AMO - Advanced Modeling and Optimization, Volume 20, Number 1, 2018

A modified measure theoretical approach for solving the optimal control problems

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Abstract

In this paper, in order to overcome the drawbacks of the Rubio's measure theoretical approach, the last step of approximation in his approach is omitted and the optimal control problems are approximated by the nonlinear programming (NLP) problems. The usefulness of the approach is confirmed by applying it on two examples.

Keywords. Measure theory, Nonlinear programming, CML model

1 Introduction

The measure theoretical-based approach for approximating an optimal control problem by a linear programming (LP) one, which has been theoretically established by Rubio [4], has been applied to optimal control of the lumped and distributed parameter systems; see [5], [3] and references therein. Nevertheless, it has some drawbacks. One of them, is the use of sequential approximations on an optimal control problem which dramatically affects the precision of the method. Moreover, there are difficulties in solving the high dimensional LP problems approximating the large scale optimal control problems. The authors in [1] have used the metaheurestic algorithms to solve the NLP problem approximation of the classic optimal control problems obtained by the Rubio's measure theoretical-based approach. Motivated by their work and in order to overcome the mentioned drawbacks, in the next section we briefly review the measure theoretical approach to approximate the optimal control problems with the NLP ones and we give a system of nonlinear equations as the necessary optimality conditions which can be easily solved by using the optimization softwares such as the MATLAB optimization toolbox. The efficiency of the method is demonstrated through the two numerical experiments, in section 3. The last section is the conclusion.

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2 The NLP approximation of optimal control problems

Let $J=[a,b]$ and $\mathcal{A}$ and $\mathcal{U}$ are, respectively, compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ and $f:\Omega\to\mathbb{R}^n$ is a continuous function, where $\Omega=J\times\mathcal{A}\times\mathcal{U}$. If $x:J\to\mathcal{A}$ is a state function and $u:J\to\mathcal{U}$ is a control function satisfying the control system

$$\dot{x}(t)=f(t,x(t),u(t)), \tag{1}$$

$$x(a)=x_a, x(b)=x_b, \tag{2}$$

then, the pair $p=(x,u)$ is called admissible. Let $g:\Omega\to\mathbb{R}$ is a continuous function. Consider an optimal control problem (OCP), minimizing the functional

$$\mathcal{J}(p)=\int_a^b g(t,x(t),u(t))dt, \tag{3}$$

over $\mathcal{P}$, the set of all admissible pairs. Denote by $C'(B)$ the set of all real valued continuously differentiable functions on B, an open ball in $\mathbb{R}^{n+1}$ containing $J\times\mathcal{A}$. From (1) and (2), an admissible pair $p=(x,u)\in\mathcal{P}$ satisfies

$$\int_a^b \phi^f(t,x(t),u(t))dt=\Delta\phi, \quad \text{for all } \phi\in C'(B), \tag{4}$$

where $\phi^f(t,x,u)=\frac{\partial\phi}{\partial x}(t,x).f(t,x,u)+\frac{\partial\phi}{\partial t}(t,x)$ and $\Delta\phi=\phi(b,x(b))-\phi(a,x(a))$. Therefore, we minimize the functional (3) subject to (4), as the first step of approximating the OCP. According to the Riesz representation theorem, there exists a unique positive Radon measure μ on Ω such that $\int_a^b F(t,x(t),u(t))dt=\int_\Omega F(t,x,u)d\mu\equiv\mu(F)$, $F\in C(\Omega)$, the space of all continuous real valued functions on Ω. Hence, we make the second approximation on the OCP as

$$\underset{\mu\in M^+(\Omega)}{Minimize} \quad \mu(g) \tag{5}$$

$$\text{s.t.} \quad \mu(\phi^f)=\Delta\phi, \quad \text{for all } \phi\in C'(B), \tag{6}$$

where $M^+(\Omega)$ is the set of all positive Radon measure μ on Ω . Note that the problem (5)-(6) is an infinite dimensional linear programming that the existence of it's optimal solution is guaranteed when $M^+(\Omega)$ is equipped with the weak*-topology [4]. Now, as the third step of approximating the OCP, we consider a finite dimensional problem as

$$\underset{\mu\in M^+(\Omega)}{Minimize} \quad \mu(g) \tag{7}$$

$$\text{s.t.} \quad \mu(\phi_j^f)=\Delta\phi_j, \quad j=1,2,...,M, \tag{8}$$

where M is a sufficiently large number and ϕ_j's are chosen from a countable dense subset of $C'(B)$. As detailed in [4], the optimal solution of the problem

