AMO-Advanced Modeling and Optimization, Volume 20, Number 1, 2018

On Duality for Semiinfinite Multiobjective Fractional Programming Problems Using Generalized (α, η, ρ)-V-Univex Functions

S.K. Porwal
Department of Mathematics
SRMS-CET, Bareilly
UP-243202, India
skpmathsdstcims@gmail.com
S.K. Mishra
Department of Mathematics
Banaras Hindu University
Varanasi-221005, India
bhu.skmishra@gmail.com

Abstract

In this paper, we consider the class of generalized (α, η, ρ)-V-univex functions for a semiinfinite multiobjective fractional programming problem and its dual models. We establish several weak, strong and converse duality results under various generalized (α, η, ρ)-V-univexity assumptions for a semiinfinite multiobjective fractional programming problem.

AMS Subject Classification. 49N15; 90C29; 90C30; 90C32; 90C34; 90C46; 49J35

Key Words and Phrases. Semiinfinite programming; multiobjective fractional programming; generalized univex functions; infinitely many equality and inequality constraints; parametric duality models; duality theorems

1 Introduction

Zalmai and Zhang [5] introduced a global semi parametric sufficient efficiency conditions for semiinfinite multiobjective fractional programming problems involving generalized (α, η, ρ)-V-invex functions and then presented a global parametric sufficient efficiency conditions introduced in [6] for semiinfinite multiobjective fractional programming problems containing generalized (α, η, ρ)-V-invex functions. Moreover, he has formulated number of parametric duality models and established numerous duality results under various generalized (α, η, ρ)-V-invexity assumptions in [7].

Consider the following semiinfinite multiobjective fractional programming problem:

AMO-Advanced Modeling and Optimization. ISSN: 1841-4311

$$(P) \qquad \text{Minimize } \varphi(x) = (\varphi_1(x), \ldots, \varphi_p(x)) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to

$$G_j(x,t) \leq 0 \text{ for all } t \in T_j, \ \ j \in \underline{q},$$
$$H_k(x,s) = 0 \text{ for all } s \in S_k, \ k \in \underline{r},$$
$$x \in \mathbb{R}^n,$$

where p, q, and r are positive integers, $\mathbb{R}^n$ is n-dimensional Euclidean space for each $j \in \underline{q} = \{1, 2, \cdots, q\}$ and $k \in \underline{r} = \{1, 2, \cdots, r\}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on $\mathbb{R}^n$, for each $j \in \underline{q}$, $G_j(., t)$ is a real-valued function defined on $\mathbb{R}^n$ for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(., s)$ is a real-valued function defined on $\mathbb{R}^n$ for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, .)$ and $H_k(x, .)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in \mathbb{R}^n$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

The feasible set (assumed to be nonempty) of the above problem (P), is

$$\mathbb{F} = \{x \in \mathbb{R}^n : G_j(x,t) \leq 0 \text{ for all } t \in T_j, \ j \in \underline{q}, \ H_k(x,s) = 0 \text{ for all } s \in S_k, \ k \in \underline{r}\}.$$

Bector et al. [1] introduced some classes of univex functions by relaxing the definition of an invex function. Optimality and duality results are also obtained for a nonlinear multiobjective programming problem in [1].

In this paper, we extend the results of Zalmai and Zhang [7] to the classes of functions introduced in Bector et al. [1]. This paper is organized as follows. In Section 2, we present a few definitions and auxiliary results which will be needed in the sequel. In Section 3, we consider two parametric duality models with somewhat limited constraint structures and prove weak, strong, and strict converse duality theorems under appropriate generalized (α, η, ρ)-V-univexity hypotheses. In Section 4, we consider two other duality models with more general constraint structures which allow for a greater variety of generalized (α, η, ρ)-V-univexity conditions under which duality can be established. We continue our discussion of duality in Sections 5 and 6 where we use two partitioning schemes and consider four generalized parametric duality models and obtain several duality results under various generalized (α, η, ρ)-V-univexity assumptions. Finally, in Section 7 we summarize our main results.

2 Preliminaries

In this section, we recall some definitions and results which will be used in the sequel.

Let the function $F = (F_1, F_2, \ldots, F_N) : \mathbb{R}^n \to \mathbb{R}^N$ be differentiable at x^*. Let X be a nonempty open subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\phi : \mathbb{R} \to \mathbb{R}$, and b :

$X \times X \times [0,1] \to \mathbb{R}_+,\ b = b\,(x,u,\lambda)$. If the function f is differentiable, then b does not depend on λ.

Throughout the paper, we use the following conventions for vectors in $\mathbb{R}^n$. For $a, b \in \mathbb{R}^n$, the following order notation will be used: $a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$; $a \geq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$; $a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$ and $a \ngeq b$ is the negation of $a \geq b$.

Consider the multiobjective problem

$$(P^*) \quad \underset{x \in \mathbb{F}}{Minimize} \quad F(x) = (F_1(x), F_2(x), \ldots, F_p(x))\,,$$

where $F_i, i \in \underline{p}$, are real valued functions defined on $\mathbb{R}^n$.

Mishra and Porwal [3] introduced the following vector versions of the concepts of univexity.

Definition 2.1. *The function F is said to be (strictly) $(\alpha, \eta, \bar{\rho})$-$V$-univex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\} \equiv (0, +\infty)$, and $\bar{\rho}_i \in \mathbb{R}, i \in \underline{N}$, such that for each $x \in \mathbb{R}^n (x \neq x^*)$,*

$$b\,(x,x^*)\,\phi\,[F_i(x) - F_i(x^*)]\,(>) \geqq \langle \alpha_i(x,x^*) \nabla F_i(x^*), \eta(x,x^*) \rangle + \bar{\rho}_i \,\|x - x^*\|^2 \,.$$

Definition 2.2. *The function F is said to be (strictly) $(\beta, \eta, \tilde{\rho})$-$V$-pseudounivex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}, i \in \underline{N}$ and $\tilde{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n (x \neq x^*)$,*

$$\left\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*) \right\rangle \geqq -\tilde{\rho}\,\|x - x^*\|^2$$

$$\Longrightarrow b\,(x,x^*)\,\phi \left[\sum_{i=1}^{N} \beta_i(x,x^*) F_i(x) - \sum_{i=1}^{N} \beta_i(x,x^*) F_i(x^*) \right] (>) \geqq 0,$$

or equivalently,

$$b\,(x,x^*)\,\phi \left[\sum_{i=1}^{N} \beta_i(x,x^*) F_i(x) - \sum_{i=1}^{N} \beta_i(x,x^*) F_i(x^*) \right] < 0$$

$$\Longrightarrow \left\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*) \right\rangle < -\tilde{\rho}\,\|x - x^*\|^2 \,.$$

Definition 2.3. *The function F is said to be (prestrictly) $(\gamma, \eta, \hat{\rho})$-$V$-quasiunivex at x^* with respect to b, ϕ and η if there exist functions b, ϕ, η and $\gamma_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}, i \in \underline{N}$ and $\hat{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$b\,(x,x^*)\,\phi \left[\sum_{i=1}^{N} \gamma_i(x,x^*) F_i(x) - \sum_{i=1}^{N} \gamma_i(x,x^*) F_i(x^*) \right] (<) \leqq 0$$

$$\Longrightarrow \left\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x, x^*) \right\rangle \leqq -\hat{\rho} \|x - x^*\|^2 .$$

Definition 2.4. *An element $x^* \in \mathbb{F}$ is said to be an efficient solution of (P^*) if there exists no $x \in \mathbb{F}$ such that $F(x) \leq F(x^*)$.*

Zalmai and Zhang [5] derived the following necessary condition which will be used in the sequel.

