AMO - Advanced Modeling and Optimization, Volume 20, Number 1, 2018

NUMERICAL METHOD FOR THE NONLINEAR VOLTERRA INTEGRAL EQUATIONS USING SIMPSON PRODUCT INTEGRATION METHOD

M. JALALVAND AND M. NABTI

ABSTRACT. In this paper we introduce and examine a numerical method for solving the Volterra integral equation of the second kind when the kernel function contains a mild singularity. This method represents the solution of proposed integral equations as a series generated by the Adomian decomposition method and coefficients are evaluated by the Simpson product integration technique. We validate the proposed method using some examples and compare numerical and analytical results to show the method's accuracy.

Key Words: Nonlinear Integral Equation; Weakly Singular Volterra; product integration method.
Mathematics Subject Classification: 49Q10, 49Q20, 49J24

1. Introduction

The mathematical model a lot of physical and chemical phenomenon such as soteriology, heat transfer, superfluidity, the reaction of heat from a semi infinite solid may be derived as nonlinear weakly singular Volterra integral equations.
In this paper we consider nonlinear Volterra integral equations of the second kind,

$$u(t) = f(t) + \int_0^t p(t,s)K(t,s,u(s))ds, \qquad s \in [0,1], \tag{1.1}$$

where the kernel p is weakly singular and the given functions f and K are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution u in $C[0,1]$ (see, for instance, [5], [7], [9], [10]). Typical forms of $p(t,s)$ are

$$\begin{array}{rcl} p(t,s) & = & (t-s)^{-\alpha},\ 0<\alpha<1, \\ or & & \\ p(t,s) & = & log(t-s). \end{array} \tag{1.2}$$

For regular Volterra integral equations the smoothness of the kernel and of the forcing function$f(t)$ determines the smoothness of the solution on the closed interval $[0,1]$. Whereas if we allow weakly singular kernels, then the resulting solutions are typically nonsmooth at the initial point of the interval. Some results concerning the behavior of the uniqe solutions of equations of type (1.1) are given in [9]. Note that the numerical solvability of weakly singular Volterra integral equations have been investigated, see for example [8,14,17,18,20,21].
In recent years the applications of the Adomian decomposition method (ADM) in

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311.

mathematical problems has been developed by scientists. This method continuously transforms a complicated problem into a sequence of simpler problems which can be easily solved. The ADM solves successfully different types of linear and nonlinear equations in deterministic and stochastic fields [3,4]. Application of ADM for solving different types of integral equations has been discussed by many authors [6,16,22]. The objective of the present paper is to approximate the solution of equation (1.1) using a new strategy of product integration, in conjunction with ADM. This paper is organized as follows. In section 2, some basic concepts of Adomian decomposition method (ADM) are presented. In section 3, we describe an algorithm based on product integration method and ADM for numerical solution of the nonlinear weakly singular Volterra integral equation (1.1). Section 4 is devoted to the numerical examples selected from the literature in connection with Volterra integral equations.

2. Adomian decomposition method

G. Adomian in [3], proposed a new and fruitful method for solving exactly nonlinear functional equations of various kinds (algebraic, differential, partial differential, integral, etc). Using this technique the solution of the nonlinear operator is presented as a series of functions. Each term of this series is a generalized polynomial called Adomian's polynomial. The Adomian technique is very simple in its principles. The difficulties consist in calculating the Adomian's polynomials and in proving the convergence of the introduced series. Some attempts to prove convergence have been made in [1,12,13,15]. The main algorithm of Adomian's decomposition method applies to a general nonlinear equation of the form

$$u = Nu + f \tag{2.1}$$

where N is a nonlinear operator from a Hilbert space H into H, and f is a known function (see, [3,12,13]). We are looking for a solution u of (2.1) belonging to H. We shall suppose that (2.1) admits a unique solution. The decomposition method consists in looking for a solution having the series form

$$u = \sum_{i=0}^{\infty} u_i. \tag{2.2}$$

The nonlinear operator N is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where the A_n are functions called Adomian's polynomials. We remark that the A_n are formally obtained from the relationship

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}. \tag{2.4}$$

For more details see [2,3,12,13]. Adomian polynomials are defined as

$$
\begin{aligned}
A_0 &= N(u_0),\\
A_1 &= u_1 N'(u_0),\\
A_2 &= u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0),\\
A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!}u_1^3 N''(u_0),\\
&\vdots
\end{aligned}
\tag{2.5}
$$