(7)-(8), say μ_M^*, tends to the optimal solution of the problem (5)-(6), as M tends to infinity. Moreover, it is known that $\mu_M^* = \sum_{k=1}^{M} \alpha_k^* \delta(z_k^*)$ with $\alpha_k^* \geq 0$ and $z_k^* \in \Omega$, where $\delta(z)$ is the unitary atomic measure characterized by $\delta(z)(F) = F(z)$, for all $F \in C(\Omega)$ and $z \in \Omega$. Therefore, the problem (7)-(8) is equivalent to

$$\underset{\alpha_j \geq 0,\ z_k \in \Omega}{Minimize} \qquad \sum_{k=1}^{M} \alpha_k g(z_k) \tag{9}$$

$$\text{s.t.} \qquad \sum_{k=1}^{M} \alpha_k \phi_j^f(z_k) = \Delta\phi_j, \quad j = 1, 2, ..., M. \tag{10}$$

Rubio approximated the NLP problem (9)-(10) with an LP problem by replacing the decision variables z_ks with the points of a grid on the set Ω, which also contains another step of approximation as well. Moreover, the numerical experiments show that the method is highly sensitive to choosing the grid points. On the other hand, if Ω is a large set in sense of measure then we have a huge number of grid points which requires enormous computational capacity. In order to overcome these difficulties, we directly focus on solving the NLP problem (9)-(10); since, nowadays there are optimization software packages to solve the NLP problems. For computational simplicity, we introduce the new decision variables β_j and w_j related to α_j and z_j through $\alpha_j = \beta_j^2$ and $z_j = \frac{M+m}{2} + \frac{M-m}{2} sin(w_j)$, where $m \leq z_j \leq M$. In this way, the NLP problem (9)-(10) is converted into an equivalent one with unbounded decision variables. To find the optimal solutions β^* and w^* we define $L(\beta, w, \lambda) = \sum_{j=1}^{M} \beta_j^2 \bar{g}_0(w_j) + \lambda_1 (\sum_{j=1}^{M} \beta_j^2 \bar{\phi}_1^f(w_j) - \Delta\phi_1) + ... + \lambda_M (\sum_{j=1}^{M} \beta_j^2 \bar{\phi}_M^f(w_j) - \Delta\phi_M)$, where $\bar{g}_0(w) = g(\frac{M+m}{2} + \frac{M-m}{2} sin(w))$ and λ is the lagrange multiplier. According to the necessary optimality conditions, there is λ^* such that the triple $(\beta^*, w^*, \lambda^*)$ satisfies a system of nonlinear equations as

$$\frac{\partial L}{\partial \beta_j} = 2\beta_j \big(\bar{g}_0(w_j) + \lambda_1 \bar{\phi}_1^f(w_j) + ... + \lambda_N \bar{\phi}_M^f(w_j)\big) = 0, \tag{11}$$

$$\frac{\partial L}{\partial w_j} = \beta_j^2 \big(\nabla \bar{g}_0(w_j) + \lambda_1 \nabla \bar{\phi}_1^f(w_j) + ... + \lambda_N \nabla \bar{\phi}_M^f(w_j)\big) = 0, \tag{12}$$

$$\frac{\partial L}{\partial \lambda_j} = \sum_{k=1}^{M} \beta_k^2 \bar{\phi}_j^f(w_k) - \Delta\phi_j = 0, \tag{13}$$

for $j = 1, ..., M$. Obviously, we can solve the equations (11)-(13) by using the *fminsearch* optimization toolbox in MATLAB. We note that the total functions in (9)-(10) can be chosen as monomials of t and x. In special case, the function $\phi(t, x, t) = t$ gives the constraint $\alpha_1 + ... + \alpha_M = b - a$, which is necessary to build the suboptimal piecewise constant control function from α_j^*, $z_j^* = (t_j^*, x_j^*, u_j^*), j = 1, ..., M$, as $u^*(t) = \sum_{k=1}^{M} u_k^* \chi_{\left[\sum_{j=0}^{k-1} \alpha_j^*,\ \sum_{j=0}^{k} \alpha_j^*\right)}(t)$, where $\alpha_0^* = a$ and χ_A is the characteristic function of a set. Moreover, we can set $\phi(t, x, t) =$

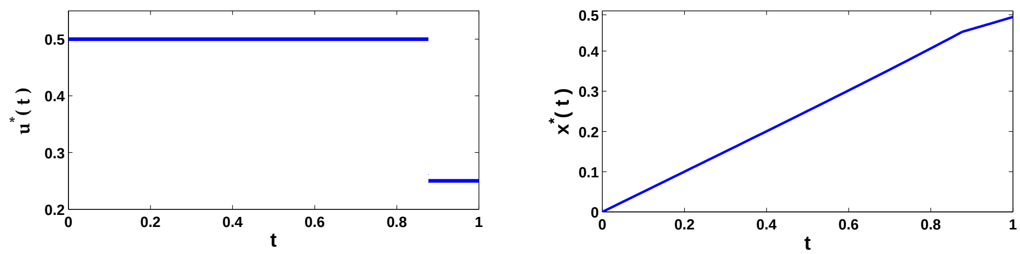

Figure 1: The control function and the state function of example 1.

