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A Note on Dual Simplex Algorithm for Linear Programming Problem with Bounded Variables

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Abstract

The present paper presents a systematic technique to solve a linear programming problem with bounded variables using dual simplex method in the case when a starting dual feasible solution is not readily available. The proposed technique involves the formulation of an augmented problem which in turn solves the original problem. Some particular cases of linear programming problem with bounded variables are discussed thereafter. Numerical illustrations are included in support of the theory.

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1 Introduction

Operations research is a discipline that deals with the application of advanced analytical methods to help make better decisions. Employing techniques from mathematical sciences such as mathematical modelling, statistical analysis and mathematical optimization, operations research arrives at optimal or near-optimal solutions to complex decision-making problems. Within this very broad subject area of optimization the most widely known and implemented technique for modelling and simulation is the methodology of linear programming. Linear programming is perhaps the core model of constrained optimization. Dantzig (1951) invented the simplex method which for the first time efficiently solved the linear programming problem (LPP) in most of the cases. Khachiyan (1980) introduced ellipsoid method and Karmarkar (1984) proposed a projective method for linear programming which were of landmark importance for establishing the polynomial-time solvability of linear programs. Lemke (1954) gave a new method called dual simplex method which can be described as a mirror image of the simplex method. While implementing the linear programming models into real life situations, one or more unknown variables are sometimes constrained by lower as well as upper bound conditions. Therefore, it is worthwhile to study linear programming problems with bounded variables. Linear programming problem with bounded variables (LPPBV) has been considered by many researchers (Dantzig, 1955; Wagner, 1958; Eisemann, 1964; Duguay, 1973; Murty, 1983; Xia and Wang, 1995; Maros, 2003a; Maros, 2003b; Dahiya, 2006; Chowdhary and Ahmad, 2012). LPPBV is a special type of linear programming problem in which one or more decision variables have lower as well as upper bounds. LPPBV can be mathematically stated as follows.

$$
\begin{aligned}
(P) \qquad & \max\ Z = c^T x = \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \qquad & Ax \le b \\
& l_j \le x_j \le u_j\ \forall\ j = 1, 2, ..., n_1 \\
& l_j \le x_j < \infty\ \forall\ j = n_1 + 1, ..., n
\end{aligned}
\tag{1}
$$

where $c = (c_1, c_2, ..., c_n)^T \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and $b = (b_1, b_2, ..., b_m)^T \in \mathbb{R}^m$.
Here, $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ with rank as m. It is assumed (without any loss of generality) that $l_j,\ u_j \ge 0\ \forall\ j$.
Any LPPBV in general form (1) can be easily expressed in the following canonical form.

$$
\begin{aligned}
(P)\ \Leftrightarrow \qquad & \max\ Z = c^T x = \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \qquad & (A\ \ I) \begin{pmatrix} x \\ x_s \end{pmatrix} = b \\
& l_j \le x_j \le u_j\ \forall\ j = 1, 2, ..., n_1 \\
& l_j \le x_j < \infty\ \forall\ j = n_1 + 1, ..., n
\end{aligned}
\tag{2}
$$

where $x_s = (x_{n+1}, x_{n+2}, ..., x_{n+m})^T \in \mathbb{R}^m$ is a vector of slack variables.
Upper bounding technique (Dantzig, 1955), based on simplex method is one of the earliest technique used to solve LPPBV. Sometimes it is much time taking to solve the LPPBV using simplex like technique because of the difficulty arising in finding the starting feasible solution due to the bound restrictions. Due to this, the dual simplex method for solving LPPBV has attracted considerable attention among the researchers (Wagner, 1958; Kostina, 2002; Huang, 2002; Maros, 2003a,b; Dahiya, 2006) over the years. Maros (2003a,b) presented dual simplex method to necessitate the design and implementation of a version of dual simplex algorithm to the problems where variables of arbitrary type are allowed. In particular, Maros' discussion treated bounded primal variables efficiently by using the concept of bound swap. However, in order to apply dual simplex algorithm, one requires a dual feasible solution to begin with. A dual feasible solution refers to a point 'x' which is a feasible solution of the dual of the given primal problem. Equivalently, a dual feasible solution is a point 'x' which is an optimal solution of the primal problem. In order to solve LPPBV by dual simplex method, the starting dual feasible solution may not be obtained by the technique employed for the case of LPP with only non-negative variables. This motivated us to investigate the case in LPPBV, where a starting dual feasible solution is not readily available and develop a systematic technique for solving it in such a case.
The current paper is organized as follows. Section 2 contains preliminaries, discussing the brief outline of the dual simplex algorithm for solving LPPBV. Section 3 develops a technique for solving LPPBV using dual simplex method when a starting dual feasible solution is not available and also discusses the situation for two particular cases of LPPBV. Section 4 presents numerical examples in support of theory. In the end, conclusions are drawn in Section 5.