We assume that N satisfies certain conditions so that equations (2.4)−(2.5) are well-defined and the corresponding series (2.2) and (2.3) are convergent, see [1](Theorem 3.1). These definitions are only formal, and nothing is proved or supposed about the convergence of the series $\sum u_i$ and $\sum A_n$. Putting (2.2) and (2.3) into (2.1) leads to the relationship

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} A_i + f \tag{2.6}$$

and Adomian's method consists of identifying the u by means of the formulae

$$
\begin{aligned}
u_0 &= f,\\
u_1 &= A_0,\\
u_2 &= A_1,\\
&\vdots\\
u_n &= A_{n-1},\\
&\vdots
\end{aligned}
\tag{2.7}
$$

3. Description of numerical procedure

In this section we introduce a three step algorithm based on the ADM and modified product integration techniques to solve equation (1.1).

Step 1. Basic idea of ADM :

In the ADM, the solution $u(t)$ of (1.1) is given by the series (2.2)

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{3.1}$$

and the nonlinear term is decomposed as

$$K(t,s,u(s)) = \sum_{n=0}^{\infty} A_n(t,s,u_0,u_1,...,u_n), \tag{3.2}$$

Using ADM yields

$$
\begin{aligned}
u_0(t) &= f(t)\\
u_{n+1}(t) &= \int_0^t p(t,s)A_n(t,s)dt,\ n \geq 0.
\end{aligned}
\tag{3.3}
$$

Step 2 . Discretization of problem :

According to the ADM, the solution of equation (1.1) may be derived using the series introduced as (3.1).

Many authors used the zeroes of Chebyshev and Legendre orthogonal polynomials as collocation points. Here we discretize equation (1.2) at the collocation nodes $\{z_{Ni}\}_{i=1}^{N} \bigcup \{1\}$, which yields using orthogonal Chelyshkov polynomials $P_{N,0}(t)$ on $[0,1]$ with the weight function 1, (see,[11]). These polynomials are defined as follows

$$P_{N,k}(t) = \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \binom{N+k+1+j}{N-k} t^{k+j}, \; k = 0, 1, ..., N. \tag{3.4}$$

The polynomials $P_{N,k}(t)$ have properties which are analogous to the properties of the classical orthogonal polynomials. In the family of orthogonal polynomials $\{P_{N,k}(t)\}_{k=0}^{N}$ every member has degree N with $N-k$ simple roots. Hence for every N, polynomial $P_{N,0}(t)$ has exactly N simple roots in $(0,1)$.
Using a quadrature which is based on $N+1$ nodal points $\{z_{Ni}\}_{i=1}^{N} \bigcup \{1\}$ and selecting collocation points to be the same as nodal points, we have

$$u_{n+1}(t_i) = \int_0^{t_i} p(t_i,s) A_n(t_i,s) ds = \sum_{k=0}^{i-2} \int_{t_k}^{t_{k+2}} p(t_i,s) A_n(t_i,s) ds, \; i = 0,1,...,N, t_0 = 0,$$

where $\{t_i\}_{i=1}^{N}$ are the roots of N^{th} degree polynomials $P_{N,0}(t)$.
Step 3 . Product integration techniques :
To construct higher order methods directly, it is necessary to use more accurate numerical integration rules. The next step is the product simpson method which is constructed by approximating $A_n(t_i,s)$ by piecewise linear functions (for more details see Linz, [18]), in particular

$$\begin{aligned} A_n(t_i,s) &= \frac{(s-t_{k+1})(s-t_{k+2})}{(t_k-t_{k+1})(t_k-t_{k+2})} A_n(t_i,t_k) + \frac{(s-t_k)(s-t_{k+2})}{(t_{k+1}-t_k)(t_{k+1}-t_{k+2})} A_n(t_i,t_{k+1}) \\ &+ \frac{(s-t_k)(s-t_{k+1})}{(t_{k+2}-t_k)(t_{k+2}-t_{k+1})} A_n(t_i,t_{k+2}), \; t_k \le t \le t_{k+2} \end{aligned} \tag{3.5}$$

This leads to the integration formula

$$\int_0^{t_i} p(t_i,s) A_n(t_i,s) ds \simeq \sum_{k=0}^{i-2} (a_{i,k+1} A_n(t_i,t_k) + b_{i,k} A_n(t_i,t_{k+1}) + c_{i,k} A_n(t_i,t_{k+2})) \tag{3.6}$$

where

$$a_{i,k} = \frac{1}{(t_k-t_{k+1})(t_k-t_{k+2})} \int_{t_k}^{t_{k+2}} (s-t_{k+1})(s-t_{k+2}) p(t_i,s) ds \tag{3.7}$$

$$b_{i,k} = \frac{1}{(t_{k+1}-t_k)(t_{k+1}-t_{k+2})} \int_{t_k}^{t_{k+2}} (s-t_k)(s-t_{k+2}) p(t_i,s) ds \tag{3.8}$$

$$c_{i,k} = \frac{1}{(t_{k+2}-t_k)(t_{k+2}-t_{k+1})} \int_{t_k}^{t_{k+2}} (s-t_k)(s-t_{k+1}) p(t_i,s) ds \tag{3.9}$$