Theorem 2.1. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $G_j(., t)$ be continuously differentiable at x^*, for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $H_k(., s)$ be be continuously differentiable at x^* for all $s \in S_k$. If x^* is an efficient solution of (P), if the generalized Guignard constraint qualification holds at x^*, and if for each $i_0 \in \underline{p}$, the set cone*

$$\left(\left\{\nabla G_j(x^*, t) : t \in \hat{T}_j(x^*), j \in \underline{q}\right\} \cup \left\{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\right\}\right) + span\left(\left\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\right\}\right)$$

is closed, then there exist

$$u^* \in U = \left\{ u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1 \right\},$$

and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*) \equiv \{t \in T_{j_m} : G_{j_m}(x^*, t) = 0\}$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, with the property that

$$\sum_{i=1}^{p} u_i^* \left[\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)\right] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m) = 0.$$

For brevity, we shall henceforth refer to an efficient solution $x^* \in \mathbb{F}$ as a normal efficient solution of (P) if the generalized Guignard constraint qualification is satisfied at x^* and for each $i_0 \in \underline{p}$,

$$set\ cone\left(\left\{\nabla G_j(x^*, t) : t \in \hat{T}_j(x^*), j \in \underline{q}\right\} \cup \left\{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\right\}\right) + span\left(\left\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\right\}\right)$$

is closed, where

$$\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)},\ i \in \underline{p}.$$

In the remaining paper, we assume that the functions $f_i, g_i, i \in \underline{p}, G_j(., t)$ and $H_k(., s)$ are continuously differentiable on $\mathbb{R}^n$ for all $t \in T_j, j \in \underline{q}$, and $s \in S_k, k \in \underline{r}$. Throughout this paper, we also assume that ϕ is linear with $\phi(x) \geq 0 \Rightarrow x \geq 0$, unless otherwise stated.

3 Duality Model I

In this section, we consider a dual problem with a relatively simple constraint structure and prove weak, strong, and strict converse duality theorems under (α, η, ρ)-V -univexity conditions. Let

$$\mathbb{H} = \Big\{(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s}) : y \in \mathbb{R}^n;\ u \in U;\ \lambda \in \mathbb{R}^p;\ 0 \leqq \nu_0 \leqq \nu \leqq n+1; \\ v \in \mathbb{R}^\nu, v_i > 0, 1 \leqq i \leqq \nu_0;\ J_{\nu_0} = (j_1, j_2, \ldots, j_{\nu_0}),\ 1 \leqq j_i \leqq q;\ K_{\nu\backslash\nu_0} = \\ (k_{\nu_0+1}, \ldots, k_\nu), 1 \leqq k_i \leqq r;\ \bar{t} = (t^1, t^2, \ldots, t^{\nu_0}),\ t^i \in T_{j_i};\ \bar{s} = (s^{\nu_0+1}, \ldots, s^\nu),\ s^i \in S_{k_i}\Big\}.$$

Consider the two problems as follows:

$$(DI) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \qquad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda_i \nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) \\ + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m) = 0, \quad (3.1)$$

$$u_i[f_i(y) - \lambda g_i(y)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m) \geqq 0, \quad i \in \underline{p}; \quad (3.2)$$

$$(\tilde{D}I) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \qquad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.2) and

$$\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda_i \nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) \\ + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m), \eta(x, y) \Big\rangle \geqq 0 \ \textit{for all}\ x \in \mathbb{F}, \quad (3.3)$$

We observe that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution to (DI) is also feasible to $(\tilde{D}I)$, while the converse may not be true.

Furthermore, we observe that (3.1) is a system of n equations, whereas (3.3) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to $(\tilde{D}I)$ because of the dependence of (3.3) on the feasible set of (P). Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for (P)-(DI) and (P)-($\tilde{D}I$) are almost identical and, therefore, we shall consider only the pair (P)-(DI).

The next two theorems show that (DI) is a dual problem for (P).

Theorem 3.1. *Let x and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s}))$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that $\lambda \geqq 0$ and that either one of the following two sets of hypotheses is satisfied:*

(a) *(i)* $(f_1, \ldots, f_p)$ *is* $(\theta, \eta, \bar{\rho})$*-$V$-univex at y with respect to b, ϕ and η;*

(ii) $(-g_1, \ldots, -g_p)$ *is* $(\xi, \eta, \tilde{\rho})$*-$V$-univex at y with respect to b, ϕ and η;*

(iii) $\left(v_1 G_{j_1}(., t^{\nu_0}), \ldots, v_{\nu_0} G_{j_{\nu_0}}(., t^{\nu_0})\right)$ *is* $(\pi, \eta, \hat{\rho})$*-V-univex at y with respect to b, ϕ and η;*

(iv) $\left(v_{\nu_0+1} H_{k_{\nu_0+1}}\left(., s^{\nu_0+1}\right), \ldots, v_\nu H_{k_\nu}(., s^\nu)\right)$ *is* $(\delta, \eta, \breve{\rho})$*-$V$-univex at y with respect to b, ϕ and η;*

(v) $\theta_i = \xi_j = \pi_k = \delta_l = \sigma$ *for all* $i, j \in \underline{p}, k \in \underline{\nu_0}$*, and* $l \in \underline{\nu}\backslash\underline{\nu_0}$*;*

(vi) $\sum_{i=1}^{p} u_i\left(\bar{\rho}_i + \lambda_i \tilde{\rho}_i\right) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \geqq 0.$

(b) The function $\left(L_1(., u, v, \lambda, \bar{t}, \bar{s}), \ldots, L_p(., u, v, \lambda, \bar{t}, \bar{s})\right)$ *is* $(\theta, \eta, 0)$*-V-pseudounivex at y with respect to b, ϕ and η where*

$$L_i(z, u, v, \lambda, \bar{t}, \bar{s}) = u_i\Big[f_i(z) - \lambda_i g_i(z) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(z, s^m)\Big], \ i \in \underline{p}.$$

Then, $\varphi(x) \nleq \lambda$.

Proof.

