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A Technical Approach to Solve A Fully Fuzzy Linear system using Hukuhara Difference

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Abstract

A fully fuzzy linear equation system is formulated by two addition and Hukuhara difference operators, is considered in this paper. Fuzzy numbers are applied in α-cuts representation. A generalized crisp system is proposed and solved to find a fuzzy solution for the original system. Numerical results confirm applicability of our technique

Keywords: *Fully fuzzy linear system, Fuzzy Solution, Hukuhara Difference, invers operator.*

1 Introduction

A wide range of life problems have been formulated as a linear equation system in which parameters are vague. Zadeh was the first researcher who introduced Fuzzy set [14]. Many researches have studied on fuzzy and especially on fuzzy systems. How to solve a fuzzy linear system is still an open question. This question comes from other famous questions like How to define equality and inequality in fuzzy space? How to subtract two fuzzy numbers in the absence of invers operator? How to compare two fuzzy numbers? There are many proposed and useful approach [1,2,3,4,5,6,7,8,10,13] to understand these wonderful space. Our proposed approach is to apply Hukuhara difference[11, 12] which is an alternative to classic difference. The structure of this paper is as follows: Section 2 presents a brief overview of necessary concepts and definitions. Section 3 introduced our proposed approach. Section 4, concentrates on a numerical example

2 Preliminaries

In this section we give some definitions and preliminaries in which needed in next sections.

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Definition 2.1 Let X denote a universal set. A fuzzy subset A of X is defined as a set of ordered pairs of element x and grade $\mu_A(\mathrm{x})$ and is written $A=\{(\mathrm{x},\mu_A(\mathrm{x})):\mathrm{x}\in \mathrm{X}\}$ where $\mu_A(\mathrm{x})$ is membership function from X to [0,1].

Definition 2.2 The α–cut set of a fuzzy set A is defined as an ordinary set A_α where $A_\alpha=\{\mathrm{x}:\mu_A(\mathrm{x})\geq\alpha\,;\alpha\in[0,1]\}$.

Among the various shapes of fuzzy number, triangular fuzzy number (TFN) is the most popular one.

Definition 2.3 (Triangular fuzzy number) A fuzzy number is represented with three points $A=(\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3)$.

Fuzzy arithmetic is based on two extension principle and α - cuts. We deal with intervals when fuzzy numbers are represented by α - cuts. A crisp interval is obtaine from a fuzzy number by the following operations. If

$$\forall\alpha\in[0,1];\quad \frac{a_1^{(\alpha)}-a_1}{a_2-a_1}=\alpha,\ \frac{a_3-a_3^{(\alpha)}}{a_3-a_2}=\alpha$$

we have

$$\begin{aligned} a_1^{(\alpha)}&=(a_2-a_1)\alpha+a_1 \\ a_3^{(\alpha)}&=-(a_3-a_2)\alpha+a_3 \end{aligned}\qquad\Rightarrow\qquad A=[a_1^{(\alpha)},a_3^{(\alpha)}] \tag{1}$$

Definition 2.4 [11,12] Let $A=[\underline{a},\overline{a}]$ and $B=[\underline{b},\overline{b}]$ be two crisp intervals, the *H- difference* is

$$A-_H B=[\underline{a},\overline{a}]-_H[\underline{b},\overline{b}]=[\underline{c},\overline{c}]\Leftrightarrow\begin{cases}\underline{a}=\underline{b}+\underline{c}\\ \overline{a}=\overline{b}+\overline{c}\end{cases}$$

Note Although the classic difference operator for intervals is not associative, it can be simply proved that H- difference has this valuable property.

3 Generalized Fully Fuzzy Linear System(FFLS)

A fully fuzzy linear system (FFLS) in defined as follows:

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1\, o_{12}\ \tilde{a}_{12}\tilde{x}_2\, o_{13} \dots\ \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1\, o_{22}\ \tilde{a}_{22}\tilde{x}_2\, o_{23} \dots\ \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \qquad\qquad \dots. \\ \tilde{a}_{n1}\tilde{x}_1\, o_{12}\ \tilde{a}_{n2}\tilde{x}_2\, o_{n3} \dots\ \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n \end{cases} \qquad (2)$$

where $\tilde{a}_{i,j}, 1 \le i, j \le n$ are fuzzy numbers, $o_{i,j}$ is a notation for operators, either addition or Hukuhara difference, $\tilde{b}$ is a vector of fuzzy numbers and $\tilde{x}_{i,j}$ are fuzzy variables. System (2) is called Generalize fuzzy linear system in order to the presence of Hukuhara difference as a operator.

