(2)}_{m,1}(0) + \sum_{l=1}^{\infty} Q_{l,1}(0)\Big], \tag{17}$$

$$\theta\tilde{B}^{(1)}_{1,n}(\theta) - B^{(1)}_{1,n}(0) = \lambda\tilde{B}^{(1)}_{1,n}(\theta) - \lambda(1-\eta)\tilde{B}^{(1)}_{1,n}(\theta) - \eta\sum_{k=1}^{n} \tilde{B}^{(1)}_{1n-k}(\theta)\lambda g_k$$
$$- \tilde{S}_1(\theta)\Big[B^{(2)}_{1,n+1}(0) + \sum_{m=a}^{b} P^{(2)}_{m,n+1}(0) + \sum_{l=1}^{\infty} Q_{l,n+1}(0)\Big], \ 1 \le n \le a-2, \tag{18}$$

$$\theta\tilde{B}^{(1)}_{1,n}(\theta) - B^{(1)}_{1,n}(0) = \lambda\tilde{B}^{(1)}_{1,n}(\theta) - \lambda(1-\eta)\tilde{B}^{(1)}_{1,n}(\theta) - \eta\sum_{k=1}^{n} \tilde{B}^{(1)}_{1n-k}(\theta)\lambda g_k, \quad n \ge a-1, \tag{19}$$

$$\theta\tilde{B}^{(2)}_{1,0}(\theta) - B^{(2)}_{1,0}(0) = \lambda\tilde{B}^{(2)}_{1,0}(\theta) - \lambda(1-\eta)\tilde{B}^{(2)}_{1,0}(\theta) - \tilde{S}_2(\theta)B^{(1)}_{10}(0), \tag{20}$$

$$\theta\tilde{B}^{(2)}_{1,n}(\theta) - B^{(2)}_{1,n}(0) = \lambda\tilde{B}^{(2)}_{1,n}(\theta) - \lambda(1-\eta)\tilde{B}^{(2)}_{1,n}(\theta) - \tilde{S}_2(\theta)B^{(1)}_{1n}(0) - \eta\sum_{k=1}^{n} \tilde{B}^{(2)}_{1n-k}(\theta)\lambda g_k, \tag{21}$$

$$\theta\tilde{P}^{(1)}_{i,0}(\theta) - P^{(1)}_{i,0}(0) = \lambda\tilde{P}^{(1)}_{i,0}(\theta) - \lambda(1-\eta)\tilde{P}^{(1)}_{i,0}(\theta) - \tilde{G}_1(\theta)\Big[\sum_{m=a}^{b} P^{(2)}_{m,i}(0) + B^{(2)}_{1,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\Big],$$
$$a \le i \le b, \tag{22}$$

$$\theta\tilde{P}^{(1)}_{i,j}(\theta) - P^{(1)}_{i,j}(0) = \lambda\tilde{P}^{(1)}_{i,j}(\theta) - \lambda(1-\eta)\tilde{P}^{(1)}_{i,j}(\theta) - \eta\sum_{k=1}^{j} \tilde{P}^{(1)}_{i,j-k}(\theta)\lambda g_k, \tag{23}$$

$$\theta\tilde{P}^{(1)}_{b,j}(\theta) - P^{(1)}_{b,j}(0) = \lambda\tilde{P}^{(1)}_{b,j}(\theta) - \lambda(1-\eta)\tilde{P}^{(1)}_{b,j}(\theta) - \eta\sum_{k=1}^{j} \tilde{P}^{(1)}_{b,j-k}(\theta)\lambda g_k$$
$$- \tilde{G}_1(\theta)\Big[\sum_{m=a}^{b} P^{(2)}_{m,b+j}(0) + B^{(1)}_{1,b+j}(0) + \sum_{l=1}^{\infty} Q_{l,b+j}(0)\Big], \ j \ge 1, \tag{24}$$

$$\theta\tilde{P}^{(2)}_{i,0}(\theta) - P^{(2)}_{i,0}(0) = \lambda\tilde{P}^{(2)}_{i,0}(\theta) - \lambda(1-\eta)\tilde{P}^{(2)}_{i,0}(\theta) - \tilde{G}_2(\theta)P^{(1)}_{i0}(0), \tag{25}$$

