AMO - Advanced Modeling and Optimization, Volume 20, Number 1, 2018

A contemporary approach for solving a historical problem: Applying Firefly Optimization (FA) algorithms to the Wald Operational Problem.

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ABSTRACT

As a member of the SRG (Statistical Research Group) at Columbia University, the mathematician Abraham Wald worked on the problem of vulnerability estimation in aircraft on combat situations.This problem became known as "operational problem", the approach proposed by Wald was used in different combat situations.This work aimed to revisit the problem and propose an alternative solution using the Swarm Intelligence method known as Firefly Algorithm (FA). As results it was possible to estimate numerically better information and with greater coherence regarding the database used.

Keywords: Wald, Survival, Firefly Algorithm.

1. INTRODUCTION

During World War II, the Mathematician Abraham Wald, at the request of the Allied forces, developed a method to estimate the probability of survival of an aircraft under enemy fire, a problem otherwise known as the "operational problem". Wald's Method was largely used by the American Army during World War II, the Korean and Vietnam War, although there were only a few of researches dedicated to analyzing its efficiency.

According to [1], the Operational Problem could be presented as such: An aircraft, returning to the base from a combat mission, had many of its parts damaged in combat. In order to raise the fighter's chance of survival, the operational commander has to decide: (i) Which tactic must be used in battle and (ii) where to apply extra platings to protect strategic points of the aircraft's hull.

The operational problem, in its entirety, is complex, especially due to the lack of information during the War's time period. The only information available to Wald was in a spreadsheet, consisting of: (i) The total number of planes on flight; (ii) the number of returning aircrafts, which also served to know how many fell in battle, and (iii) the number of shots a surviving aircraft received. Nothing could be known about the missing planes, for they could have been shot down by the enemy or have crashed down due to some sort of mechanical failure, a common fact during the war.

Therefore, this research has as objective the solution of the operational problem, utilizing the Firefly Swarm Method, based on Wald's original approach. Currently, swarming algorithms, as well as evolutionary algorithms and other stochastic methods, have proved very useful in solving a wide array of problems. Therefore, it is considered that firefly swarm algorithms (FA - Firefly Algorithm) stand out in relation to other stochastic optimization techniques. Studies show that FA is efficient in numerical optimization [5 – 7]. The algorithm can locate the global optimal, as well as all the local optimal points, simultaneously and very effectively.

An additional advantage of the FA is that different fireflies work almost independently, therefore being particularly adequate to parallel implementation. It is a method superior to the Genetic Algorithms (GA) and Particle Swarm Optimization (PSO) since the fireflies aggregate themselves closely to each optimal point. Interactions between distinct sub-regions are minimal in parallel implementations. [6,7] Among its many applications, the Firefly Algorithm is used in: Annual planning of harvests [11], parts design problems [10,13] and process management problems.[9] The FA presented good results in all of its applications, thus proving to be a robust and reliable problem-solving technique.

2. THEORY REVIEW

2.1 Wald's approach to the Operational Problem

2.1.1 Defining the survival probability

Initially, Wald considered the following variables to characterize the problem: The total number of aircrafts in flight (N), the number of surviving aircrafts (S), the number of aircrafts shot down by the enemy (L), the number of aircrafts that survived *i* enemy shots (Si), the number of aircrafts that were shot down by *i* shots (Li), and the ratio of surviving planes to the total number of aircrafts, denoted by *s*.

In order to begin the problem modeling, it was necessary to assume the following assumptions: (i)$L0 = 0$, (ii)$L=\sum Li=N-S$ and (iii)$Li+Si=Ni$, where *i* is the number of successful shots in the aircrafts. Given the first assumption, it is considered that all missing planes were shot down, in order to ignore mechanical failures, for example. The goal is to find the probability *pi*, that is the conditional chance of an aircraft crash, after receiving *i* number of shots. This probability was defined by Wald as:

$$p_i = \frac{L_i}{N_i} \tag{01}$$

Therefore,

$$L_i = p_i N_i \tag{02}$$

The following progression was defined,

$$L_0 = 0 \tag{03}$$

$$L_1 = p_1(N - S_0) \tag{04}$$

$$L_2 = p_2(N - S_0 - S_1 - L_1) \tag{05}$$

$$L_3 = p_3(N - S_0 - S_1 - S_2 - L_1 - L_2) \tag{06}$$

$$L_i = p_i\,(N - \sum_{m=0}^{i-1} S_m - \sum_{m=1}^{i-1} L_m) \tag{07}$$

Thus,

$$q_i = 1 - p_i \tag{08}$$

By manipulating algebraically the presented progression, it is possible to arrive at the following expression, also known as Wald Basic Equation:

$$\sum \frac{s_i}{\prod q_i} = 1 - s_0 \tag{09}$$

For more details, see [2]. In this way it is possible to determine the values of qi by solving the following nonlinear programming problem:

Minimize $$\prod q_i \tag{10}$$

Subject to $$\sum \frac{s_i}{\prod q_i} = 1 - s_0 \tag{11}$$

$$0 \leq q_i \leq 1 \tag{12}$$

To Wald, the basic solution would be to consider that all values of qi are equal, therefore all possibilities of p would also be equal. Thus, the problem is summarized in finding the real roots of the following equation:

Applying numerical methods, such as Newton's method, it is possible to define the value of q. The simplification employed is justified by the fact that computational tools were not so developed at the time. However, employing such simplification may lead to erroneous conclusions, which is.

why the Firefly Algorithm (FA) method will be applied to the multinomial optimization problem originally proposed

2.1.2 Subdividing the aircraft into areas of equal vulnerability

Having determined how to estimate the probability of survival, it would be necessary to subdivide the aircraft into areas of equal vulnerability and thereby define the probability of survival of an aircraft being shot i times in j areas, defined here as qi. For Wald, this probability is estimated as follows:

$$q_j = \frac{\eta_j}{\theta_j} q_i \tag{14}$$

Where qi is the chance of survival previously estimated, θj is the ratio of area j by the total area, thus being a parameter dependent on the physical aspect of the aircraft and therefore defined by:

$$\theta_j = \frac{A_j}{A_T} \tag{15}$$

η_j is a parameter from field observations and is related to the pattern observed in the received shots and the total number of shots received by the aircraft, its estimation is not as direct as the parameter θj. Wald observed that the distribution of the shots was characterized by the following equation:

$$h_j = j\eta_j h_t \tag{16}$$

where ht is the total number of shots received by the Si surviving aircrafts and hj is the number of shots observed in j areas, therefore:

$$\eta_j = \frac{h_j}{jh_t} \tag{17}$$

And finally the probability of crashing is defined in a manner analogous to equation (8):

$$q_j = 1 - p_j \tag{18}$$

As noted, the estimation of the probability of survival qi has an important role in the total solution of the problem and therefore its adequate estimation can significantly alter the results of the study. In the next section, the FA will be formally presented.

2.2. Firefly Algorithm (FA)

Swarm Intelligence is a set of techniques based on the collective behavior of self-organized, distributed, autonomous, flexible, and dynamic systems. For [3], the purpose of computational swarm intelligence models is to model the simple behaviors of individuals and the local interactions with the environment and neighboring individuals, in order to obtain more complex behaviors that can be used to solve optimization problems.

These systems are made of a population of simple computational agents capable of perceiving and modifying their environment locally. This capability makes communication possible between agents, who capture environmental changes generated by their counterparts' behaviors. Although there is no centralized control structure that establishes how agents should behave, and even if there is no explicit model of the environment, local interactions between agents usually lead to the emergence of a global behavior approaching the solution of the problem [4].

Given such characteristics, those systems could have been inspired by: (i) The behavior of social insects, such as ants, bees, termites, and wasps; Or (ii) the ability of human societies to process knowledge. The Firefly Algorithms (FA) fit the first case. Summarizing, FA are metaheuristic algorithms inspired by the intermittent behavior of fireflies, social insects widely known for their luminescence and abdominal flashes, which are mostly used for mating.

In an FA, the main purpose of a firefly's flash is to serve as a signaling system to attract other fireflies. Yang [5 – 7] formulated the algorithm assuming that: (i) All fireflies are unisex, meaning that a firefly can be attracted to all the other fireflies in the swarm; (ii) Attractiveness is proportional to the flash's brightness, therefore, between two fireflies, the one with the strongest flash will attract the dimmer one. The perceived brightness of an abdominal flash is diminished in proportion to distance, though, and if no other is brighter than a certain firefly, this agent will move randomly; (iii) The firefly's brightness is associated with the objective function.