2 Dual Simplex Algorithm for Bounded Variable LPPs

This section briefly discusses the dual simplex algorithm for solving an LPPBV. For the simplicity of notations, the canonical form (2) of LPPBV is restated in the following form (3) in this section.

$$\begin{aligned} & \max\ Z = c^T x \\ \text{subject to}\quad & Ax = b \\ & l_j \le x_j \le u_j\ \forall\ j = 1, 2, ..., n_1 \\ & l_j \le x_j < \infty\ \forall\ j = n_1 + 1, ..., n \end{aligned} \tag{3}$$

2.1 Notations

$J = \{1, 2, ..., n\}$.
$B = \{b_1, b_2, ..., b_m\}$ is the index set of basic variables.
$N_1 = \{j \in J \setminus B$ such that $x_j = l_j\}$ is the index set of non-basic variables which are at their lower bounds.
$N_2 = \{j \in J \setminus B$ such that $x_j = u_j\}$ is the index set of non-basic variables which are at their upper bounds.

$A = (A_B \ A_{N_1} \ A_{N_2})$ where A_B, A_{N_1} and A_{N_2} are the sub matrices of A corresponding to the index sets B, N_1 and N_2 respectively.
Then, $Ax = b \Rightarrow A_B x_B + A_{N_1} x_{N_1} + A_{N_2} x_{N_2} = b$ where x_B, x_{N_1} and x_{N_2} are the vectors of variables corresponding to the index sets B, N_1 and N_2 respectively and $x = (x_B, \ x_{N_1}, \ x_{N_2})$ is the basic solution.
$y_j = A_B^{-1} a_j \ \forall \ j \in J$ (A_B is non-singular being the basis matrix).
$c = (c_B, \ c_{N_1}, \ c_{N_2})$ is the corresponding partition of c such that $c^T x = c_B^T x_B + c_{N_1}^T x_{N_1} + c_{N_2}^T x_{N_2}$ and $Z_j - c_j = c_B^T y_j - c_j \ \forall \ j \in J$.

2.2 Optimality criteria for LPPBV

Following theorem states the optimality criteria when an LPPBV is solved using simplex method.

Theorem 1. *A basic feasible solution $x = (x_B \ x_{N_1} \ x_{N_2})^T$ will be an optimal basic feasible solution of the problem* (3) *iff $Z_j - c_j \geq 0 \ \forall \ j \in N_1$ and $Z_j - c_j \leq 0 \ \forall \ j \in N_2$.*

Proof. Proof Refer to Murty (1983). □

Note 1. A basic solution of an LPPBV which satisfies the optimality criteria as given in Theorem 1 is called a primal optimal or in other words, a dual feasible solution. The corresponding basis A_B is called a dual feasible basis. If such a solution is a feasible solution of primal problem as well, then it is a primal as well as dual feasible solution.

The basic idea of dual simplex approach is to begin with an initial dual feasible but primal infeasible basis and then traverse through adjacent dual feasible basic solutions to a terminal basis which is primal feasible as well as dual feasible.

2.3 Algorithm

The main steps of the dual simplex algorithm (Dahiya, 2006; Maros, 2003b) for solving LPPBV are as follows:
Step I. Start with a dual feasible basis and create a corresponding simplex tableau.

Step II. If all basic variables are within bounds then the process must be terminated since the current basic vector x_B is an optimal solution. Otherwise pick the basic variable which is not within its bounds. Let it be x_{B_r} and corresponding column of A_B be a_{b_r}. Let $a_{b_r} = a_r$. If x_{B_r} is not unique, then select x_{B_r} which deviates maximum from its nearest bound. Depart x_{B_r} and enter some non-basic variable, say x_j, which is selected as in Step III.

Step III. If x_{B_r} is below its lower bound, i.e., $x_{B_r} < l_{B_r}$, then select entering variable x_j corresponding to which the ratio $\dfrac{Z_j - c_j}{y_{rj}}$ is as follows:

$$\frac{Z_j - c_j}{y_{rj}} = \min \left\{ \left| \frac{Z_j - c_j}{y_{rk}} \right| : y_{rk} < 0, k \in N_1 : y_{rk} > 0, k \in N_2 \right\}$$

Here, if $y_{rk} \geq 0 \; \forall \; k \in N_1$ and $y_{rk} \leq 0 \; \forall \; k \in N_2$, then there does not exist any feasible solution to the given problem.
On the other hand, if x_{B_r} is above its upper bound, i.e., $x_{B_r} > u_{B_r}$, then x_j is to be selected for which the ratio $\dfrac{Z_j - c_j}{y_{rj}}$ is as follows:

$$\frac{Z_j - c_j}{y_{rj}} = \min\left\{\left|\frac{Z_j - c_j}{y_{rk}}\right| : y_{rk} > 0, k \in N_1 : y_{rk} < 0, k \in N_2\right\}$$

If $y_{rk} \leq 0 \; \forall \; k \in N_1$ and $y_{rk} \geq 0 \; \forall \; k \in N_2$, then there does not exist any feasible solution to the given problem.
Note that such a choice of entering variable maintains the optimality of the basic solution.