(a) For $u > 0$ and $\lambda \geqq 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{p} u_i^* b(x,y)\,\phi\left[f_i(x) - \lambda_i g_i(x)\right] \\
&\quad = \sum_{i=1}^{p} u_i^* b(x,y)\,\phi\left\{\left[f_i(x) - f_i(y)\right] - \lambda_i\left[g_i(x) - g_i(y)\right]\right\} \quad (since\ \lambda = \phi(x)) \\
&= \sum_{i=1}^{p} u_i \left\{b(x,y)\,\phi\left[f_i(x) - f_i(y)\right] - \lambda_i b(x,y)\,\phi\left[g_i(x) - g_i(y)\right]\right\} \quad (by\ the\ linearity\ of\ \phi) \\
&\geqq \sum_{i=1}^{p} u_i \left[\sigma(x,y)\left\langle \nabla f_i(y) - \lambda_i \nabla g_i(y), \eta(x,y)\right\rangle + (\bar{\rho}_i + \lambda_i \tilde{\rho}_i)\left\|x-y\right\|^2\right] \quad (by (i), (ii)\ and (v)) \\
&\quad = -\sigma(x,y)\left\langle \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y,s^m), \eta(x,y)\right\rangle \\
&\qquad + \sum_{i=1}^{p} u_i (\bar{\rho}_i + \lambda_i \tilde{\rho}_i)\left\|x-y\right\|^2 \quad (by (3.1)) \\
&\quad \geqq \sum_{m=1}^{\nu_0} v_m \left[G_{j_m}(y,t^m) - G_{j_m}(x,t^m)\right] + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y,s^m) \\
&\qquad + \left[\sum_{i=1}^{p} u_i (\bar{\rho}_i + \lambda_i \tilde{\rho}_i) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m\right]\left\|x-y\right\|^2 \\
&\qquad (by\ (iii), (iv), (v)\ and\ the\ primal\ feasibility\ of\ x) \\
&\geqq \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m)\,(by\ (vi)\ and\ the\ primal\ feasibility\ of\ x).
\end{aligned}
$$

Therefore, in view of (3.2), we have

$$
\sum_{i=1}^{p} u_i^* b(x,y)\,\phi\left[f_i(x) - \lambda_i g_i(x)\right] \geqq 0. \tag{3.4}
$$

Since $u^* > 0$, and by the assumption on ϕ the above inequality implies that

$$
\left(f_1(x) - \lambda_1^* g_1(x), \ldots, f_p(x) - \lambda_p^* g_p(x)\right) \nleq (0, \ldots, 0),
$$

which in turn implies that

$$
\varphi(x) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right) \nleq \left(\lambda_1^*, \ldots, \lambda_p^*\right) = \varphi(x^*).
$$

This completes the proof.

(b) By our $(\theta, \eta, 0)$-V-pseudounivexity assumption, (3.1) implies that

$$b(x,y)\,\phi\Big[\sum_{i=1}^{p}\theta_i(x,y)\left\{u_i\left[f_i(x)-\lambda_i g_i(x)\right]+\sum_{m=1}^{\nu_0}v_m G_{j_m}(x,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m H_{k_m}(x,s^m)\right\}$$
$$-\sum_{i=1}^{p}\theta_i(x,y)\left\{u_i\left[f_i(y)-\lambda_i g_i(y)\right]+\sum_{m=1}^{\nu_0}v_m G_{j_m}(y,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m H_{k_m}(y,s^m)\right\}\Big] \geqq 0.$$

Because of (3.2), the right-hand side of this inequality is greater than or equal to zero, and so we have that

$$\sum_{i=1}^{p}\theta_i(x,y)\,b(x,y)\,\phi\Big\{u_i\left[f_i(x)-\lambda_i g_i(x)\right]+\sum_{m=1}^{\nu_0}v_m G_{j_m}(x,t^m)$$
$$+\sum_{m=\nu_0+1}^{\nu}v_m H_{k_m}(x,s^m)\Big\} \geqq 0.$$

But $x \in \mathbb{F}$ and $v_m > 0$ for each $m \in \underline{\nu_0}$, and hence the above inequality reduces to

$$\sum_{i=1}^{p}u_i\theta_i(x,y)\,b(x,y)\,\phi\left[f_i(x)-\lambda_i g_i(x)\right] \geqq 0. \tag{3.5}$$

Since u and $\theta_i(x,y) > 0$, and by the assumption on ϕ, the above inequality implies that

$$\left(f_1(x)-\lambda_1 g_1(x),\ldots,f_p(x)-\lambda_p g_p(x)\right) \nleq (0,\ldots,0),$$

which in turn implies that

$$\varphi(x)=\left(\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right) \nleq (\lambda_1,\ldots,\lambda_p)=\lambda.$$

Hence, proved.

□

Theorem 3.2. *(Strong Duality). Let x^* be a normal efficient solution of (P). Assume that for each feasible solution $(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s}))$ of (DI), either one of the two sets of conditions specified in Theorem 3.1 is satisfied. Then there exist $u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*$ such that $\left(x^*,u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*\right)$ is an efficient solution of (DI) and $\varphi(x^*)=\lambda^*$.*

Proof. Since x^* is a normal efficient solution of (P), by Theorem 2.1, there exist $u^*,v^*,\lambda^*\left(=\varphi(x^*)\right),\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*$ such that $\left(x^*,u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*\right)$ is a feasible solution of (DI). Since $\varphi(x^*)=\lambda^*$, the efficiency of $\left(x^*,u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*\right)$ for (DI) follows from Theorem 3.1.

We also have the following converse duality result for (P)-(DI).

Theorem 3.3. *(Strict Converse Duality). Let x^* and $\left(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\setminus\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)$ be arbitrary feasible solutions of (P) and (DI), respectively, such that*

$$\sum_{i=1}^{p} \tilde{u}_i b\left(x^*, \tilde{x}\right) \phi\left[f_i(x^*) - \tilde{\lambda}_i g_i(x^*)\right] = 0. \tag{3.6}$$

Furthermore, assume that either one of the following two sets of hypotheses is satisfied:

(a) The assumptions of part (a) of Theorem 3.1 are satisfied for the feasible solution $\left(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\setminus\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)$ of (DI), $(f_1, \ldots, f_p)$ is strictly $(\theta, \eta, \bar{\rho})$-$V$-univex at $\tilde{x}$ with respect to b, ϕ and η; or $(-g_1, \ldots, -g_p)$ is strictly $(\xi, \eta, \tilde{\rho})$-$V$-univex at $\tilde{x}$ with respect to b, ϕ and η; or $\left(\tilde{v}_1 G_{j_1}\left(., \tilde{t}^1\right), \ldots, \tilde{v}_{\tilde{\nu}_0} G_{j_{\tilde{\nu}_0}}\left(., \tilde{t}^{\tilde{\nu}_0}\right)\right)$ is strictly $(\pi, \eta, \hat{\rho})$-V-univex at $\tilde{x}$ with respect to b, ϕ and η; or $\left(\tilde{v}_{\tilde{\nu}_0+1} H_{k_{\tilde{\nu}_0+1}}\left(., \tilde{s}^{\tilde{\nu}_0+1}\right), \ldots, \tilde{v}_{\tilde{\nu}} H_{k_{\tilde{\nu}}}\left(., \tilde{s}^{\tilde{\nu}}\right)\right)$ is strictly $(\delta, \eta, \breve{\rho})$-$V$-univex at $\tilde{x}$ with respect to b, ϕ and η; or

$$\sum_{i=1}^{p} \tilde{u}_i \left(\bar{\rho}_i + \tilde{\lambda}_i \tilde{\rho}_i\right) + \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m \hat{\rho}_m + \sum_{m=\tilde{\nu}_0+1}^{\tilde{\nu}} \breve{\rho}_m > 0.$$

(b) The function $L\left(., \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)$ is strictly $(\theta, \eta, 0)$-V-pseudounivex at $\tilde{x}$ with respect to b, ϕ and η.