According to relation (1) for given α, system (2) is converted to the following interval system:

$$\begin{cases} [\underline{a},\overline{a}]_{11}[\underline{x},\overline{x}]_1\, o_{12}\ [\underline{a},\overline{a}]_{12}[\underline{x},\overline{x}]_2\, o_{13} \dots\ [\underline{a},\overline{a}]_{1n}[\underline{x},\overline{x}]_n = [\underline{b},\overline{b}]_1 \\ [\underline{a},\overline{a}]_{21}[\underline{x},\overline{x}]_1\, o_{22}\ [\underline{a},\overline{a}]_{22}[\underline{x},\overline{x}]_2\, o_{23} \dots\ [\underline{a},\overline{a}]_{2n}[\underline{x},\overline{x}]_n = [\underline{b},\overline{b}]_2 \\ \qquad\qquad \dots. \\ [\underline{a},\overline{a}]_{n1}[\underline{x},\overline{x}]_1\, o_{12}\ [\underline{a},\overline{a}]_{n2}[\underline{x},\overline{x}]_2\, o_{n3} \dots\ [\underline{a},\overline{a}]_{nn}[\underline{x},\overline{x}]_n = [\underline{b},\overline{b}]_n \end{cases} \qquad (3)$$

Definition 2.5 Corresponds to (3), for all $1 \le i, j \le n$ we define a coefficients matrix $A = [\mathrm{a}_{i,j}]_{n\times n}$,an operator matrix $O = [\mathrm{o}_{i,j}]_{n\times n}$, where $o_{i,j} \in \{+, -_H\}$ and a representation matrix $A^o = [\mathrm{a}^1, \mathrm{o}^2, \mathrm{a}^2, \mathrm{o}^3, \dots, \mathrm{o}^n, \mathrm{a}^n]$ in which a^j and o^j are the columns of matrices A and O, respectively.

For A^o , a crisp matric $\overline{O} = [\overline{o}_{ij}]$ is defined as follows

$$\overline{o}_{ij} = \begin{cases} +, & if\ o_{ij} = + \\ -, & if\ o_{ij} = -_H \end{cases} ; 1 \le i, j \le n.$$

We extend the generalize FLS to a $2n \times 2n$ crisp matrix as follows:

$$\begin{cases} \bar{a}_{11}\underline{x}_1(\alpha)\bar{o}_{12}\ \ \bar{a}_{12}\underline{x}_2(\alpha)\bar{o}_{13}\dots\bar{o}_{1n}\ \ \bar{a}_{1n}\underline{x}_n(\alpha)=\underline{b}_1(\alpha) \\ \bar{a}_{21}\underline{x}_1(\alpha)\bar{o}_{22}\ \ \bar{a}_{22}\underline{x}_2(\alpha)\bar{o}_{23}\dots\bar{o}_{2n}\ \ \bar{a}_{2n}\underline{x}_n(\alpha)=\underline{b}_2(\alpha) \\ \dots \\ \bar{a}_{n1}\underline{x}_1(\alpha)\bar{o}_{n2}\ \ \bar{a}_{n2}\underline{x}_2(\alpha)\bar{o}_{n3}\dots\bar{o}_{nn}\ \ \bar{a}_{nn}\underline{x}_n(\alpha)=\underline{b}_n(\alpha) \\ \\ \underline{a}_{11}\bar{x}_1(\alpha)\bar{o}_{12}\ \ \underline{a}_{12}\bar{x}_2(\alpha)\bar{o}_{13}\dots\bar{o}_{1n}\ \ \underline{a}_{1n}\bar{x}_n(\alpha)=\bar{b}_1(\alpha) \\ \underline{a}_{21}\bar{x}_1(\alpha)\bar{o}_{22}\ \ \underline{a}_{22}\bar{x}_2(\alpha)\bar{o}_{23}\dots\bar{o}_{2n}\ \ \underline{a}_{2n}\bar{x}_n(\alpha)=\bar{b}_2(\alpha) \\ \dots \\ \underline{a}_{n1}\bar{x}_1(\alpha)\bar{o}_{n2}\ \ \underline{a}_{n2}\bar{x}_2(\alpha)\bar{o}_{n3}\dots\bar{o}_{nn}\ \ \underline{a}_{nn}\bar{x}_n(\alpha)=\bar{b}_n(\alpha) \end{cases}$$