Step IV. Once a_j is selected, then obtain the new simplex tableau having $\hat{x}_B$ as the basic solution.
If $x_{B_r} < l_{B_r}$, then

$$\hat{x}_B = \begin{cases} \hat{x}_{B_i} = x_{B_i} - \Delta_j y_{ij} & \forall \; i = 1, 2, ..., m; \; i \neq r \\ \hat{x}_{B_r} = \hat{x}_j = l_j + \Delta_j & \qquad\qquad\qquad\qquad \text{if } j \in N_1 \\ \hat{x}_r = l_{B_r} \end{cases}$$

$$\text{and} \qquad \hat{x}_B = \begin{cases} \hat{x}_{B_i} = x_{B_i} + \Delta_j y_{ij} & \forall \; i = 1, 2, ..., m; \; i \neq r \\ \hat{x}_{B_r} = \hat{x}_j = u_j - \Delta_j & \qquad\qquad\qquad\qquad \text{if } j \in N_2 \\ \hat{x}_r = l_{B_r} \end{cases}$$

where $\Delta_j = \dfrac{l_{B_r} - x_{B_r}}{|y_{rj}|}$ and rest of the non-basic variables remain same.

If $x_{B_r} > u_{B_r}$, then

$$\hat{x}_B = \begin{cases} \hat{x}_{B_i} = x_{B_i} - \Delta_j^{'} y_{ij} & \forall \; i = 1, 2, ..., m; \; i \neq r \\ \hat{x}_{B_r} = \hat{x}_j = l_j + \Delta_j^{'} & \qquad\qquad\qquad\qquad \text{if } j \in N_1 \\ \hat{x}_r = u_{B_r} \end{cases}$$

$$\text{and} \qquad \hat{x}_B = \begin{cases} \hat{x}_{B_i} = x_{B_i} + \Delta_j^{'} y_{ij} & \forall \; i = 1, 2, ..., m; \; i \neq r \\ \hat{x}_{B_r} = \hat{x}_j = u_j - \Delta_j^{'} & \qquad\qquad\qquad\qquad \text{if } j \in N_2 \\ \hat{x}_r = u_{B_r} \end{cases}$$

where $\Delta_j^{'} = \dfrac{x_{B_r} - u_{B_r}}{|y_{rj}|}$ and rest of the non-basic variables remain same.

Update the corresponding rows of the tableau by applying the pivot operations as in traditional simplex algorithm. Calculate $\hat{Z}_j - \hat{c}_j \; \forall \; j$.

Step V. In the new simplex tableau, if all $\hat{x}_{B_i}$ are with in their bounds, then the optimal basic feasible solution is found. Otherwise, repeat the process until either an optimal feasible solution has been obtained (in a finite number of steps) or there is an indication of non-existence of a primal feasible solution.

3 Method to solve LPPBV when a starting dual feasible solution is not available

Dual simplex algorithm begins with an in hand dual feasible basis (ref. Step I of Algorithm presented in Subsection 2.3) and then traverses to a terminal basis which is primal as well as dual feasible. However, if an initial dual feasible basis is not available, then it is not possible to apply dual simplex algorithm to the given problem directly. This section discusses such a case in detail for an LPPBV.

3.1 Theoretical development

LPPBV in canonical form (2) can be re-written as follows.

$$\max\ Z \qquad \text{where}\ \ Z = c^T x = \sum_{j=1}^{n} c_j x_j \tag{4}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i,\ \ i = 1, 2, ..., m \tag{5}$$

$$l_j \leq x_j \leq u_j\ \forall\ j = 1, 2, ..., n_1 \tag{6}$$

$$l_j \leq x_j < \infty\ \forall\ j = n_1 + 1, ..., n \tag{7}$$

Let $B = \{n+1, n+2, ..., n+m\}$ be the index set of basic variables so that $A_B = (a_{n+1}\ a_{n+2}\ ...\ a_{n+m}) = I_{m \times m}$ is the basis matrix. Let $N = \{1, 2, ..., n\}$ be the index set of non-basic variables out of which $N_1 = \{j \in N : x_j = l_j\}$ and $N_2 = \{j \in N : x_j = u_j\}$ are the index sets of non-basic variables which are at their lower and upper bounds respectively.
Let x_B be the basic solution yielded by the above choice of A_B, A_{N_1} and A_{N_2}. The case of interest occurs if x_B is neither a dual feasible nor a primal feasible basic solution.
The next subsection discusses the augmentation of an artificial constraint to the original problem which leads to a dual feasible solution of the augmented problem.

3.1.1 Formulation of an augmented problem P_{aug}

Consider the constraint

$$\sum_{j \in N_1} (x_j - l_j) + \sum_{j \in N_2} (u_j - x_j) \leq M \tag{8}$$

where $M > 0$ is a sufficiently large number, larger than any finite number with which it will be compared in the computations.
Since the constraint added depends on the choice of B, N_1 and N_2, therefore, the added constraint is a solution dependent constraint.