Then, $\tilde{x} = x^*$ and $\varphi\left(x^*\right) = \tilde{\lambda}$.

Proof. (a) Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of Theorem 3.1 (with x replaced by x^* and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\setminus\nu_0}, \bar{t}, \bar{s})$ by $\left(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\setminus\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)$) and using any of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i b\left(x^*, \tilde{x}\right) \phi\left[f_i(x^*) - \tilde{\lambda}_i g_i(x^*)\right] > 0,$$

which is contradiction to (3.6). Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi\left(x^*\right) = \tilde{\lambda}$.

(b) The proof is similar to that of part (a).

4 Duality Model II

In this section, we consider certain variants of (DI) and $(\tilde{D}I)$ that allow for a greater variety of generalized (α, η, ρ)-V -univexity conditions under which duality can be established. These duality models have the following forms:

$$(DII) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \qquad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.1) and

$$\sum_{i=1}^{p} u_i[f_i(y) - \lambda_i g_i(y)] \geqq 0, \quad i \in \underline{p}, \tag{4.1}$$

$$G_{j_m}(y, t^m) \geqq 0, \quad m \in \underline{\nu_0}, \tag{4.2}$$

$$v_m H_{k_m}(y, s^m) \geqq 0, \quad m \in \underline{\nu} \backslash \underline{\nu_0}; \tag{4.3}$$

$$(\tilde{D}II) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \quad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.3) and (4.1)-(4.3).

The remarks and observations made earlier about the relationships between (DI) and $(\tilde{D}I)$ are, of course, also valid for (DII) and $(\tilde{D}II)$ Since the constraint inequalities of (DII) are formed by splitting the inequality (3.2) into three inequalities (4.1)-(4.3), it is clear that Theorems 3.1-3.3 are valid for the pair (P)-(DII). Below, we shall establish some duality results in which various generalized (α, η, ρ)-V-univexity requirements will be placed on the vector function $(\varepsilon_1(., \lambda, u), \ldots, \varepsilon_p(., \lambda, u))$, where for each $i \in \underline{p}$ the component function $\varepsilon_i(., \lambda, u)$ is defined, for fixed λ and u, on $\mathbb{R}^n$ by

$$\varepsilon_i(z, \lambda, u) = u_i b(z, y) \phi[f_i(z) - \lambda_i g_i(z)].$$

Theorem 4.1. *Let x and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) *(i)* $(\varepsilon_1(., \lambda, u), \ldots, \varepsilon_p(., \lambda, u))$, *is (θ, η, ρ)-V-pseudounivex at y with respect to b, ϕ and η;*

(ii) $\left(v_1 G_{j_1}(., t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(., t^{\nu_0})\right)$ *is $(\pi, \eta, \tilde{\rho})$-$V$-quasiunivex at y with respect to b, ϕ and η;*

(iii) $\left(v_{\nu_0+1} H_{k_{\nu_0+1}}(., s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(., s^\nu)\right)$ *is $(\delta, \eta, \hat{\rho})$-$V$-quasiunivex at y with respect to b, ϕ and η;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geqq 0$;

(b) *(i)* $(\varepsilon_1(., \lambda, u), \ldots, \varepsilon_p(., \lambda, u))$, *is prestrictly (θ, η, ρ)-V-quasiunivex at y with respect to b, ϕ and η;*

(ii) $\left(v_1 G_{j_1}(., t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(., t^{\nu_0})\right)$ *is $(\pi, \eta, \tilde{\rho})$-$V$-quasiunivex at y with respect to b, ϕ and η;*

(iii) $\left(v_{\nu_0+1} H_{k_{\nu_0+1}}(., s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(., s^\nu)\right)$ *is $(\delta, \eta, \hat{\rho})$-$V$-quasiunivex at y with respect to b, ϕ and η;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} > 0$;

(c) *(i)* $(\varepsilon_1(., \lambda, u), \ldots, \varepsilon_p(., \lambda, u))$, *is prestrictly (θ, η, ρ)-V-quasiunivex at y with respect to b, ϕ and η;*

(ii) $\left(v_1 G_{j_1}\left(., t^1\right), \ldots, v_{\nu_0} G_{j_{\nu_0}}\left(., t^{\nu_0}\right)\right)$ *is strictly* $(\pi, \eta, \tilde{\rho})$*-V-pseudounivex at* y *with respect to* b, ϕ *and* η;

(iii) $\left(v_{\nu_0+1} H_{k_{\nu_0+1}}\left(., s^{\nu_0+1}\right), \ldots, v_\nu H_{k_\nu}\left(., s^\nu\right)\right)$ *is* $(\delta, \eta, \hat{\rho})$*-V-quasiunivex at* y *with respect to* b, ϕ *and* η;

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geqq 0$;

(d) *(i)* $\left(\varepsilon_1(., \lambda, u), \ldots, \varepsilon_p(., \lambda, u)\right)$, *is prestrictly* (θ, η, ρ)*-V-quasiunivex at* y *with respect to* b, ϕ *and* η;

(ii) $\left(v_1 G_{j_1}\left(., t^1\right), \ldots, v_{\nu_0} G_{j_{\nu_0}}\left(., t^{\nu_0}\right)\right)$ *is* $(\pi, \eta, \tilde{\rho})$*-V-quasiunivex at* y *with respect to* b, ϕ *and* η;

(iii) $\left(v_{\nu_0+1} H_{k_{\nu_0+1}}\left(., s^{\nu_0+1}\right), \ldots, v_\nu H_{k_\nu}\left(., s^\nu\right)\right)$ *is* $(\delta, \eta, \hat{\rho})$*-V-pseudounivex at* y *with respect to* b, ϕ *and* η;

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geqq 0$.

Then, $\varphi(x) \nleq \lambda$.