$$\theta\tilde{P}^{(2)}_{i,j}(\theta) - P^{(2)}_{i,j}(0) = \lambda\tilde{P}^{(2)}_{i,j}(\theta) - \lambda(1-\eta)\tilde{P}^{(2)}_{i,j}(\theta) - \tilde{G}_2(\theta)P^{(1)}_{ij}(0) - \eta\sum_{k=1}^{j} \tilde{P}^{(2)}_{i,j-k}(\theta)\lambda g_k,$$
$$a \le i \le b, \tag{26}$$

$$\theta\tilde{C}_0(\theta) - C_0(0) = \lambda\tilde{C}_0(\theta) - \lambda(1-\eta)C_0(\theta) - \tilde{C}(\theta)\Big[B^{(2)}_{10}(0) + \sum_{m=a}^{b} P^{(2)}_{m0}(0)\Big], \tag{27}$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda\tilde{C}_n(\theta) - \lambda(1-\eta)C_n(\theta) - \eta\sum_{k=1}^{n} \tilde{C}_{n-k}(\theta)\lambda g_k, \ n \ge 1, \tag{28}$$

$$\theta\tilde{Q}_{1,0}(\theta)-Q_{1,0}(0) = \lambda\tilde{Q}_{1,0}(\theta)-\lambda(1-\eta)Q_{1,0}(\theta)-\tilde{V}(\theta)C_0(0), \quad (29)$$

$$\theta\tilde{Q}_{1,n}(\theta)-Q_{1,n}(0) = \lambda\tilde{Q}_{1,n}(\theta)-\lambda(1-\eta)Q_{1,n}(\theta)-\tilde{V}(\theta)C_n(0)-\eta\sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)\lambda g_k, \quad n\geq 1, \quad (30)$$

$$\theta\tilde{Q}_{j,0}(\theta)-Q_{j,0}(0) = \lambda\tilde{Q}_{j,0}(\theta)-\lambda(1-\eta)Q_{j,0}(\theta)-\tilde{V}(\theta)Q_{j-1,0}(0), \quad (31)$$

$$\theta\tilde{Q}_{j,n}(\theta)-Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\theta)-\lambda(1-\eta)Q_{j,n}(\theta)-\eta\sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda g_k. \quad (32)$$

To find the probability generating function (PGF) of queue size, we define the following PGFs:

$$\begin{aligned}
\tilde{P}_i^{(1)}(z,\theta) &= \sum_{j=0}^{\infty}\tilde{P}_{i,j}^{(1)}(\theta)z^j, \quad P_i^{(1)}(z,0)=\sum_{j=0}^{\infty}P_{i,j}^{(1)}(0)z^j, \quad a\leq i\leq b,\\
\tilde{P}_i^{(2)}(z,\theta) &= \sum_{j=0}^{\infty}\tilde{P}_{i,j}^{(2)}(\theta)z^j, \quad P_i^{(2)}(z,0)=\sum_{j=0}^{\infty}P_{i,j}^{(2)}(0)z^j, \quad a\leq i\leq b,\\
\tilde{B}^{(1)}(z,\theta) &= \sum_{j=0}^{\infty}\tilde{B}_{1,j}^{(1)}(\theta)z^j, \quad B^{(1)}(z,0)=\sum_{j=0}^{\infty}B_{1,j}^{(1)}(0)z^j,\\
\tilde{B}^{(2)}(z,\theta) &= \sum_{j=0}^{\infty}\tilde{B}_{1,j}^{(2)}(\theta)z^j, \quad B^{(2)}(z,0)=\sum_{j=0}^{\infty}B_{1,j}^{(2)}(0)z^j,\\
\tilde{C}(z,\theta) &= \sum_{n=0}^{\infty}\tilde{C}_n(\theta)z^n, \quad C(z,0)=\sum_{n=0}^{\infty}C_n(0)z^n,\\
\tilde{Q}_j(z,\theta) &= \sum_{n=0}^{\infty}\tilde{Q}_{j,n}(\theta)z^n, \quad Q_j(z,0)=\sum_{n=0}^{\infty}Q_{j,n}(0)z^j, \quad j\geq 1. \qquad (33)
\end{aligned}$$

By multiplying the equations from (17) to (32) by suitable power of z^n and summing over n, ($n=0$ to ∞) and using (33),

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{Q}_1(z,\theta) = Q_1(z,0)-C(z,0)\tilde{V}(\theta), \quad (34)$$