Equivalently, the constraint (8) can be written as

$$\sum_{j\in N_1}(x_j-l_j)+\sum_{j\in N_2}(u_j-x_j)+x_o=M \quad (x_o\geq 0) \tag{9}$$

or

$$\sum_{j\in N_1}x_j-\sum_{j\in N_2}x_j+x_o=M_1 \quad (x_o\geq 0) \tag{10}$$

where $M_1=M+\sum_{j\in N_1}l_j-\sum_{j\in N_2}u_j>0$ is again a sufficiently large number. Now, consider the augmented problem $\widehat{P}$ obtained by adding constraint (8) to problem P.

$$(\widehat{P}) \qquad \max Z=c^Tx=\sum_{j=1}^{n}c_jx_j$$

subject to the constraints (5), (6), (7) and (9).
Clearly, a basic solution to $\widehat{P}$ is given by

$$\begin{aligned} x_{n+i}&=b_i\ \forall\ i=1,2,...,m,\\ x_o&=M,\\ x_j&=l_j\ \forall\ j\in N_1,\\ \text{and}\ \ x_j&=u_j\ \forall\ j\in N_2. \end{aligned}$$

in which $x_{n+1},...,x_{n+m},x_o$ are the basic variables.
Next, search for k such that

$$|c_k|=\max\{\ |c_j|:c_j>0,\ j\in N_1;\ c_j<0,\ j\in N_2\} \tag{11}$$

Two cases arise. Either $k\in N_1$ or $k\in N_2$.

Case 1. $k\in N_1$
Then, (10) implies

$$x_k=M_1-x_o-\sum_{\substack{j\in N_1\\ j\neq k}}x_j+\sum_{j\in N_2}x_j \tag{12}$$

Replacing x_k by its value, (4) reduces to

$$Z=c_kM_1-c_kx_o+\sum_{\substack{j\in N_1\\ j\neq k}}(c_j-c_k)x_j+\sum_{j\in N_2}(c_j+c_k)x_j \tag{13}$$

and (5) reduces to

$$\Leftrightarrow\ -a_{ik}x_o+\sum_{\substack{j\in N_1\\ j\neq k}}(a_{ij}-a_{ik})x_j+\sum_{j\in N_2}(a_{ij}+a_{ik})x_j+x_{n+i}=b_i-a_{ik}M_1,\ i=1,2,...,m \tag{14}$$

Then, the augmented problem $\widehat{P}$ is equivalent to the augmented problem P_{aug}.

$$(P_{aug}) \quad \max\ Z=c_kM_1-c_kx_o+\sum_{\substack{j\in N_1\\ j\neq k}}(c_j-c_k)x_j+\ \sum_{j\in N_2}(c_j+c_k)x_j$$

subject to the constraints (6), (7), (10) and (14).

Case 2. $k \in N_2$
In this case, (10) implies

$$x_k = -M_1 + x_o + \sum_{j\in N_1} x_j - \sum_{\substack{j\in N_2\\ j\neq k}} x_j \tag{15}$$

Substituting this value of x_k, (4) reduces to

$$Z = -c_k M_1 + c_k x_o + \sum_{j\in N_1}(c_j + c_k)x_j + \sum_{\substack{j\in N_2\\ j\neq k}}(c_j - c_k)x_j \tag{16}$$

and (5) becomes

$$a_{ik}x_o + \sum_{j\in N_1}(a_{ij} + a_{ik})x_j + \sum_{\substack{j\in N_2\\ j\neq k}}(a_{ij} - a_{ik})x_j + x_{n+i} = b_i + a_{ik}M_1,\ i = 1, 2, ..., m \tag{17}$$

Thus, in this case, the equivalent augmented problem P_{aug} is

$$(P_{aug}) \quad \max Z = -c_k M_1 + c_k x_o + \textstyle\sum_{j\in N_1}(c_j + c_k)x_j + \sum_{\substack{j\in N_2\\ j\neq k}}(c_j - c_k)x_j$$

subject to the constraints (6), (7), (10) and (17).

Remark 1. For the non-basic variables with upper bound '∞' we always set those non basic variables at their lower bounds.

Next, we prove that there exists a ready starting dual feasible solution for the augmented problem P_{aug} as discussed in the next Subsection.

3.1.2 Dual feasible solution for problem P_{aug}

Theorem 2. *There exists a dual feasible solution for the augmented problem* P_{aug}.

Proof. Proof According to equation (8), as discussed in previous subsection,

$$\sum_{j\in N_1}(x_j - l_j) + \sum_{j\in N_2}(u_j - x_j) \leq M$$

Depending on the case under which 'k' falls, a dual feasible solution for P_{aug} can be obtained as explained below.

Case 1. $k \in N_1$.