Proof. (a) Because of (4.2) and the primal feasibility of x, we have $G_{j_m}(x, t^m) \leqq 0 = G_{j_m}(x^*, t^m)$, and hence

$$b(x, y)\,\phi\left[\sum_{m=1}^{\nu_0} v_m \pi_m(x, y)\, G_{j_m}(x, t^m) - \sum_{m=1}^{\nu_0} v_m \pi_m(x, y)\, G_{j_m}(y, t^m)\right] \leqq 0,$$

which in view of (ii) implies that

$$\left\langle \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m), \eta(x, y)\right\rangle \leqq -\tilde{\rho}\,\|x - y\|^2. \tag{4.4}$$

Similarly, we can show that our assumptions in (iii) combined with the feasibility of x and (4.3) lead to the following inequality:

$$\left\langle \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m), \eta(x, y)\right\rangle \leqq -\hat{\rho}\,\|x - y\|^2. \tag{4.5}$$

Now because of (4.4), (4.5) and (iv), (3.1) reduces to

$$\left\langle \sum_{i=1}^{p} u_i\left[\nabla f_i(y) - \lambda_i \nabla g_i(y)\right], \eta(x, y)\right\rangle \leqq -\rho\,\|x - y\|^2, \tag{4.6}$$

which in view of (i) implies that

$$b(x, y)\,\phi\left[\sum_{i=1}^{p} u_i^* \theta_i(x, y)\left[f_i(x) - \lambda_i g_i(x)\right] - \sum_{i=1}^{p} u_i \theta_i(x, y)\left[f_i(y) - \lambda_i g_i(y)\right]\right] \geqq 0, \tag{4.7}$$

where the equality follows from (4.1). In the proof of part (b) of Theorem 3.1, it was shown that this inequality leads to the desired conclusion that $\varphi(x) \nleq \lambda$. (b)-(d) The proofs are similar to that of part (a).

□

The proof of the following theorem follows along the line of the proof of Theorem 3.2.

Theorem 4.2. *(Strong Duality). Let x^* be a normal efficient solution of (P). Assume that for each feasible solution $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ of (DII), any one of the four sets of conditions specified in Theorem 4.1 is satisfied. Then there exist $u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*$ such that $\left(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*\right)$ is an efficient solution of (DII) and $\varphi(x^*) = \lambda^*$.*

We also have the following converse duality result for (P)-(DII).

Theorem 4.3. *(Strict Converse Duality). Let x^* and $\tilde{\omega} \equiv \left(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)$ be arbitrary feasible solutions of (P) and (DII), respectively, such that*

$$\sum_{i=1}^{p} \tilde{u}_i b\left(x^*, \tilde{x}\right) \phi \left[f_i(x^*) - \tilde{\lambda}_i g_i(x^*)\right] = 0. \tag{4.8}$$

(a) The assumptions specified in part (a) of Theorem 4.1 are satisfied for the feasible solution $\tilde{\omega}$, and the function $\left(\varepsilon_1\left(., \tilde{\lambda}, \tilde{u}\right), \ldots, \varepsilon_p\left(., \tilde{\lambda}, \tilde{u}\right)\right)$, is strictly (θ, η, ρ)-V-pseudo univex at $\tilde{x}$ with respect to b, ϕ and η;

(b) The assumptions specified in part (b) of Theorem 4.1 are satisfied for the feasible solution $\tilde{\omega}$, and the function $\left(\varepsilon_1\left(., \tilde{\lambda}, \tilde{u}\right), \ldots, \varepsilon_p\left(., \tilde{\lambda}, \tilde{u}\right)\right)$, is (θ, η, ρ)-V-quasiunivex at $\tilde{x}$ with respect to b, ϕ and η;

(c) The assumptions specified in part (c) of Theorem 4.1 are satisfied for the feasible solution $\tilde{\omega}$, and the function $\left(\varepsilon_1\left(., \tilde{\lambda}, \tilde{u}\right), \ldots, \varepsilon_p\left(., \tilde{\lambda}, \tilde{u}\right)\right)$, is (θ, η, ρ)-V-quasiunivex at $\tilde{x}$ with respect to b, ϕ and η;

(d) The assumptions specified in part (d) of Theorem 4.1 are satisfied for the feasible solution $\tilde{\omega}$, and the function $\left(\varepsilon_1\left(., \tilde{\lambda}, \tilde{u}\right), \ldots, \varepsilon_p\left(., \tilde{\lambda}, \tilde{u}\right)\right)$, is (θ, η, ρ)-V-quasiunivex at $\tilde{x}$ with respect to b, ϕ and η.

Then, $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

Proof. The proof is similar to that of Theorem 3.3.

5 Duality Model III

In this section, we discuss several families of duality results under various generalized (α, η, ρ)-V-univexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by employing a partitioning scheme which was originally proposed in [2] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let ν_0 and ν be integers, with $1 \leqq \nu_0 \leqq \nu \leqq n+1$, and let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$ respectively; thus,$J_i \subseteq \underline{\nu_0}$ for each $i \in \underline{M} \bigcup \{0\}$, $J_i \bigcap J_j = \phi$ for each $i, j \in \underline{M} \bigcup \{0\}$ with $i \neq j$, and $\bigcup_{i=0}^{m} J_i = \underline{\nu_0}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$ Moreover, if m_1 and m_2 are the members of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \phi$ or $K_i = \phi$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i\left(., \lambda, u, v, \bar{t}, \bar{s}\right)$, $i \in \underline{p}$, and $\Lambda_\tau\left(., v, \bar{t}, \bar{s}\right)$, $\tau \in \underline{M}$, defined for fixed $u, v, \lambda, \bar{t} \equiv \left(t^1, t^2, \ldots, t^{\nu_0}\right)$, and $\bar{s} \equiv \left(s^{\nu_0+1}, s^{\nu_0+2}, \ldots, s^{\nu}\right)$, on $\mathbb{R}^n$ as follows:

$$\Phi_i\left(z, \lambda, u, v, \bar{t}, \bar{s}\right) = u_i \left[f_i\left(z\right) - \lambda_i g_i\left(z\right) + \sum_{m \in J_0} v_m G_{j_m}\left(z, t^m\right) + \sum_{m \in k_0} v_m H_{k_m}\left(z, s^m\right) \right], \; i \in \underline{p},$$

$$\Lambda_\tau\left(z, v, \bar{t}, \bar{s}\right) = \sum_{m \in J_\tau} v_m G_{j_m}\left(z, t^m\right) + \sum_{m \in k_\tau} v_m H_{k_m}\left(z, s^m\right), \; \tau \in \underline{M}.$$

Making use of the sets and functions defined above, we can state our general duality models as follows:

$$(DIII) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s}) \in \mathbb{H}} \qquad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.1) and

$$f_i(y) - \lambda g_i(y) + \sum_{m \in J_0} v_m G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m H_{k_m}(y, s^m) \geqq 0, \; i \in \underline{p}, \tag{5.1}$$

$$\sum_{m \in J_\tau} v_m G_{j_m}\left(z, t^m\right) + \sum_{m \in k_\tau} v_m H_{k_m}\left(z, s^m\right) \geqq 0, \; \tau \in \underline{M}. \tag{5.2}$$

$$(\tilde{D}III) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s}) \in \mathbb{H}} \qquad \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.3), (5.1) and (5.2).