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{Q}_j(z,\theta) = Q_j(z,0)-\tilde{V}(\theta)Q_{j-1,0}(0), \ j\geq 2, \quad (35)$$

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{C}(z,\theta) = C(z,0)-\tilde{C}(\theta)\left[B^{(2)}{}_{10}(0)+\sum_{m=a}^{b}P_{m,0}^{(2)}(0)\right], \quad (36)$$

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{B}^{(1)}(z,\theta) = B^{(1)}(z,0) - \tilde{S}_1(\theta)\left[\sum_{n=1}^{a-1}\left[B_{1n}^{(2)}(0)+\sum_{m=a}^{b}P_{m,n}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,n}(0)\right]z^{n-1}\right], \quad (37)$$

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{B}^{(2)}(z,\theta) = B^{(2)}(z,0)-\tilde{S}_2(\theta)B^{(1)}(z,0), \quad (38)$$

$$(\theta-\lambda\eta+\lambda\eta X(z))\tilde{P}_i^{(1)}(z,\theta) = P_i^{(1)}(z,0)-\tilde{G}_1(\theta)\left[B_{1,i}^{(2)}(0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\right], \quad a\leq i\leq b-1, \quad (39)$$

$$(\theta - \lambda\eta + \lambda\eta X(z))\tilde{P}_b^{(1)}(z,\theta) = P_b^{(1)}(z,0) - \frac{\tilde{G}_1(\theta)}{z^b}\Big[B^{(2)}(z,0) - \sum_{j=0}^{b-1} B_{1,j}^{(2)}(0)z^j + \sum_{m=a}^{b} P_m^{(2)}(z,0) - \sum_{m=a}^{b}\sum_{j=0}^{b-1} P_{m,j}^{(2)}(0)z^j + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{l=1}^{\infty}\sum_{j=0}^{b-1} Q_{l,j}(0)z^j\Big], \quad (40)$$

$$(\theta - \lambda\eta + \lambda\eta X(z))\tilde{P}_i^{(2)}(z,\theta) = P_i^{(2)}(z,0) - \tilde{G}_2(\theta)P_i^{(1)}(z,0),\ a \le i \le b. \quad (41)$$

By substituting $\theta = \lambda\eta - \lambda\eta X(z)$ in (34) to (41), we get

$$Q_1(z,0) = \tilde{V}(\lambda\eta - \lambda\eta X(z))C(z,0), \quad (42)$$

$$Q_j(z,0) = \tilde{V}(\lambda\eta - \lambda\eta X(z))\sum_{n=0}^{a-1} Q_{j-1,0}(0),\ j \ge 2, \quad (43)$$

$$C(z,0) = \tilde{C}(\lambda\eta - \lambda\eta X(z)))\Big[B_{10}^{(2)}(0) + \sum_{m=a}^{b} P_{m,0}^{(2)}(0)\Big], \quad (44)$$

$$P_i^{(1)}(z,0) = \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\Big[B_{1i}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{li}(0) + \sum_{m=a}^{b} P_{m,i}^{(2)}(0)\Big] \quad a \le i \le b-1, \quad (45)$$

$$z^b P_b^{(1)}(z,0) = \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\Big[B^{(2)}(z,0) - \sum_{j=0}^{b-1} B_{1,j}^{(2)}(0)z^j + \sum_{m=a}^{b} P_m^{(2)}(z,0) - \sum_{j=0}^{b-1}\sum_{m=a}^{b} P_{m,j}^{(2)}(0)z^j + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1}\sum_{l=1}^{\infty} Q_{l,j}(0)z^j\Big]. \quad (46)$$

Solving for $P_b^{(1)}(z,0)$,

$$P_b^{(1)}(z,0) = \frac{\tilde{G}_1(\lambda\eta - \lambda\eta X(z))f(z)}{z^b - \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\tilde{G}_2(\lambda\eta - \lambda\eta X(z))}, \quad (47)$$

where

$$f(z) = B^{(2)}(z,0) - \sum_{n=0}^{b-1} B_{1n}^{(2)}(0)z^n + \tilde{G}_2(\lambda\eta - \lambda\eta X(z))\sum_{m=a}^{b-1}(P_m^{(1)}(z,0) - \sum_{j=0}^{b-1} P_{mj}^{(2)}z^j) + \sum_{l=1}^{\infty}(Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}z^j), \quad (48)$$

$$P_i^{(2)}(z,0) = \tilde{G}_2(\lambda\eta - \lambda\eta X(z))P_i^{(1)}(z,0), \quad (49)$$

$$B^{(1)}(z,0) = \tilde{S}_1(\lambda\eta - \lambda\eta X(z))\sum_{n=1}^{a-1}\Big[B_{1,n}^{(2)}(0) + \sum_{m=a}^{b} P_{m,n}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{l,n}(0)\Big]z^{n-1}, \quad (50)$$

$$B^{(2)}(z,0) = \tilde{S}_2(\lambda\eta - \lambda\eta X(z))B^{(1)}(z,0), \quad (51)$$