Equation can be simplified as follows

$$\begin{cases} \sum_{j=1}^{n} d_{ij}\underline{x}_j(\alpha)+\sum_{j=1}^{n} d_{i(j+n)}\bar{x}_j(\alpha)=\underline{b}_i(\alpha) \\ \sum_{j=1}^{n} d_{(i+n)j}\underline{x}_j(\alpha)+\sum_{j=1}^{n} d_{(i+n)(j+n)}\bar{x}_j(\alpha)=\bar{b}_i(\alpha) \end{cases} ;0\le i\le n$$

in which

$$\begin{cases} d_{ij}=d_{(i+n)(j+n)}=\bar{a}_{ij} \\ d_{(i+n)j}=d_{i(j+n)}=0 \end{cases};\quad if\ \ \bar{o}_{ij}=+\ and\ \bar{a}_{ij}>0$$

$$\begin{cases} d_{ij}=d_{(i+n)(j+n)}=0 \\ d_{(i+n)j}=d_{i(j+n)}=\bar{a}_{ij} \end{cases};\quad if\ \ \bar{o}_{ij}=+\ and\ \bar{a}_{ij}<0$$

$$\begin{cases} d_{ij}=d_{(i+n)(j+n)}=-\underline{a}_{ij} \\ d_{(i+n)j}=d_{i(j+n)}=0 \end{cases};\quad if\ \ \bar{o}_{ij}=-\ and\ \underline{a}_{ij}>0$$

$$\begin{cases} d_{ij}=d_{(i+n)(j+n)}=0 \\ d_{(i+n)j}=d_{i(j+n)}=-\underline{a}_{ij} \end{cases};\quad if\ \ \bar{o}_{ij}=-\ and\ \underline{a}_{ij}>0$$

The path we passed to reach D is to adopt and generalize what Freidman[9] proposed for solving a fuzzy linear system. Based on what Freidman and many other authors who followed him achieved, we employ α -cuts and L-R function for solving GFLS. Meanwhile we are interested to obtain more simple representation of $D=[\mathrm{d}_{ij}]_{2n\times 2n}$. To this end we prefer to define a binary matrix $P=[\mathrm{p}_{ij}]_{n\times n}$ where

$$p_{ij}=\begin{cases}1, & 1\le i\le n,\ j=1\\ 1, & o_{ij}=+, 1\le i\le n, 2\le j\le n\\ -1, & o_{ij}=-_H, 1\le i\le n, 2\le j\le n\end{cases}$$

Proposition 6. Let A be a coefficients matrix in a GFLS and P be its binary matrix. Then D can be computed as follows

$$D=\begin{bmatrix} A.P.\overline{S}(\mathrm{A}) & A.P.\underline{S}(-\mathrm{A})\\ A.P.\overline{S}(-\mathrm{A}) & A.P.\underline{S}(\mathrm{A})\end{bmatrix}$$

where . is Hadamard product and S is defined as

$$\begin{array}{ccc} \overline{S}:R^{n\times n}\to R^{n\times n} & & \underline{S}:R^{n\times n}\to R^{n\times n}\\ A\to\overline{S}(\mathrm{A}) & , & A\to\underline{S}(\mathrm{A})\\ (\overline{S}(\mathrm{A}))_{ij}=\begin{cases}1, \overline{a}_{ij}\ge 0\\ 0, \overline{a}_{ij}<0\end{cases} & & (\underline{S}(\mathrm{A}))_{ij}=\begin{cases}1, \underline{a}_{ij}\ge 0\\ 0, \underline{a}_{ij}<0\end{cases}\end{array}$$

If D is invertible, the solution of the system obtains as follows

$$x=[\underline{x};\overline{x}]=\mathrm{D}^{-1}[\underline{b};\overline{b}]$$

4 Example

Let

$$\begin{cases}[-1+2\alpha,2-\alpha]x_1-_H[4+3\alpha,8-\alpha]x_2=[-8+\alpha,-5-2\alpha]\\ \quad [2+\alpha,5-2\alpha]x_1+[7+\alpha,9-\alpha]x_2=[10+4\alpha,16-2\alpha]\end{cases}$$

Table 1 represents the obtained results by proposed algorithm.

5 Conclusion

In this paper a fully fuzzy linear equation system is considered. Hukuhara difference operator is applied instead of classic difference operator and a numerical method is proposed to find α-cuts solution of the system for given. Meanwhile the main problem is transformed to a generalized crisp linear equation system.

Table 1. Obtaining lower and upper bound for the system solution for given α

α		$\underline{x}$	$\overline{x}$
$\alpha=0$	$\underline{A}$ **=[-1,4 ; 2,7]**		
	$\overline{A}$ **=[2,8; 5 9]**		
	$\underline{b}$ **=[-8;10]**	**-0.4892**	**1.3858**
	$\overline{b}$ **=[-5; 16]**	**0.5226**	**0.7101**
	$\underline{S}$ **=[0,1;1,1]**		
	$\overline{S}$ **=[1,1;1,1]**		
$\alpha=0.5$	$\underline{A}$ **=[0,5.5 ; 2.5,7.5]**		
	$\overline{A}$ **=[1.5,7.5; 4, 8.5]**		
	$\underline{b}$ **=[-7.5;12]**	**-0.2112**	**1.4555**
	$\overline{b}$ **=[-6; 15]**	**0.3410**	**0.8410**
	$\underline{S}$ **=[1,1;1,1]**		
	$\overline{S}$ **=[1,1;1,1]**		

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