The remarks made earlier about the relationships between (DI) and $(\tilde{D}I)$ are, of course, also valid for $(DIII)$ and $(\tilde{D}III)$.

Theorem 5.1. *Let x and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $(\Phi_1(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Phi_p(.,u,v,\lambda,\bar{t},\bar{s}))$ *is* $(\theta,\eta,\bar{\rho})$*-*V*-pseudounivex at* y *with respect to* b,ϕ *and* η*;*

(ii) $(\Lambda_1(.,v,\bar{t},\bar{s}),\ldots,\Lambda_M(.,v,\bar{t},\bar{s}))$ *is* $(\pi,\eta,\tilde{\rho})$*-*V*-quasiunivex at* y *with respect to* b,ϕ *and* η*;*

(iii) $\bar{\rho}+\tilde{\rho} \geqq 0$*;*

(b) *(i)* $(\Phi_1(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Phi_p(.,u,v,\lambda,\bar{t},\bar{s}))$ *is prestrictly* $(\theta,\eta,\bar{\rho})$*-*V*-quasiunivex at* y *with respect to* b,ϕ *and* η*;*

(ii) $(\Lambda_1(.,v,\bar{t},\bar{s}),\ldots,\Lambda_M(.,v,\bar{t},\bar{s}))$ *is* $(\pi,\eta,\tilde{\rho})$*-*V*-quasiunivex at* y *with respect to* b,ϕ *and* η*;*

(iii) $\bar{\rho}+\tilde{\rho} > 0$*;*

(c) *(i)* $(\Phi_1(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Phi_p(.,u,v,\lambda,\bar{t},\bar{s}))$ *is prestrictly* $(\theta,\eta,\bar{\rho})$*-*V*-quasiunivex at* y *with respect to* b,ϕ *and* η*;*

(ii) $(\Lambda_1(.,v,\bar{t},\bar{s}),\ldots,\Lambda_M(.,v,\bar{t},\bar{s}))$ *is strictly* $(\pi,\eta,\tilde{\rho})$*-*V*-pseudounivex at* y *with respect to* b,ϕ *and* η*;*

(iii) $\bar{\rho}+\tilde{\rho} \geqq 0$*.*

Then, $\varphi(x) \nleq \lambda$ *.*

Proof. (a) It is clear that (3.1) can be expressed as follows:

$$\sum_{i=1}^{p} u_i \left[\nabla f_i(y) - \lambda_i \nabla g_i(y) + \sum_{m\in J_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in K_0} v_m \nabla H_{k_m}(y,s^m)\right] + \sum_{\tau=1}^{M}\left[\sum_{m\in J_\tau} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in K_\tau} v_m \nabla H_{k_m}(y,s^m)\right] = 0. \quad (5.3)$$

Since $x, x^* \in \mathbb{F}$ and $m \in \underline{\nu_0}$, and (5.2) hold, it follows that for each $\tau \in \underline{M}$,

$$\begin{aligned}\Lambda_\tau(x,v,\bar{t},\bar{s}) &= \sum_{m\in J_\tau} v_m G_{j_m}(x,t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(x,s^m) \leqq 0 \\ &= \sum_{m\in J_\tau} v_m G_{j_m}(y,t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(y,s^m) \\ &= \Lambda_\tau(y,v,\bar{t},\bar{s}),\end{aligned}$$

and hence

$$b(x,y)\,\phi\left[\sum_{\tau=1}^{M} \pi_\tau(x,y)\Lambda_\tau(x,v,\bar{t},\bar{s}) - \sum_{\tau=1}^{M} \pi_\tau(x,y)\Lambda_\tau(y,v,\bar{t},\bar{s})\right] \leqq 0,$$

which because of (ii) implies that

$$\left\langle \sum_{\tau=1}^{M}\left[\sum_{m\in J_\tau} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in K_\tau} v_m \nabla H_{k_m}(y,s^m)\right], \eta(x,y)\right\rangle \leqq -\tilde{\rho}\,\|x-y\|^2, \quad (5.4)$$

Combining (5.3) and (5.4), and using (iii) we get

$$\left\langle \sum_{i=1}^{p} u_i \left[\nabla f_i(y) - \lambda_i \nabla g_i(y) + \sum_{m\in J_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in K_0} v_m \nabla H_{k_m}(y,s^m) \right], \eta(x,y) \right\rangle \geqq \tilde{\rho}\|x-y\|^2 \geqq \bar{\rho}\|x-y\|^2, \quad (5.5)$$

which by virtue of (i) implies that

$$b(x,y)\,\phi\left[\sum_{i=1}^{p} \theta_i(x,y)\Phi_i(x,u,v,\lambda,\bar{t},\bar{s}) - \sum_{i=1}^{p} \theta_i(x,y)\Phi_i(y,u,v,\lambda,\bar{t},\bar{s})\right] \geqq 0. \quad (5.6)$$

Since $x \in \mathbb{F}, v_m > 0, m \in \underline{\nu_0}$, and (5.2) holds, it follows that for each

$$\sum_{i=1}^{p} u_i\theta_i(x,y)\,b(x,y)\,\phi\left[f_i(x) - \lambda_i g_i(x)\right] \geqq 0.$$

Now using this inequality, as in the proof of Theorem 3.1, we obtain $\varphi(x) \nleq \lambda$.

(b) Proceeding in exactly the same manner as in the proof of part (a), we obtain (5.5) in which the second inequality is strict. By (i), this implies that (5.6) holds and, therefore, the rest of the proof is identical to that of part (a).

(c) The proof is similar to those of parts (a) and (b).

The proof of the following theorem follows along the line of the proof of Theorem 3.2.

Theorem 5.2. *(Strong Duality). Let x^* be a normal efficient solution of (P). Assume that for each feasible solution $(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})$ of (DIII), any one of the three sets of conditions specified in Theorem 5.1 is satisfied. Then there exist $u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*$ such that $\left(x^*,u^*,v^*,\lambda^*,\nu^*,\nu_0^*,J_{\nu_0^*},K_{\nu^*\setminus\nu_0^*},\bar{t}^*,\bar{s}^*\right)$ is an efficient solution of (DIII) and $\varphi(x^*) = \lambda^*$.*

We also have the following converse duality result for (P)-(DIII).