Substitute (42) to (50) in (34) to (41), we get

$$\tilde{Q}_1(z,\theta) = \frac{\left[\tilde{V}(\lambda\eta - \lambda\eta X(z)) - \tilde{V}(\theta)\right] C(z,0)}{(\theta - \lambda\eta + \lambda\eta X(z))}, \tag{52}$$

$$\tilde{Q}_j(z,\theta) = \frac{\left[\tilde{V}(\lambda\eta - \lambda\eta X(z)) - \tilde{V}(\theta)\right] \sum_{l=1}^{\infty} Q_{l0}(0)}{(\theta - \lambda\eta + \lambda\eta X(z))}, \; j \geq 2, \tag{53}$$

$$\tilde{C}(z,\theta) = \frac{\left(\tilde{C}(\lambda\eta - \lambda\eta X(z)) - \tilde{C}(\theta)\right) (\sum_{m=a}^{b} P_{m0}^{(2)}(0) + B_{10}^{(2)}(0)}{(\theta - \lambda\eta + \lambda\eta X(z))}, \tag{54}$$

$$\tilde{P}_i^{(1)}(z,\theta) = \frac{\left(\tilde{G}_1(\lambda\eta - \lambda\eta X(z)) - \tilde{G}_1(\theta)\right) \left[B_{1i}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{li}(0) + \sum_{m=a}^{b} P_{m,i}^{(2)}(0)\right]}{(\theta - \lambda\eta + \lambda\eta X(z))}, \tag{55}$$

$$\tilde{P}_b^{(1)}(z,\theta) = \frac{\left[\tilde{G}_1(\lambda\eta - \lambda\eta X(z)) - \tilde{G}_1(\theta)\right] f(z)}{(\theta - \lambda\eta + \lambda\eta X(z)) \left[z^b - \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\tilde{G}_2(\lambda\eta - \lambda\eta X(z))\right]}, \tag{56}$$

where

$$f(z) = B^{(2)}(z,0) - \sum_{n=0}^{b-1} B_{1n}^{(2)}(0)z^n + \tilde{G}_2(\lambda\eta - \lambda\eta X(z)) \sum_{m=a}^{b-1} (P_m^{(1)}(z,0) - \sum_{j=0}^{b-1} P_{mj}^{(2)} z^j)$$
$$+ \sum_{l=1}^{\infty} (Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj} z^j), \tag{57}$$

$$\tilde{P}_i^{(2)}(z,\theta) = \frac{\left(\tilde{G}_2(\lambda\eta - \lambda\eta X(z)) - \tilde{G}_2(\theta)\right) P_i^{(1)}(z,0)}{(\theta - \lambda\eta + \lambda\eta X(z))}, \tag{58}$$

$$\tilde{B}^{(1)}(z,\theta) = \frac{\left(\tilde{S}_1(\lambda\eta - \lambda\eta X(z)) - \tilde{S}_1(\theta)\right) \sum_{n=1}^{a-1} \left[B_{1,n}^{(2)}(0) + \sum_{m=a}^{b} P_{m,n}^{(2)}(0) + \sum_{l=1}^{\infty} Q_{l,n}(0)\right] z^{n-1}}{(\theta - \lambda\eta + \lambda\eta X(z))}, \tag{59}$$

$$\tilde{B}^{(2)}(z,\theta) = \frac{\left(\tilde{S}_2(\lambda\eta - \lambda\eta X(z)) - \tilde{S}_2(\theta)\right) B^{(1)}(z,0)}{(\theta - \lambda\eta + \lambda\eta X(z))}. \tag{60}$$

Let

$$p_i^{(1)} = \sum_{m=a}^{b} P_{m,i}^{(1)}(0), \; p_i^{(2)} = \sum_{m=a}^{b} P_{m,i}^{(2)}(0), \; q_i = \sum_{l=1}^{\infty} Q_{l,i}(0),$$
$$b_i^{(1)} = B_{1,i}^{(1)}(0), \; b_i^{(2)} = B_{1,i}^{(2)}(0), \; d_i = p_i^{(2)} + b_i^{(2)}, \; k_i = d_i + q_i. \tag{61}$$

3 Probability generating function of queue size

In this section, the PGF, $P(z)$ of the queue size at an arbitrary time is derived.