In this case, $c_k > 0$. The augmented problem P_{aug} is

$$(P_{aug}) \qquad \max\ Z = \sum_{j=0}^{n} \hat{c}_j x_j + c_k M_1$$

subject to

$$-a_{ik}x_o + \sum_{\substack{j\in N_1\\ j\neq k}} (a_{ij} - a_{ik})x_j + \sum_{j\in N_2} (a_{ij} + a_{ik})x_j + x_{n+i} = b_i - a_{ik}M_1,\ i = 1,2,...,m,$$

$$\sum_{\substack{j\in N_1\\ j\neq k}} x_j - \sum_{j\in N_2} x_j + x_o + x_k = M_1,$$

$l_j \leq x_j \leq u_j\ \forall\ j = 1,2,...,n_1,\ l_j \leq x_j < \infty\ \forall\ j = n_1+1,...,n$ and $0 \leq x_o < \infty$.

$$\text{where} \quad \hat{c}_j = \left\{ \begin{array}{cc} -c_k & \text{if } j = 0 \\ (c_j - c_k) & \text{if } j \in N_1,\ j \neq k \\ (c_j + c_k) & \text{if } j \in N_2 \\ 0 & \text{if } j = k \end{array} \right\}$$

Consider $x_{n+1},\ x_{n+2}, ..., x_{n+m},\ x_k$ as the basic variables. That is, consider $\hat{B} = B \cup k$ as the index set of basic variables so that $\hat{A}_B = I_{(m+1)\times(m+1)}$ is the basis matrix corresponding to index set $\hat{B}$. Take $\hat{N}_1 = \{j \in N_1 : j \neq k\} \cup \{0\}$ as the index set of non-basic variables at lower bounds and $\hat{N}_2 = \{j \in N_2\} = N_2$ as the index set of non-basic variables at upper bounds so that $\hat{N} = \hat{N}_1 \cup \hat{N}_2$ is the index set of all non-basic variables.
Let $\hat{x}_B$ denotes the basic solution obtained by the above choice of $\hat{B}$, $\hat{N}_1$ and $\hat{N}_2$.

$$\text{Then,} \qquad \hat{c}_j = \left\{ \begin{array}{cc} -c_k & \text{if } j \in \hat{N}_1,\ j = 0 \\ (c_j - c_k) & \text{if } j \in \hat{N}_1,\ j \neq 0 \\ (c_j + c_k) & \text{if } j \in \hat{N}_2 \end{array} \right\}$$

From the choice of c_k, it follows that $\hat{c}_j \leq 0\ \forall\ j \in \hat{N}_1$ and $\hat{c}_j \geq 0\ \forall\ j \in \hat{N}_2$.
Here, cost of each basic variable is 0 so that cost vector for the new basic solution is 0, i.e., $\hat{c}_B = 0$.
Therefore, at the basic solution $\hat{x}_B$,

$$\hat{Z}_j - \hat{c}_j = \hat{c}_B \hat{y}_j - \hat{c}_j = 0 - \hat{c}_j = -\hat{c}_j\ \forall\ j \in \hat{N}$$
$$\Rightarrow \hat{Z}_j - \hat{c}_j \geq 0\ \forall\ j \in \hat{N}_1 \ \text{ and } \ \hat{Z}_j - \hat{c}_j \leq 0\ \forall\ j \in \hat{N}_2$$

Hence, in view of Theorem 1, $\hat{x}_B$ is a dual feasible solution of P_{aug}.

Case 2. $k \in N_2$. It is clear that $c_k < 0$ in this case.

Here, the augmented problem P_{aug} is

$$(P_{aug}) \qquad \max\ Z = \sum_{j=0}^{n} \hat{c}_j x_j - c_k M_1$$

subject to

$$a_{ik}x_o + \sum_{j\in N_1}(a_{ij}+a_{ik})x_j + \sum_{\substack{j\in N_2\\ j\neq k}}(a_{ij}-a_{ik})x_j + x_{n+i} = b_i + a_{ik}M_1,\ i=1,2,...,m,$$

$$\sum_{j\in N_1} x_j - \sum_{\substack{j\in N_2\\ j\neq k}} x_j + x_o - x_k = M_1 \ \Leftrightarrow\ -x_o - \sum_{j\in N_1} x_j + \sum_{\substack{j\in N_2\\ j\neq k}} x_j + x_k = -M_1,$$

$l_j \le x_j \le u_j\ \forall\ j=1,2,...,n_1,\ l_j \le x_j < \infty\ \forall\ j=n_1+1,...,n$ and $0 \le x_o < \infty$.

$$\text{where} \quad \hat{c}_j = \left\{ \begin{array}{cc} c_k & \text{if } j=0 \\ (c_j+c_k) & \text{if } j\in N_1 \\ (c_j-c_k) & \text{if } j\in N_2,\ j\neq k \\ 0 & \text{if } j=k \end{array} \right\}$$

Taking $\hat{B} = B\cup\{k\}$ as the index set of basic variables, $\hat{N}_1 = N_1\cup\{0\}$ as the index set of non-basic variables at lower bounds and $\hat{N}_2 = \{j\in N_2 : j\neq k\}$ as the index set of non-basic variables at upper bounds, we get $\hat{A}_B = I_{(m+1)\times(m+1)}$ as the basis matrix and $\hat{x}_B$ as the basic solution of P_{aug}.