Theorem 5.3. *(Strict Converse Duality). Let x^* and $\tilde{\omega} \equiv \left(\tilde{x},\tilde{u},\tilde{v},\tilde{\lambda},\tilde{\nu},\tilde{\nu}_0,J_{\tilde{\nu}_0},K_{\tilde{\nu}\setminus\tilde{\nu}_0},\tilde{\bar{t}},\tilde{\bar{s}}\right)$ be arbitrary feasible solutions of (P) and (DIII), respectively, such that*

$$\sum_{i=1}^{p} \tilde{u}_i b(x^*,\tilde{x})\,\phi\left[f_i(x^*) - \tilde{\lambda}_i g_i(x^*)\right] = 0. \quad (5.7)$$

Furthermore, assume that any one of the following three sets of hypotheses is satisfied:

(a) The assumptions specified in part (a) of Theorem 5.1 are satisfied for the feasible solution $\tilde{\omega}$, and the function $\left(\Phi_1\left(.,\tilde{u},\tilde{v},\tilde{\lambda},\tilde{\bar{t}},\tilde{\bar{s}}\right),\ldots,\Phi_p\left(.,\tilde{u},\tilde{v},\tilde{\lambda},\tilde{\bar{t}},\tilde{\bar{s}}\right)\right)$ is strictly $(\theta,\eta,\bar{\rho})$-V-pseudounivex at $\tilde{x}$ with respect to b,ϕ and η.

(b) *The assumptions specified in part (b) of Theorem 5.1 are satisfied for the feasible solution* $\tilde{\omega}$, *and the function* $\left(\Phi_1\left(., \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\tilde{t}}, \bar{\tilde{s}}\right), \ldots, \Phi_p\left(., \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)\right)$ *is* $(\theta, \eta, \bar{\rho})$-$V$-*quasiunivex at* $\tilde{x}$ *with respect to* b, ϕ *and* η.

(c) *The assumptions specified in part (c) of Theorem 5.1 are satisfied for the feasible solution* $\tilde{\omega}$, *and the function* $\left(\Phi_1\left(., \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\tilde{t}}, \bar{\tilde{s}}\right), \ldots, \Phi_p\left(., \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\tilde{t}}, \bar{\tilde{s}}\right)\right)$ *is* $(\theta, \eta, \bar{\rho})$-$V$-*quasiunivex at* $\tilde{x}$ *with respect to* b, ϕ *and* η.

Then, $\tilde{x} = x^*$ *and* $\varphi(x^*) = \tilde{\lambda}$.

Proof. The proof is similar to that of Theorem 3.3.

Each one of the three sets of results given in Theorem 5.1 can be viewed as a family of duality results whose members can easily be identified by appropriate choices of the partitioning sets and J_μ and $K_\mu, \mu \in \underline{M} \bigcup \{0\}$.

6 Duality Model IV

In this section, we discuss another collection of duality results for (P) which are different from those stated in Theorem 5.1. In the formulations of these duality results, we utilize a partition of $\underline{p}$ in addition to those of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$ It appears that this partitioning scheme was first proposed in [4] for a multiobjective fractional programming problem with a finite number of constraints. In our theorems, we impose appropriate generalized (α, η, ρ)-V-univexity requirements on certain vector functions whose components comprise some combinations of the functions involving $\varepsilon_i(., \lambda, u), i \in \underline{p}, G_j, j \in \underline{q}$ and $H_k, k \in \underline{r}$. Let $\{I_0, I_1, \ldots, I_d\}$, $\{J_0, J_1, \ldots, J_e\}$ and $\{K_0, K_1, \ldots, K_e\}$, be partitions of $\underline{p}, \underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, such that $D = \{0, 1, \ldots, d\} \subset E = \{0, 1, \ldots, e\}$, and let the function $\Pi_\tau(., u, v, \lambda, \bar{t}, \bar{s}) : \mathbb{R}^n \to \mathbb{R}$ be defined, for fixed $u, v, \lambda, \bar{t}$ and $\bar{s}$, by

$$\Pi_\tau(z, u, v, \lambda, \bar{t}, \bar{s}) = \sum_{i \in I_\tau} u_i\left[f_i(z) - \lambda_i g_i(z)\right] + \sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(z, s^m), \ \tau \in D.$$

Making use of the sets and functions defined above, we consider the following problems:

$$(DIV) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s}) \in \mathbb{H}} \lambda = (\lambda_1, \ldots, \lambda_p)$$

subject to (3.1) and

$$\sum_{i \in I_\tau} u_i\left[f_i(y) - \lambda g_i(y)\right] + \sum_{m \in J_\tau} v_m G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(y, s^m) \geqq 0, \ \tau \in D, \tag{6.1}$$

$$\sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in k_\tau} v_m H_{k_m}(z, s^m) \geqq 0, \ \tau \in E\backslash D. \tag{6.2}$$

$$(\tilde{D}IV) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \quad \lambda=(\lambda_1,\ldots,\lambda_p)$$

subject to (3.3), (6.1) and (6.2).

The remarks and observations made earlier about the relationships between (DI) and $(\tilde{D}I)$ are, of course, also valid for (DIV) and $(\tilde{D}IV)$.

The next two theorems show that (DIV) is a dual problem for (P).

Theorem 6.1. *Let x and $(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})$ be arbitrary feasible solutions of (P) and (DIV), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $(\Pi_0(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Pi_d(.,u,v,\lambda,\bar{t},\bar{s}))$ *is $(\theta,\eta,\bar{\rho})$-V-pseudounivex at y with respect to b,ϕ and η;*

(ii) $(\Lambda_{d+1}(.,v,\bar{t},\bar{s}),\ldots,\Lambda_e(.,v,\bar{t},\bar{s}))$ *is $(\pi,\eta,\tilde{\rho})$-V-quasiunivex at y with respect to b,ϕ and η;*

(iii) $\bar{\rho}+\tilde{\rho}\geqq 0$;

(b) *(i)* $(\Pi_0(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Pi_d(.,u,v,\lambda,\bar{t},\bar{s}))$ *is prestrictly $(\theta,\eta,\bar{\rho})$-V-quasiunivex at y with respect to b,ϕ and η;*

(ii) $(\Lambda_{d+1}(.,v,\bar{t},\bar{s}),\ldots,\Lambda_e(.,v,\bar{t},\bar{s}))$ *is strictly $(\pi,\eta,\tilde{\rho})$-V-pseudounivex at y with respect to b,ϕ and η;*

(iii) $\bar{\rho}+\tilde{\rho}\geqq 0$;

(c) *(i)* $(\Pi_0(.,u,v,\lambda,\bar{t},\bar{s}),\ldots,\Pi_d(.,u,v,\lambda,\bar{t},\bar{s}))$ *is prestrictly $(\theta,\eta,\bar{\rho})$-V-quasiunivex at y with respect to b,ϕ and η;*

(ii) $(\Lambda_{d+1}(.,v,\bar{t},\bar{s}),\ldots,\Lambda_e(.,v,\bar{t},\bar{s}))$ *is $(\pi,\eta,\tilde{\rho})$-V-quasiunivex at y with respect to b,ϕ and η;*

(iii) $\bar{\rho}+\tilde{\rho}>0$.

Then, $\varphi(x)\nleq\lambda$.