3.1 PGF of queue size at an arbitrary time

If P(z) be the PGF of the queue size at an arbitrary time, then

$$\begin{aligned} P(z) &= \sum_{m=a}^{b-1} \tilde{P}_m^{(1)}(z,0) + \tilde{P}_b^{(1)}(z,0) + \sum_{m=a}^{b} \tilde{P}_m^{(2)}(z,0) + \tilde{B}^{(1)}(z,0) + \tilde{B}^{(2)}(z,0) \\ &\quad + \tilde{C}(z,0) + \sum_{l=1}^{\infty} \tilde{Q}_l(z,0). \end{aligned} \tag{62}$$

By substituting $\theta = 0$ into the equations (52) to (60), then the equation (62) becomes

$$P(z) = \frac{\left[\begin{array}{l} \Big[(z^b-1)(\tilde{S}_1(\lambda\eta-\lambda\eta X(z))\tilde{S}_2(\lambda\eta-\lambda\eta X(z))-1)-(z-1)(\tilde{G}_1(\lambda\eta-\lambda\eta X(z)) \\ \quad \tilde{G}_2(\lambda\eta-\lambda\eta X(z))-1)\Big]\sum_{n=1}^{a-1} k_n z^n + z(\tilde{G}_1(\lambda\eta-\lambda\eta X(z))\tilde{G}_2(\lambda\eta-\lambda\eta X(z))-1) \\ \quad \sum_{n=a}^{b-1}(z^b-z^n)k_n + z(\tilde{V}(\lambda\eta-\lambda\eta X(z))\tilde{C}(\lambda\eta-\lambda\eta X(z))-1)(z^b-1)d_0 \\ \quad + z(\tilde{V}(\lambda\eta-\lambda\eta X(z))-1)(z^b-1)q_0 \end{array}\right]}{(-\lambda\eta+\lambda\eta X(z))z\left[z^b-\tilde{G}_1(\lambda\eta-\lambda\eta X(z))\tilde{G}_2(\lambda\eta-\lambda\eta X(z))\right]}. \tag{63}$$

Equation (63) has $b+1$ unknowns $k_1, k_2, \ldots, k_{b-1}, d_0, q_0$. Using the following result, we express q_0 in terms of d_0 in such a way that numerator has only b constants. Now equation (63) gives the PGF of the number of customers involving only 'b' unknowns. By Rouche's theorem of complex variables, it can be proved that $(z^b - \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\tilde{G}_2(\lambda\eta - \lambda\eta X(z)))$ has $b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations in b unknowns. These equations can be solved by any suitable numerical technique.

3.2 Steady-state condition

Using $P(1) = 1$, the steady state condition is derived as $\rho = \lambda\eta E(X)\left[E(G_1) + E(G_2)\right]/b$.

Theorem 1 *Let q_0 can be expressed in terms of d_0 as*

$$q_0 = \frac{\gamma_0\delta_0 d_0}{1-\gamma_0}, \tag{64}$$

Proof: From equations (42) and (43), we have

$$\begin{aligned} \sum_{n=0}^{\infty} q_n z^n &= \tilde{V}(\lambda\eta-\lambda\eta X(z))\left[\tilde{C}(\lambda\eta-\lambda\eta X(z))d_0 + q_0\right], \\ &= \sum_{n=0}^{\infty}\gamma_n z^n\left[\sum_{n=0}^{\infty}\delta_i z^i[d_0+q_0]\right] \end{aligned} \tag{65}$$

Equating constant term, we get

$$q_0 = \frac{\gamma_0 \delta_0 d_0}{1 - \gamma_0}$$

3.3 Particular case

When there is no second phase of service, no closedown and no balking

$$P(z) = \frac{\left[\begin{array}{r} (z^b - 1)\left(\tilde{S}(\lambda - \lambda X(z)) - 1\right) - (z-1)\left(\tilde{G}(\lambda - \lambda X(z)) - 1\right)\sum_{n=1}^{a-1} k_n z^n \\ +z\left(\tilde{G}(\lambda - \lambda X(z)) - 1\right)\sum_{n=a}^{b-1}(z^b - z^n)k_n \\ + z\left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)(z^b - 1)k_0 \end{array}\right]}{(-\lambda + \lambda X(z))z(z^b - \tilde{G}(\lambda - \lambda X(z)))}, \quad (66)$$

which coincides with the PGF of Jayakumar and Arumuganathan [11].