$$\text{Here,} \quad \hat{c}_j = \left\{ \begin{array}{cc} c_k & \text{if } j\in\hat{N}_1,\ j=0 \\ (c_j+c_k) & \text{if } j\in\hat{N}_1,\ j\neq 0 \\ (c_j-c_k) & \text{if } j\in\hat{N}_2 \end{array} \right\}$$

Clearly, $\hat{c}_j \le 0\ \forall\ j\in\hat{N}_1$ and $\hat{c}_j \ge 0\ \forall\ j\in\hat{N}_2$ in this case as well.
Also, $\hat{c}_B = 0$ so that at the basic solution $\hat{x}_B$,

$$\hat{Z}_j - \hat{c}_j = c_{\hat{B}}\hat{y}_j - \hat{c}_j = 0 - \hat{c}_j = -\hat{c}_j\ \forall\ j\in\hat{N}$$
$$\Rightarrow \hat{Z}_j - \hat{c}_j \ge 0\ \forall\ j\in\hat{N}_1 \ \text{ and }\ \hat{Z}_j - \hat{c}_j \le 0\ \forall\ j\in\hat{N}_2$$

which implies that $\hat{x}_B$ is a dual feasible solution of P_{aug}. □

Thus, a starting dual feasible solution can be obtained for the augmented problem P_{aug} and dual simplex algorithm (ref. Subsection 2.3) can be applied to solve P_{aug}. The next subsection describes how to find an optimal feasible solution (if exists) of problem P after solving P_{aug}. It may be noted that problem P may be infeasible or unbounded, which can also be interpreted by solving P_{aug} as discussed in the following subsection.

3.1.3 Finding an optimal solution of P from the augmented problem P_{aug}

In this subsection, we discuss the various situations arising in the termination of dual simplex algorithm applied to solve P_{aug}. After obtaining a dual feasible solution (ref. Subsection 3.1.2) for P_{aug}, dual simplex algorithm is applied to P_{aug} whose termination results into the following three possibilities.

Possibility I. P_{aug} has no feasible solution.
Then, the problem P also does not have a feasible solution.
This is because of the reason that every feasible solution $(x_1, x_2, ..., x_n)^T$ to P yields a feasible solution $(x_o, x_1, x_2, ..., x_n)^T$ to P_{aug}, where

$$x_o = M_1 - \sum_{j\in N_1} x_j + \sum_{j\in N_2} x_j.$$

Possibility II. P_{aug} has an optimal basic feasible solution, say $\tilde{x}$ and x_o is a basic variable in it. Let $\tilde{B}$ be the index sets of basic variables other than x_o and $\tilde{N}_1$ and $\tilde{N}_2$ be the index sets of non-basic variables at lower bounds and upper bounds respectively corresponding to $\tilde{x}$.
Then, from the constraints (5),

$$(\tilde{A}_B\tilde{x}_B)_i + \sum_{j\in\tilde{N}_1} a_{ij}l_j + \sum_{j\in\tilde{N}_2} a_{ij}u_j = b_i, \quad i = 1, 2, ..., m$$

$$\text{or} \qquad (\tilde{A}_B\tilde{x}_B)_i = b_i - \sum_{j\in\tilde{N}_1} a_{ij}l_j - \sum_{j\in\tilde{N}_2} a_{ij}u_j = \bar{b}_i, \quad i = 1, 2, ..., m$$

Therefore, the values of the basic variable x_o and x_B in the optimal basic feasible solution are given by

$$\begin{pmatrix} 1 & \hat{e} \\ 0 & \tilde{B} \end{pmatrix} \begin{pmatrix} x_o \\ \tilde{x}_B \end{pmatrix} = \begin{pmatrix} M_1 \\ \bar{b}_i \end{pmatrix} = \begin{pmatrix} M + \sum_{j\in\tilde{N}_1} l_j - \sum_{j\in\tilde{N}_2} u_j \\ b_i - \sum_{j\in\tilde{N}_1} a_{ij}l_j - \sum_{j\in\tilde{N}_2} a_{ij}u_j \end{pmatrix}$$

where $\hat{e}$ is an m-vector with entries as +1 or -1 (ref. equation (10)).

$$\begin{aligned} \Rightarrow \begin{pmatrix} x_o \\ \tilde{x}_B \end{pmatrix} &= \begin{pmatrix} 1 & \hat{e} \\ 0 & \tilde{B} \end{pmatrix}^{-1} \begin{pmatrix} M_1 \\ \bar{b}_i \end{pmatrix} \\ &= \begin{pmatrix} 1 & \hat{e}(\tilde{B})^{-1} \\ 0 & (\tilde{B})^{-1} \end{pmatrix} \begin{pmatrix} M_1 \\ \bar{b}_i \end{pmatrix} \\ \Rightarrow x_o &= M_1 + (\text{a constant independent of } M) \\ &= M + (\text{a constant independent of } M) \\ \text{and} \qquad \tilde{x}_B &= (\tilde{B})^{-1}\bar{b}_i \end{aligned}$$

That is, all the basic variables are independent of M.
Since, $x_o > 0$, we have at the optimal basic feasible solution,

$$\sum_{j\in\tilde{N}_1} (x_j - l_j) + \sum_{j\in\tilde{N}_2} (u_j - x_j) < M \tag{18}$$

From (18), it follows that the values of the variables $\tilde{x}_B$ in the optimal basic feasible solution of P_{aug} are finite and constitute an optimal feasible solution to the original program P. Because if not so, then there exist a feasible solution x of P such that $cx > \tilde{c}_B\tilde{x}_B$. This implies that x will also be a feasible solution of P_{aug} yielding the value of objective function greater than the value corresponding to $\tilde{x}_B$, contradicting the optimality of $\tilde{x}_B$.