The numerical method for solving (1.1) is then

$$u_{n+1}(t_i) = \sum_{k=0}^{i-2} (a_{i,k+1} A_n(t_i,t_k) + b_{i,k} A_n(t_i,t_{k+1}) + c_{i,k} A_n(t_i,t_{k+2})), \tag{3.10}$$

Hence we obtain the following numerical values by ADM as

$$\begin{aligned} u_0(t_i) = u_{0,i} &= f(t_i),\ i = 1, ..., N, \\ u_1(t_i) \cong u_{1,i} &= \sum_{k=0}^{i-2} (a_{i,k+1} A_0(t_i, t_k) + b_{i,k} A_0(t_i, t_{k+1}) + c_{i,k} A_0(t_i, t_{k+2})),\ i = 1, ..., N, \\ &\vdots \\ u_{n+1}(t_i) \cong u_{n,i} &= \sum_{k=0}^{i-2} (a_{i,k+1} A_n(t_i, t_k) + b_{i,k} A_n(t_i, t_{k+1}) + c_{i,k} A_n(t_i, t_{k+2})),\ i = 1, ..., N. \end{aligned} \tag{3.11}$$

Therefore the approximation of $u(t_i)$ may be obtained using the M-term partial sum of the Adomian decomposition series solution as follows

$$u(t_i) \simeq u_{0,i} + u_{1,i} + ... + u_{n,i},\ n = 1, ..., M. \tag{3.12}$$

4. Numerical examples

We evaluate the efficiency of our method using some examples by comparing the numerical results with the analytical solution of the problem. Our benchmark for accuracy is the error given by

$$E^N = \|u - \hat{u}\| = \max_{0 \leq i \leq N} |u_i - \hat{u}_i| \tag{4.1}$$

where u_i denotes the exact solution and $\hat{u}_i$ denotes the approximate solution at the nodes $t_i,\ i = 0, 1, 2, ..., N$. We note that this error formula represents a reasonable measure of the accuracy. To examine the accuracy of the algorithm proposed in previous section, in the following examples, different degrees of Chelyshkov polynomials $\{P_{N0}(t)\}$ are considered ($N = 8, 16, 32, 64$). In addition in our computations, we consider a fixed $M = 15$. We evaluate the efficiency of our method using some examples by comparing the numerical results with the analytical solution of the problem.

***Example* 1.** Consider the following integral equation

$$u(t) = \frac{\sqrt[3]{t}}{15}(t - 15) + \int_0^t \frac{y^3(s)}{\sqrt[3]{t - s}} ds.$$

One may see that $u(t) = \sqrt[3]{t}$ is the solution of this equation. Table 1. shows the errors for different values of N.

N	E^N
8	$1.54E-2$
16	$5.32E-3$
32	$1.54E-4$
64	$5.32E-5$

Table 1. Results of Example 1.

***Example* 2.** Consider the following integral equation

$$u(t) = \sqrt{t} - \frac{1}{2} t^2 lnt + \frac{3}{4} t^2 + \int_0^t ln(t - s) u^2(s) ds.$$

One can obtain $u(x) = \sqrt{t}$ as the solution of this equation. Table 2. shows the error between exact and approximate solutions at the nodes $t_i,\ i = 1, \cdots, N$ for different values of N.

N	E^N
8	$1.3E-3$
16	$2.6E-4$
32	$1.1E-5$
64	$9.1E-6$

Table 2. Results of Example 2.

Example 2 is solved in [20] with variable transformation methods in combination with the trapezoidal quadrature rule and the absolute error between the exact and the approximate solution evaluated at the mesh points is presented. In comparisons with this method, our proposed method is very simple and the accuracy of the numerical results obtained with this method is considerable.

***Example* 3.** As the final example consider the following nonlinear integral equation

$$u(t) = t + \frac{11}{15}t^3 - \frac{1}{3}t^3 lnt + \int_0^t ln(t-s)u^2(s)ds.$$

One may show that the function $u(t) = t$ is the solution of this equation.

N	E^N
8	$1.44E-6$
16	$1.34E-7$
32	$3.50E-8$
64	$7.45E-10$

Table 3. Results of Example 3.