4 Performance measures

4.1 Expected queue length

The expected queue length $E(Q)$ at an arbitrary epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$E(Q) = \frac{\left[\begin{array}{r} f_1(X, S_1, S_2, G_1, G_2)\left[\sum_{n=1}^{a-1} k_n\right] + f_2(X, S_1, S_2, G_1, G_2)\sum_{n=1}^{a-1} n k_n \\ +f_3(X, G_1, G_2)\sum_{n=a}^{b-1}(b-n)k_n + f_4(X, G_1, G_2)\left[\sum_{n=a}^{b-1}[b(b-1) - n(n-1)]\,k_n\right] \\ + f_5(X, G_1, G_2, V)(d_0 + q_0) + f_6(X, G_1, G_2, V, C)d_0 \end{array}\right]}{2.\left[(\lambda.\eta.X_1).(b - G_1^{(1)} - G_2^{(1)})\right]^2}, \quad (67)$$

$$\begin{aligned}
f_1(X, , S_1, S_2, G_1, G_2) =& \Big[b(b-1).(S_1^{(1)} + S_2^{(1)}) + b(S_1^{(2)} + S_2^{(2)} + 2.S_1^{(1)}.S_2^{(1)}) \\
& - (G_1^{(2)} + G_2^{(2)} + 2.G_1^{(1)}.G_2^{(1)})\Big].H_1 - \Big[b.(S_1^{(1)} + S_2^{(1)}) - G_1^{(1)} - G_2^{(1)}\Big].H_2 \\
f_2(X, S_1, S_2, G_1, G_2) =& 2.\Big[b.(S_1^{(1)} + S_2^{(1)}) - G_1^{(1)} - G_2^{(1)}\Big].H_1, \\
f_3(X, G_1, G_2) =& \Big[2.(G_1^{(1)} + G_2^{(1)}) + G_1^{(2)} + G_2^{(2)} + 2.G_1^{(1)}.G_2^{(1)}\Big].H_1 - (G_1^{(1)} + G_2^{(1)}).H_2 \\
f_4(X, G_1, G_2) =& (G_1^{(1)} + G_2^{(1)}).H_1,
\end{aligned}$$

$$f_5(X, G_1, G_2, V) = \Big[2.b.V^{(1)} + b.V^{(2)} + b(b-1).V^{(1)}\Big].H_1 - b.V^{(1)}.H_2$$
$$f_6(X, G_1, G_2, V, C) = \Big[2.b.C^{(1)} + b.(C^{(2)} + 2.C^{(1)}.V^{(1)}) + b(b-1).C^{(1)}\Big].H_1 - b.C^{(1)}.H_2$$

where

$$\begin{aligned} H_1 =&(\lambda.\eta.X_1).\left(b - G_1^{(1)} - G_2^{(1)}\right), \\ H_2 =&(\lambda.\eta.X_2).\left(b - G_1^{(1)} - G_2^{(1)}\right) \\ &+ (\lambda.\eta.X_1).\left(b.(b-1) - G_1^{(2)} - G_2^{(2)} - 2.G_1^{(1)}.G_2^{(1)}\right) + 2(\lambda.\eta.X_1).\left(b - G_1^{(1)} - G_2^{(1)}\right) \end{aligned}$$