Possibility III. P_{aug} has an optimal basic feasible solution x^* and x_o is not a basic variable in it.
If B^*, N_1^* and N_2^* are the index sets of basic variables, non-basic variables at lower bounds and non-basic variables at upper bounds respectively corresponding to x^*, then at the optimal basic feasible solution x^*,

$$\sum_{j \in N_1^*} (x_j - l_j) + \sum_{j \in N_2^*} (u_j - x_j) = M \tag{19}$$

Consequently, the values of the basic variables are functions of M.
Now, two possibilities arise according as the optimal value of Z, (say Z^o) depends upon M or not.
(a) If Z^o is an explicit function of M for all M greater than some fixed value M_1.
Then, $Z^o \to +\infty$ as $M \to +\infty$.
Note that Z^o cannot approach $-\infty$ because there is a feasible solution to P_{aug} which yields a finite value of Z. Therefore, P_{aug} has an unbounded solution.
Since, both P and P_{aug} have the same objective function and every feasible solution to P_{aug} yields a feasible solution to P.
Therefore, P also has an unbounded solution.
(b) If Z^o is independent of M.
In this case, when M varies and is larger than M_1, the hyperplane (19) is displaced parallel to itself and the optimal vertex which is lying on this hyperplane ($x_o = 0$) moves out to an infinite edge of the polyhedron represented by the set of feasible solutions to P_{aug}. As Z^o is not a function of M, the objective hyperplane $c^T x = Z^o$ contains this edge and therefore all the points on this edge are optimal feasible solutions. In particular, there exists an optimal basic feasible solution to P represented by the origin of this infinite edge which is obtained by decreasing M until one of the variables which is a function of M vanishes.
Thus, when a starting dual feasible solution cannot be obtained for an P, then it can be solved by applying dual simplex algorithm to P_{aug} rather than P itself. The complete technique for the various possibilities is illustrated through numerical examples in Section 4.

3.2 Particular cases of LPPBV

1. If both upper as well as lower bounds on all the variables of LPPBV are finite and x_B is not dual feasible, then dual feasible solution can be obtained directly for the LPPBV. In this case, flip the bounds of the non basic variables corresponding to x_B where optimality is hampered and update the values of the basic variables accordingly. Such a flip of bounds will readily provide the new basic feasible solution which will be dual feasible as explained in Numerical example 3 in Section 4.

2. If all variables have upper bound ∞ and lower bound 0, then the LPPBV reduces to an ordinary LPP with non-negative variables. In that case, $N_2 = \phi$, $N_1 = N$ and $l_j = 0$ for all $j \in N$. Therefore, cut (8) reduces to the following simplified form

$$\sum_{j \in N} x_j \leq M.$$

4 Numerical Examples

Example 1. Consider the following problem

$$\begin{aligned} (P1) \quad & \max\ Z = 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 5, \\ & x_1 + 5x_2 + x_3 \ge 10, \\ & 0 \le x_1 < \infty,\ 1 \le x_2 \le 4,\ 2 \le x_3 \le 8 \end{aligned} \tag{20}$$

The given problem $P1$ can be expressed in canonical form as

$$\begin{aligned} & \max\ Z = 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 5, \\ & -x_1 - 5x_2 - x_3 + x_5 = -10, \\ & 0 \le x_1 < \infty,\ 1 \le x_2 \le 4,\ 2 \le x_3 \le 8 \\ & 0 \le x_4 < \infty, 0 \le x_5 < \infty \end{aligned}$$

where x_4 and x_5 are slack variables.
Let $A_B = (a_4 \ \ a_5) = I$, $A_N = (a_1 \ \ a_2 \ \ a_3)$, $A_{N_1} = (a_1 \ \ a_3)$, $A_{N_2} = (a_2)$.
This gives the basic solution $x_4 = -1,\ x_5 = 12$ and corresponding net evaluations of the non-basic variables as $Z_1 - c_1 = 0 - 2 = -2$, $Z_2 - c_2 = 0 - 1 = -1$ and $Z_3 - c_3 = -3 < 0$.
Theorem 1 implies that this solution is not dual feasible.
Consider the constraint

$$\begin{aligned} \sum_{j \in N_1} (x_j - l_j) + \sum_{j \in N_2} (u_j - x_j) \le M \\ \Leftrightarrow (x_1 - 0) + (4 - x_2) + (x_3 - 2) \le M \\ \Leftrightarrow (x_1 - 0) + (4 - x_2) + (x_3 - 2) + x_o = M \\ \Leftrightarrow x_1 - x_2 + x_3 + x_o = M \end{aligned} \tag{21}$$