Table 3. shows the error approximation solutions at the nodes $t_i,\ i = 1, \cdots, N$ for different values of N. Khater et al. [17] have been solved the example 3, by Chebyshev polynomials expansion. Comparing results reported in [17] show that for $N = 128$ and 64 issued maximum errors of this problem are $O(10^{-6})$ and $O(10^{-9})$, respectively. Looking at Table 3, we can observe an improvement of the accuracy for $N = 64$ in the case of our algorithm respect to methods in [17].

5. Conclusion

In this work, a class of nonlinear weakly singular Volterra integral equations of the second kind is investigated by using an algorithm based on Adomian decomposition method and product integration approaches. The method of product integration is constructed with respect to a new family of orthogonal polynomials, named Chelyshcov polynomials. The new orthogonal polynomials keep distinctively of the classical orthogonal polynomials and give more accurate quadratures and hence better results.

References

[1] K. Abbaoui and Y. Cherruault, Convergence of Adomian's method applied to differential equations, Mathl. Comput. Modelling 28 (5), 103-110.

[2] G. Adomian, A review of decomposition method and some recent results for nonlinear equations, Mathl. Comput. Modelling 13 (7), (1990) 17-43 .

[3] G. Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer, Dordrech, Holland, (1989).
[4] G. Adomian, Solving frontier problems modelled by nonlinear partial differential equations, Comput. Math. Anal.(1991) 91-94.
[5] K. E. Atkinson, An existence theorem for Abel integral equations, SIAM J. Math. Anal. 5 (1974) 729-736.
[6] E. Babolian, A. Davari, Numerical implementation of Adomian decomposition method for linear Volterra integral equations of the second kind, Appl. Math. Comput. (2005) 223-227.
[7] C. T. H. Baker, The numerical treatment of integral equations, Clarendon Press, Oxford,1977.
[8] P.Baratella, A.P. Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, J. Comput. Appl. Math, (2003) 401-418.
[9] H. Brunner and P. J. van der Houwen, The numerical solution of Volterra equations, North- Holland, Amsterdam, 1986.
[10] B. Cahlon, Numerical solution of non-linear Volterra integral equations, J. Comput.Appl.Math. 7 (1981) 121-128.
[11] V.S. Chelyshkov, Alternative orthogonal polynomials and quadratures Electron, Trans. Numer. Anal. (2006) 17-26.
[12] Y. Cherruault, Convergence of Adomian's method, Kybernetes 18 (2),(1989) 31-38.
[13] Y. Cherruault, G. Saccomandi and B. Som6, New results for convergence of Adomian's method applied to integral equations, Mathl. Comput. Modelling 16 (2), (1992) 85-93.
[14] A. Galperin, E.J. Kansa, Application of global optimazation and radial basis functions to numerical solution of weakly singular Volterra equations. J.Comp. Appl. Math., (2002) 491-499.
[15] S. Guellal and Y. Cherruault, Practical formulae for calculation of Adomian's polynomials and application to the convergence of the decomposition method, Int. J. of Biomedical Comp. 36 (1994) 223-228.
[16] Sh. Javadi, A. Davari, E. Babolian, Numerical implementation of the Adomian decomposition method for nonlinear Volterra integral equations of the second kind, Int. J. Comput. Math. (2007) 75-79.
[17] A.H. Khater, A.B. Shamardan, D.K. Callebaut, M.R.A. Sakran, Solving integral equations with logarithmic kernels by Chebyshev polynomials. Numer. Algorithms 47, (2008) 81-93.
[18] P. Linz, Analytical and Numerical Methods for Volterra Equations, SIAM, Philadelphia 1985.
[19] M. Rasty, M. Hadizadeh, A Product Integration Approach Based on New Orthogonal Polynomials for NonlinearWeakly Singular Integral Equations,Acta Appl Math, (2008).
[20] L. Tao, H. Yong, Extrapolation method for solving weakly singular nonlinear Volterra integral equations of second kind. J. Math. Anal. Appl. 324, (2006) 225-237.
[21] A.M.Wazwaz, A First Course In Integral Equations, World Scientific Publishing Company, New jersey, 1997.

M. Jalalvand
Department of Mathematics,, Faculty of Mathematical Sciences & Computer,, Shahid Chamran University of Ahvaz, Ahvaz, Iran
E-mail address: m.jalalvand@scu.ac.ir

M. Nabati
Department of Basic Sciences,, Abadan Faculty of Petroleum Engineering,, Petroleum University of Technology, Abadan, Iran,
E-mail address: nabati@put.ac.ir