and

$$\begin{aligned} X_1 &= E(X),\ S_1^{(1)} = \lambda.\eta.X_1.E(S_1),\ S_2^{(1)} = \lambda.\eta.X_1.E(S_2), \\ V^{(1)} &= \lambda.\eta.X_1.E(V),\ C^{(1)} = \lambda\eta.X_1.E(C), \\ G_1^{(1)} &= \lambda.\eta.X_1.E(G_1),\ G_2^{(1)} = \lambda.\eta.X_1.E(G_2), \\ S_1^{(2)} &= \lambda.\eta.X_2.E(S_1) + \lambda^2.\eta^2.(E(X))^2.E(S_1^2), \\ S_2^{(2)} &= \lambda.X_2.E(S_2) + \lambda^2.\eta^2.(E(X))^2.E(S_2^2), \\ G_1^{(2)} &= \lambda.\eta.X_2.E(G_1) + \lambda^2.\eta^2.(E(X))^2.E(G_1^2), \\ G_2^{(2)} &= \lambda.X_2.E(G_2) + \lambda^2.\eta^2.(E(X))^2.E(G_2^2), \\ V^{(2)} &= \lambda.\eta.X_2.E(V) + \lambda^2.\eta^2.(E(X))^2.E(V^2), \\ C^{(2)} &= \lambda.\eta.X_2.E(C) + \lambda^2.\eta^2.(E(X))^2.E(C^2). \end{aligned}$$

4.2 Expected waiting time

The expected waiting time is obtained by using Little's formula as:

$$E(W) = \frac{E(Q)}{\lambda E(X)},$$

where $E(Q)$ is given in (67).

4.3 Expected length of busy period

Theorem 2 *Let B be the busy period random variable. Then the expected length of busy period is*

$$E(B) = \frac{E(T)}{d_0}, \tag{68}$$

where

$$E(T) = E(S_1) + E(S_2) + E(G_1) + E(G_2).$$

Proof: Let T be the residence time that the server is rendering single service or bulk service.

$$E(T) = E(S_1) + E(S_2) + E(G_1) + E(G_2).$$

Define a random variable J_1 as

$$J_1 = \begin{cases} 0, & \text{if the server finds no customer after the residence time,} \\ 1, & \text{if the server finds at least one customer after the residence time.} \end{cases}$$

Now the expected length of the busy period is given by

$$\begin{aligned} E(B) &= E(B/J_1 = 0)P(J_1 = 0) + E(B/J_1 = 1)P(J_1 = 1) \\ &= E(T)P(J_1 = 0) + [E(T) + E(B)]\, P(J_1 = 1). \end{aligned}$$

Solving for $E(B)$, we get

$$E(B) = \frac{E(T)}{P(J_1 = 0)} = \frac{E(T)}{d_0}.$$

4.4 Expected length of idle period

Theorem 3 *Let I be the idle period random variable. Then the expected length of idle period is given by*

$$E(I) = E(C) + E(I_1), \tag{69}$$

where

$$E(I_1) = \frac{E(V)}{1 - \gamma_0\delta_0 d_0}, \tag{70}$$

I_1 is the idle period due to multiple vacation process, $E(C)$ is the expected closedown time.

Proof: Define a random variable J_2 as

$$J_2 = \begin{cases} 0, & \text{if the server finds atleast one customer after the first vacation,} \\ 1, & \text{if the server finds no customer after the first vacation.} \end{cases}$$

The expected length of idle period due to multiple vacations $E(I_1)$ is given by

$$\begin{aligned} E(I_1) &= E(I_1/J_2 = 0)P(J_2 = 0) + E(I_1/J_2 = 1)P(J_2 = 1) \\ &= E(V)P(J_2 = 0) + [E(V) + E(I_1)]\, P(J_2 = 1). \end{aligned}$$

On solving, we get

$$E(I_1) = \frac{E(V)}{P(J_2 = 0)} = \frac{E(V)}{1 - P(J_2 = 1)} = \frac{E(V)}{1 - Q_{10}(0)}. \tag{71}$$

From equation (45), we get $Q_{10}(0)$ = coefficient of z^n in $Q_1(z, 0)$

$$\begin{aligned} Q_1(z, 0) =& \tilde{V}(\lambda\eta - \lambda\eta X(z))\tilde{C}(\lambda\eta - \lambda\eta X(z))d_0 \left[\sum_{n=0}^{a-1} g_n z^n\right] \\ =& \sum_{n=0}^{\infty} \gamma_n z^n \sum_{i=0}^{\infty} \delta_i z^i d_0. \end{aligned}$$

Equating the coefficient of z^0 on both sides, we get

$$Q_{10}(0) = \gamma_0\delta_0 d_0.$$

Substitute in (71), we get (70).