$x_j\psi(t),\ j=1,...,n$, where ψ is a differentiable function with compact support in (a,b) such as $\psi(t)=sin\big(\frac{2\pi(t-a)}{b-a}\big)$ and $\psi(t)=1-cos\big(\frac{2\pi(t-a)}{b-a}\big)$.

3 Numerical Results

Example 1. Minimize the functional $J=\int_0^1 u^2(t)dt$ subject to $\dot{x}=x^2sin(x)+u,\ x(0)=0,\ x(1)=0.5$. We have solved this OCP with $\mathcal{A}=[0,0.5],\ \mathcal{U}=[0.25,0.5],\ \phi_1=x,\ \phi_2=x^2$ and $\phi_3=xsin(2\pi t)$. The resulting piecewise control and the state function are depicted in figure 1.
Example 2. Consider a mathematical model for chronic myelogenous leukemia (CML) introduced in [2] as

$$\begin{aligned}\frac{dx_1}{dt}&=s_1-u_2d_1x_1-k_1x_1\left(\frac{x_3}{x_3+\eta}\right),\\ \frac{dx_2}{dt}&=\alpha_1k_1x_1\left(\frac{x_3}{x_3+\eta}\right)+\alpha_2x_2\left(\frac{x_3}{x_3+\eta}\right)-u_2d_2x_2-\gamma_2x_3x_2,\\ \frac{dx_3}{dt}&=(1-u_1)r_3x_3ln\left(\frac{x_{max}}{x_3}\right)-u_2d_3C-\gamma_3x_3x_2,\end{aligned}$$

where $x_3(t)$, $x_1(t)$ and $x_2(t)$, respectively, denote the cancer cell population, the naive T cell population and the effector T cell population at time t and $u_1(t)$ and $u_2(2)$ are time dependent drug efficacy. The interpretation of the model and the value of parameters are presented in [2]. The goal of cancer treatments like chemotherapy and radiation therapy is to destroy cancerous cells in the body while minimizing the systemic costs to the body of drugs. Therefore, we minimize the objective functional as $J=\int_0^{250}x_3(t)+w_1u_1^2(t)+w_2u_2^2(t)dt$, where w_1 and w_2 are weight coefficients denoting the relative importance of the drugs cost. We have solved this OCP with $\mathcal{A}=[0,1510]\times[0,40]\times[0,10000]$, $\mathcal{U}=[0,0.9]\times[1,1.47]$, $\phi_j=x_j,\ \phi_{3+j}=sin\big(\frac{\pi t}{125}\big)x_j,\ j=1,2,3$. The resulting control functions and the corresponding state functions are depicted in figure 2.

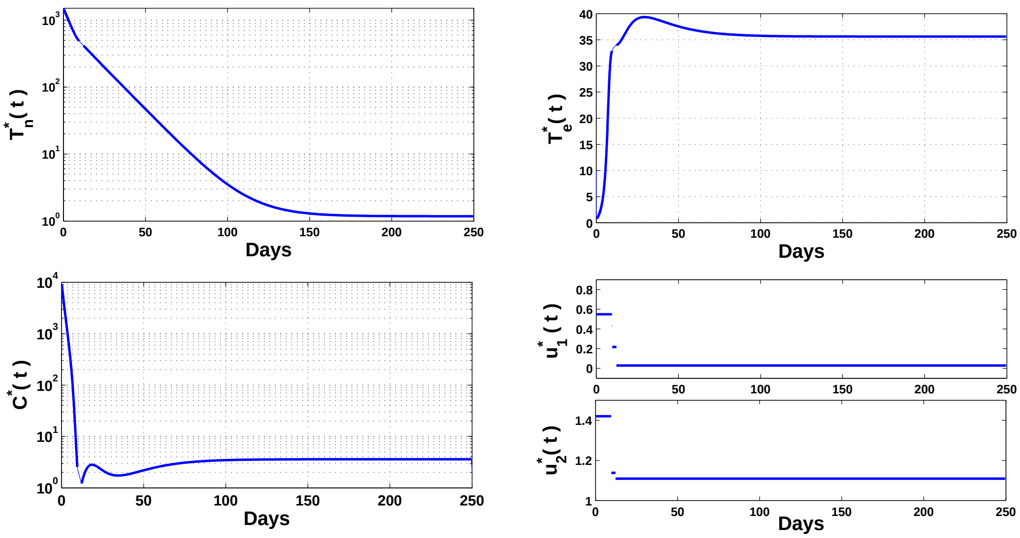

Figure 2: The state function and the control function of example 2.

4 Conclusion

In this paper we omitted the last approximation in Rubio's measure theoretical approach and approximated the optimal control problems by the NLP ones, which can be solved by the nonlinear programming softwares developed in recent years. Numerical results shows that the necessary optimality conditions which are a system of nonlinear equations can be solved by using the fminserach optimization toolbox in MATLAB.

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