Since that, for those agents, attractiveness is proportional to luminous intensity, a firefly's attractiveness is defined in terms of the cartesian distance between a firefly j and a firefly k. In this case, the motion of the firefly k attracted by a brighter firefly j is determined by:

$$x_j = x_j + \beta_0 e^{-\gamma r_{jk}^2}\left(x_k - x_j\right) + \alpha(rand - \frac{1}{2}) \tag{19}$$

where xj and xk are the positions of fireflies j and k, respectively, r is the Euclidean distance between them, β_0 is the attractiveness among the agents when r equals 0, γ characterizes the variation of attractiveness, α is a random, binary parameter and rand is a random number generated by a uniform distribution, ranging from 0 to 1.

It can be observed that, when $\gamma = 0$, the attractiveness will not be dependent of distance, making all fireflies flock to the global optimal point, a behavior usually seen in the Particle Swarm Optimization Algorithm (PSO). On the other hand, if $\gamma \rightarrow \infty$, the agents' actions become purely random, characterizing a Random Search Algorithm. Therefore, the Firefly Algorithm will present a behavior between those two extremes, depending of the value of γ, this being a critical parameter for the method's good performance. The FA implementation procedure can be summarized as the pseudocode shown in figure 1:

1) Define algorithm parameters, including the γ factor;

2) Generate an initial population of m fireflies xj (j=1, 2, ... , m), using uniform distribution in the search area;

3) Evaluate the objective function F for all fireflies;

4) Light intensity Ij in firefly xj is determined by F;

5) Initialize generation count by setting *Generation* = 0;

6) **While** *Generation* < *Max Generation*

```
        Generation = Generation + 1
        For j = 1, 2, ..., m
                For k = 1, 2, ... , m
                If (Ik > Ij), Move firefly j to proximity k according to (19)
                Evaluate new solution and update I
                End
        End
End
```

Figure 1. Pseudo code for implementation of the FA. **Source**: Adapted from [7].

3. RESULTS

3.1. Materials and Methods

Algorithms and analyses were implemented on MATLAB version R2014.a. For the development of this study, the following database was considered:

Table 1: Database used in the study, adapted from [1].

	Number	**Ratio**
Aircrafts in Flight	$N = 400$	1
Returning Aircrafts	$S = 380$	0.95
Missing Aircrafts	$L = 20$	0.05
Undamaged Aircrafts	$S_0 = 320$	$s_0 = 0.80$
Aircrafts that Survived 1 Shot	$S_1 = 32$	$s_1 = 0.08$
Aircrafts that Survived 2 Shot	$S_2 = 20$	$s_2 = 0.05$
Aircrafts that Survived 3 Shot	$S_3 = 4$	$s_3 = 0.01$
Aircrafts that Survived 4 Shot	$S_4 = 2$	$s_4 = 0.005$
Aircrafts that Survived 5 Shot	$S_5 = 2$	$s_5 = 0.005$

Wald's original study was implemented with the planes available at that time, which had its weak spots subdivided in 4 regions, as shown in Table 2:

Table 2: Database used in the study (cont.), adapted from [1].

j	**Área (square foot)**	$\boldsymbol{\theta_j}$	$\boldsymbol{h_j}$	$\boldsymbol{\eta_j}$
1 – Motors	35	0.269	20	0.193
2 – Fuselage	45	0.346	39	0.386
3 – Fuel System	20	0.154	16	0.154
4 – Other parts	30	0.231	27	0.267
Total	130	1	102	–

Given the information on Table 1 and applying the expression (10), the objective function *F*, minimized, is defined as:

$$F(q_1, q_2, q_3, q_4, q_5,) = \prod_{i=1}^{5} q_i + \kappa \tag{20}$$

Where *k* is the penalty parameter used to treat the constraint function *R*, defined by applying Table 1 data into expression (11), therefore:

$$R(q_1, q_2, q_3, q_4, q_5,) = \frac{0{,}08}{q_1} + \frac{0{,}05}{q_1 q_2} + \frac{0{,}01}{q_1 q_2 q_3} + \frac{0{,}005}{q_1 q_2 q_3 q_4} + \frac{0{,}005}{q_1 q_2 q_3 q_4 q_5} \tag{21}$$

The Value of *k* is defined as:

$$\kappa = \begin{cases} 0;\ if\ 0.19 < R < 0.21 \\ 10^3; otherwise \end{cases} \tag{22}$$

The purpose of this approach is to eliminate the equality constraint from the original formulation. The 0.19<R<0.21 condition is due to the fact that, for the case in study, 1− s_0= 0.2.