Proof. (a) Suppose to the contrary that $\varphi(\bar{x})\leq\lambda$. Then

$$f_i(x)-\lambda_i g_i(x)\leqq 0,\ i\in\underline{p},$$

with strict inequality holding for at least one index $i\in\underline{p}$. Since $u>0$, we see that for each $\tau\in D$,

$$\sum_{i\in I_\tau} u_i\left[f_i(x)-\lambda_i g_i(x)\right]\leqq 0, \tag{6.3}$$

with strict inequality holding for at least one index $\tau\in D$.

Now using this inequality, we see that

$$\begin{aligned}
&\Pi_\tau\left(x,u,v,\lambda,\bar{t},\bar{s}\right) \\
&\quad = \sum_{i\in I_\tau} u_i\left[f_i(x)-\lambda_i g_i(x)\right] + \sum_{m\in J_\tau} v_m G_{j_m}\left(x,t^m\right) + \sum_{m\in K_\tau} v_m H_{k_m}\left(x,s^m\right) \\
&\leqq \sum_{i\in I_\tau} u_i\left[f_i(x)-\lambda_i g_i(x)\right] \ \left(by\ the\ feasibility\ of\ x\ and\ positivity\ of v_m, m\in \underline{\nu_0}\right) \\
&\qquad\qquad \leqq 0,\ (by (6.3)) \\
&\quad = \sum_{i\in I_\tau} u_i\left[f_i(y)-\lambda_i g_i(y)\right] + \sum_{m\in J_\tau} v_m G_{j_m}\left(y,t^m\right) + \sum_{m\in K_\tau} v_m H_{k_m}\left(y,s^m\right) \\
&\qquad\qquad = \Pi_\tau\left(y,u,v,\lambda,\bar{t},\bar{s}\right),
\end{aligned}$$

with strict inequality holding for at least one index $\tau \in D$, and hence

$$b\left(x,y\right)\phi\left[\sum_{\tau\in D}\theta_\tau(x,y)\Pi_\tau\left(x,u,v,\lambda,\bar{t},\bar{s}\right) - \sum_{\tau\in D}\theta_\tau(x,y)\Pi_\tau\left(y,u,v,\lambda,\bar{t},\bar{s}\right)\right] < 0,$$

which in view of (i) implies that

$$\left\langle \sum_{i=1}^{p} u_i\left[\nabla f_i\left(y\right)-\lambda_i\nabla g_i\left(y\right)\right] + \sum_{\tau\in D}\left[\sum_{m\in J_\tau} v_m\nabla G_{j_m}\left(y,t^m\right) + \sum_{m\in K_\tau} v_m\nabla H_{k_m}\left(y,s^m\right)\right] \right.$$
$$\left. ,\eta\left(x,y\right)\right\rangle < -\bar{\rho}\left\|\bar{x}-y\right\|^2. \quad (6.4)$$

As shown in the proof of Theorem 5.1, for each $\tau \in E\backslash D$, $\Lambda_\tau\left(x,v,\bar{t},\bar{s}\right) \leqq \Lambda_\tau\left(y,v,\bar{t},\bar{s}\right)$, and hence

$$b\left(x,y\right)\phi\left[\sum_{\tau\in E\backslash D}\pi_\tau(x,y)\Lambda_\tau\left(x,v,\bar{t},\bar{s}\right) - \sum_{\tau\in E\backslash D}\pi_\tau(x,y)\Lambda_\tau\left(y,v,\bar{t},\bar{s}\right)\right] \leqq 0,$$

which in view of (ii) implies that

$$\left\langle \sum_{\tau\in D}\left[\sum_{m\in J_\tau} v_m\nabla G_{j_m}\left(y,t^m\right) + \sum_{m\in K_\tau} v_m\nabla H_{k_m}\left(y,s^m\right)\right],\eta\left(x,y\right)\right\rangle \leqq -\tilde{\rho}\left\|x-y\right\|^2. \quad (6.5)$$

Now combining (6.4) and (6.5) and using (iii), we see that

$$\left\langle \sum_{i=1}^{p} u_i\left[\nabla f_i\left(y\right)-\lambda_i\nabla g_i\left(y\right)\right] + \sum_{m=1}^{\nu_0} v_m\nabla G_{j_m}\left(y,t^m\right) + \sum_{m\nu_0+1}^{\nu} v_m\nabla H_{k_m}\left(y,s^m\right),\eta\left(x,y\right)\right\rangle$$
$$< -\left(\bar{\rho}+\tilde{\rho}\right)\left\|x-y\right\|^2 \leqq 0,$$

which contradicts (3.1). Therefore, $\varphi\left(x\right) \nleq \lambda$.
(b) and (c): The proofs are similar to that of part (a).

□

The proof of the following theorem follows along the line of the proof of Theorem 3.2.

Theorem 6.2. *(Strong Duality). Let x^* be a normal efficient solution of (P). Assume that for each feasible solution $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\setminus\nu_0}, \bar{t}, \bar{s})$ of (DIV), any one of the three sets of conditions specified in Theorem 6.1 is satisfied. Then there exist $u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\setminus\nu_0^*}, \bar{t}^*, \bar{s}^*$ such that $\left(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\setminus\nu_0^*}, \bar{t}^*, \bar{s}^*\right)$ is an efficient solution of (DIV) and $\varphi(x^*) = \lambda^*$.*

7 Concluding Remarks

In this paper, we have established several weak, strong and converse duality results under various generalized (α, η, ρ)-V-univexity hypotheses for a semiinfinite multiobjective fractional programming problem. It indicates that all these results are new in the area of semiinfinite programming.

References

[1] Bector, C.R., Suneja, S.K. and Gupta, S. Univex functions and Univex nonlinear programming, In: Proceedings of the Administrative Sciences Association of Canada, 115-124 (1992).

[2] Mond, B., Weir, T. Generalized concavity and duality. In: Generalized Concavity in Optimization and Economics, ed. by Schaible, S, Ziemba, W.T., Academic Press, New York, 1981.

[3] Mishra, S.K., Porwal, S.K. On sufficiency for semiinfinite multiobjective fractional programming problems using generalized (α, η, ρ)-V-univex functions. Communications on Applied Nonlinear Analysis, 22(4): 29-48 (2015).

[4] Yang, X. Generalized convex duality for multiobjective fractional programs. Opsearch, 31: 155-163 (1994).

[5] Zalmai, G.J., Zhang, Q. Global semiparametric sufficient efficiency conditions for semiinfinite multiobjective fractional programming problems involving generalized (α, η, ρ)-V-invex functions. Southeast Asian Bull.Math., 32: 573-599 (2008).

[6] Zalmai, G.J., Zhang, Q. Global parametric sufficient efficiency conditions for semiinfinite multiobjective fractional programming problems containing generalized (α, η, ρ)-V-invex functions. Acta Math. Appl. Sinica, 29(1): 63-78 (2012).

[7] Zalmai, G.J., Zhang, Q. Parametric duality models for semiinfinite multiobjective fractional programming problems containing generalized (α, η, ρ)-V-invex functions. Acta Math. Appl. Sinica, 29(2): 225-240 (2013).