5 Cost Model

We derive the expression for finding the total average cost with the following assumptions. C_s - Start up cost, C_v - Reward per unit time due to vacation, C_h - Holding cost per customer, C_o - Operating cost per unit time, C_u - closedown cost per unit time. The length of cycle is the sum of the idle period and busy period. Now the expected length of the cycle $E(T_c)$ is obtained as

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(J_2 = 0)} + E(C) + \frac{E(T)}{d_0}.$$

Total Average Cost = Start-up cost + closedown cost per unit time + holding cost of number of customers in the queue per unit time + operating cost per unit time × ρ - reward due to vacation per unit time.

$$TAC = \left[C_s + C_u.E(c) - C_v.\frac{E(V)}{P(J_2 = 0)}\right].\frac{1}{E(T_c)} + C_h.E(Q) + C_o.\rho,$$

where

$$\rho = \lambda\eta E(X)\left[E(G_1) + E(G_2)\right]/b.$$

6 Numerical illustration

In this section, various performance measures which are computed in earlier sections are verified numerically. Numerical example is analyzed using MATLAB, the zeros of the function $(z^b - \tilde{G}_1(\lambda\eta - \lambda\eta X(z))\tilde{G}_2(\lambda\eta - \lambda\eta X(z)))$ are obtained and simultaneous equations are solved. A numerical example is analyzed with the following assumptions:

1. Batch size distribution of the arrival is Geometric with mean two.
2. Single service time distribution is Exponential and service rate for first phase service is μ_1, second phase service is μ_2.
3. Bulk service time distribution is Erlang - k with $k = 2$ and service rate for first phase service is μ_1^*, second phase service is μ_2^*.
4. Vacation time and closedown time are exponential with parameter $\eta = 9$ and $\beta = 7$ respectively.
5. η=0.8
6. Start-up cost : Rs.3
7. Holding cost per customer: Rs. 0.50
8. Operating cost per unit time: Rs.2
9. Reward per unit time due to vacation: Rs.3
10. Closedown cost per unit time: Rs. 0.25.

Table 1, 2and 3 show the performance of various measures like $E(Q), E(B), E(I)$ and $E(W)$ with the increment of arrival rate λ for the values of $\mu_1 = 4$, $\mu_2 = 4.5, \mu_1^* = 5, \mu_2^* = 5.5$, $\mu_1 = 5$, $\mu_2 = 5.5, \mu_1^* = 6, \mu_2^* = 6.5$ and $\mu_1 = 6$, $\mu_2 = 6.5, \mu_1^* = 7, \mu_2^* = 7.5$ respectively. It is also evident that the average queue length increases as the increase of arrival rate. However, average queue length decreases as the increase of service rate.

λ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
0.5	1.5081	0.3770	0.2186	0.1103
1.0	1.9831	0.4958	0.2278	0.1095
1.5	2.2576	0.5644	0.2321	0.1052
2.0	2.7119	0.6780	0.2377	0.1011
2.5	3.3632	0.8408	0.2413	0.0096
3.0	3.8053	0.9513	0.2496	0.0064

Table 1: Arrival rate vs performance measures $\mu_1 = 4$, $\mu_2 = 4.5, \mu_1^* = 5, \mu_2^* = 5.5$

λ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
0.5	1.3276	0.3319	0.2113	0.1172
1.0	1.6314	0.4079	0.2195	0.1138
1.5	1.9927	0.4982	0.2238	0.1105
2.0	2.3265	0.5816	0.2276	0.1088
2.5	2.8964	0.7241	0.2322	0.1041
3.0	3.1275	0.7819	0.2369	0.1013

Table 2: Arrival rate vs performance measures $\mu_1 = 5$, $\mu_2 = 5.5, \mu_1^* = 6, \mu_2^* = 6.5$

λ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
0.5	1.2084	0.3021	0.2009	0.1198
1.0	1.3175	0.3294	0.2083	0.1153
1.5	1.6321	0.4080	0.2168	0.1121
2.0	1.8673	0.4668	0.2199	0.1100
2.5	2.2016	0.5504	0.2241	0.1075
3.0	2.8897	0.7224	0.2285	0.1057

Table 3: Arrival rate vs performance measures $\mu_1 = 6$, $\mu_2 = 6.5, \mu_1^* = 7, \mu_2^* = 7.5$

7 Conclusion and future work

In this paper, we have derived the PGF of the queue size for batch arrival single and bulk service queue with two phases of service, closedown, multiple vacation and balking under the steady-state case. Also we have obtained various performance measures and verified them numerically. In future this work may be extended into a queueing model with multi stages of service and modified vacation.

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