To implement the FA, the following parameters were considered: (i) *Generations = 300;* (ii) Number of fireflies *(m=30);* (iii) β_0 = γ = 1 and (iv) α = 0.5. A total of 30 experiments were performed. The averages, minimum and maximum values of the best *F* values found for each experiment were calculated, as well as all the *q* values found. In this research, it was also considered that:

$$\frac{\eta_j}{\theta_j} \sim N(\mu, \sigma) \tag{23}$$

in order to calculate *qj*, the following approximation was defined:

$$q_j = Z\left(\frac{\eta_j}{\theta_j}\right) q_i \tag{24}$$

Where *Z* is the table value of the Gaussian distribution for the values estimated by the ratio originally proposed by Wald. Thus, the following values are defined for the ratios:

Table 3: Z values for the ratio $\frac{\eta_j}{\theta_j}$.

j	$\frac{\eta_j}{\theta_j}$	**Z**
1	0.717472	0.764238
2	1.115607	0.868643
3	1	0.84135
4	1.155844	0.874928

3.2. Results and Discussion

After 30 experiments, the following results for *F* and *qi* were found:

Table 4: qi estimated.

	Minimum	**Mean**	**Maximum**	**Best**
F	0.091725966	0.099735070	0.135162729	0.091725966
q_1	0.999985151	0.999999365	1	1
q_2	0.978148896	0.998956622	1	1
q_3	0.678725257	0.920668095	0.999845780	0.995287776
q_4	0.228620686	0.723091596	0.997379809	0.923692426
q_5	0.099773737	0.175586004	0.612209372	0.099773737

Calculating the values of *pi,* according to the Expression (8), the following results could be reached:

Table 5: pi estimated.

	Minimum	**Mean**	**Maximum**	**Best**
p_1	0.000014849	0.0000006	0	0
p_2	0.021851104	0.0010434	0	0
p_3	0.321274743	0.0793319	0.000154220	0.004712224
p_4	0.771379314	0.2769084	0.002620191	0.076307574
p_5	0.900226263	0.8244140	0.387790628	0.900226263

As observed, the probability of an aircraft crash after one or two shots is minimal. The variability of responses increases considerably with 3 shots, although being relatively low (not so much in comparison to the earlier cases, but still low.)

With 4 shots, the situation presents great uncertainty, since the results show the greatest variability among the others. Either more information is needed or implementing fine attunements to the algorithm could reduce the variability observed in this case. Finally, in the case of 5 shots, there was some variability in the estimates, however, a consensus can be reached that this is the case with the highest chance of slaughter among the available data set. According to the results, it can be inferred that most of the *L* felled aircrafts suffered five or more shots.

Applying the progression (7) to the best-estimated answer and considering only integer values, the following table can be assembled:

Table 6: Estimated L based on the progression (7).

L = 20	**Estimation**
L_1	0
L_2	0
L_3	0
L_4	2
L_5	18

As can be seen, the highest estimated number of airplanes shot down occurs when they suffer five or more shots. Ergo, it is possible to define the risk zone for crashing as between 5 or more shots. It is worth noting that there is still a chance of an aircraft surviving after receiving 5 shots, albeit it is very small.

Comparatively, it is reasonable to apply Wald's approach in this data set. Initially, the following equation must be solved:

$$\frac{0.08}{q}+\frac{0.05}{q^2}+\frac{0.01}{q^3}+\frac{0.005}{q^4}+\frac{0.005}{q^5}=0.20 \quad (23)$$

Using a numerical method, such as Newton's Method, the value of q is found as $q = 0.851$, and, therefore, $p = 0.149$. Applying those results in Progression (7), while considering integer numbers only, the following table was assembled,

Table 7: Estimated L based on the progression (7) with Wald's original approach.

L = 20	**Estimation**
L_1	12
L_2	5
L_3	2
L_4	1
L_5	0

Based on the simplified approach, it is verified that the risk zone for crashing occurs when receiving a shot. Therefore, it is concluded that the largest portion of aircrafts crashed due to a single, fatal shot. By numerically evaluating the two results, it becomes clear that both meet the restriction $R = 0,20.$ However, with different F values, Wald's original approach reached $F = 0,446$, while the optimal estimated answer was $F = 0,091.$ Thus, it is pertinent to state that the results obtained by the method proposed by this study are more numerically reliable than the former.

Data analysis clearly shows that there are considerably more aircrafts surviving a single shot than those that survived 5 shots, a fact in consonance with data observed in the battlefield. Evaluating the random variable p estimated by both methods, it is seen that, in Wald's approach, considering the same value of q in all cases, the probability of survival has, obviously, a constant behavior, therefore characterizing a uniform, discrete distribution. In contrast pi values, estimated by the FA method, clearly showing a behavior closer to that of an exponential distribution.

Wald's original approach, despite having a lower numerical accuracy, is consistent with its assumption that q has the same value for all cases. In other words, what matters to this method is that at the moment a given aircraft is shot, on average, it has a survival chance of 0.851, a result quite close to the FA's estimations, calculated as 0.803. For decision making analyses and total solution of the Operational Problem, though, the information estimated by the FA approach is far more relevant, due to its numerical reliability and practical coherence.

Carrying on, with new information being estimated, it will be possible to continue the problem's resolution by defining where a certain aircraft should receive shielding reinforcement. Applying the data present in Tables 2 and 3, together, into Equations (14) and (20), with the estimated values of *qi*, it was possible to estimate the values of *pj.*

Table 8: pj estimated.

	1 Shot	**2 Shots**	**3 Shots**	**4 Shots**	**5 Shots**
Area 1	0.235762	0.235762	0.239363	0.294079	0.923749
Area 2	0.131357	0.131357	0.13545	0.197641	0.913332
Area 3	0.15865	0.15865	0.162615	0.222851	0.916055
Area 4	0.125072	0.125072	0.129195	0.191836	0.912705

Analyzing Table 8's content, it is safe to assume that the most vulnerable weak point will always be area 1, where the engines are located, followed by area 3, where the fuel system lies. Ergo, the shielding should be strengthened around the engines and, if possible, the fuel systems.

The results are in agreement with the following reasoning: "the region that must be strengthened is precisely the one that suffered minimal damage to surviving airplanes." Observing Table 2, it is very noticeable that the areas that suffered the least damage in the studied aircraft consist exactly in the areas considered of greater risk by the estimates made.

Table 8: pj estimated with Wald’s original approach.

j	**pj**
1	0.349633
2	0.260785
3	0.284011
4	0.255436

Wald's approach, applied in this data set, reaches similar results, although not with the same information depth. As Table 8 shows, it can be stated without a shadow of a doubt that, most of the time and independent of how many shots were received, area 1 will be a permanent source of risk, while area 3 becomes one only after being shot 3 times. The other areas have minimal probabilities of survival after receiving 5 shots.

Reviewing the results estimated by the Firefly Algorithm, it is observed a need for the analysis to be performed horizontally, evaluating the probability of each number of shots fired in a determined region. Therefore, in order to choose the area to be fortified, the evolution of the crashing chance over time should be used as a criterion, not just its isolated value. The reason being, according to Table 5, most of the destroyed planes received 5 or more shots and the probability of the aircraft crashing, after receiving 5 shots or more, in all vulnerable regions, is very close. In other words, the probability value by itself is not a sufficient criterion to make a decision.

4. CONCLUSION

The operational problem aims to define the best strategy to increase the survival rate of a given aircraft in a scenario of war. The objective of this work was to solve a part of this problem by means of a contemporary swarm intelligence technique called Firefly Swarm Algorithm (FA). The problem under study was presented, partially demonstrating the origin of the functions and equations used, as well as the original approach used to solve part of the problem during World War II. Concepts related to the FA method and Swarm Intelligence were also discussed.

With a field data set, survival estimates were calculated using the FA method along with the original approach, proving the efficiency of the former in estimating results with greater numerical accuracy and consistency in comparison to the original.

It is important to note that Wald's method was proven as able to estimate areas of risk that should be strengthened. When using the FA approach to the original multinomial problem, however, certain flaws were identified, mainly due to the ratio $\frac{\eta_j}{\theta_j}$. Firstly, this ratio was applied directly, as presented in Equation (14), but the approach ends up generating several numerical errors in the probabilities computation. To solve this problem, the ratio needed to be considered as it truly is: a random variable that could be approximated by a Gaussian distribution. Such an approach does not alter the calculations deeply, however it gives a greater numerical reliability to the achieved results.

For future research, it is proposed: (i) To continue the resolution of the Original Problem, now taking into account the type of anti-aircraft artillery used at the time; (ii) To make comparisons with other methods of Swarm Intelligence or Evolutionary Algorithms; And (iii) to use Fuzzy Logic in order to solve the problem of estimating survival probabilities.

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