where $x_o \ge 0$ and $M > 0$ is a sufficiently large number.
Then the augmented problem $\widehat{P1}$ is

$$\begin{aligned} (\widehat{P1}) \quad & \max\ Z = 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 5 \\ & -x_1 - 5x_2 - x_3 + x_5 = -10 \\ & x_1 - x_2 + x_3 + x_o = M \\ & 0 \le x_1 < \infty,\ 1 \le x_2 \le 4,\ 2 \le x_3 \le 8 \\ & 0 \le x_4 < \infty,\ 0 \le x_5 < \infty,\ 0 \le x_o < \infty \end{aligned}$$

Here $|c_k| = \max\{|c_j| : c_j > 0,\ j \in N_1;\ c_j < 0,\ j \in N_2\} = \max\{\ |2|,\ |3|\ \} = 3 = |c_3|$.
From (21), $x_2 = M + x_1 - x_o$.
Substituting this value of x_2 in $\widehat{P1}$, we get the equivalent augmented problem

Table 1: Initial dual feasible solution for $P1_{aug}$

		$c_j \rightarrow$	-3	-1	4	0	0	0
		Bounds	l_o	l_1	u_2	b	b	b
c_B	B	x_B	a_o	a_1	a_2	a_3	a_4	a_5
0	$a_{b_1} = a_4$	$-3 - M$	-1	0	2	0	1	0
0	$a_{b_2} = a_5$	$M + 14$	1	0	-6	0	0	1
0	$a_{b_3} = a_3$	$M + 4$	1	1	-1	1	0	0
	$Z = 3M + 16$	$Z_j - c_j$	3	1	-4	0	0	0

Table 2: Termination simplex table for $P1_{aug}$

		$c_j \rightarrow$	-3	-1	4	0	0	0
		Bounds	b	l_1	b	b	l_4	l_5
c_B	B	x_B	a_o	a_1	a_2	a_3	a_4	a_5
0	$a_{b_1} = a_2$	5/4	0	0	1	0	-1/4	-1/4
0	$a_{b_2} = a_o$	$M - 5/2$	1	0	0	0	-3/2	-1/2
0	$a_{b_3} = a_3$	15/4	0	1	0	1	5/4	1/4
	$Z = 50/4$	$Z_j - c_j$	0	1	0	0	7/2	1/2

$P1_{aug}$ as

$$
\begin{aligned}
(P1_{aug}) \qquad & \max Z = 3M - x_1 + 4x_2 - 3x_o \\
\text{subject to} \qquad & -x_o + 2x_2 + x_4 = 5 - M \\
& x_o - 6x_2 + x_5 = -10 + M \\
& x_o + x_1 - x_2 + x_3 = M \\
& 0 \leq x_1 < \infty, \ 1 \leq x_2 \leq 4, \ 2 \leq x_3 \leq 8 \\
& 0 \leq x_4 < \infty, \ 0 \leq x_5 < \infty, \ 0 \leq x_o < \infty
\end{aligned}
$$

Now, taking $\hat{A}_B = (a_4 \ \ a_5 \ \ a_3)$, $\hat{A}_{N_1} = (a_o \ \ a_1)$, $\hat{A}_{N_2} = (a_2)$, obtain the initial basic solution of $P1_{aug}$ which will be dual feasible as given in the simplex table (Table 1).

It is clear from Theorem 1 that the above solution is dual feasible (primal optimal) but not primal feasible. Further, to obtain the primal optimal feasible solution, apply dual simplex method (Subsection 2.3) to problem $P1_{aug}$. The termination simplex table thus obtained is as given in Table 2.

The corresponding basic solution is an optimal feasible solution of $P1_{aug}$ and x_o is a basic variable in it. Therefore, in view of Subsection 3.1.3, rest of the basic variables x_2 and x_3 constitute an optimal basic feasible solution of original problem $P1$. The optimal solution thus obtained is as follows.

$$x_1 = 0, \ x_2 = \frac{5}{4}, \ x_3 = \frac{15}{4} \text{ and } Z = \frac{50}{4}.$$

Example 2. Consider the following problem

$$
\begin{aligned}
(P2) \qquad & \max\ Z = 2x_1 + 4x_2 \\
\text{subject to} \qquad & x_1 - x_2 \geq -1, \\
& -x_1 + 2x_2 \leq 4, \\
& 1 \leq x_1 < \infty,\ 3 \leq x_2 \leq 5
\end{aligned} \tag{22}
$$

The given problem $P2$ can be expressed in canonical form as

$$
\begin{aligned}
& \max\ Z = 2x_1 + 4x_2 \\
\text{subject to} \qquad & -x_1 + x_2 + x_3 = 1, \\
& -x_1 + 2x_2 + x_4 = 4, \\
& 1 \leq x_1 < \infty,\ 3 \leq x_2 \leq 5 \\
& 0 \leq x_3 < \infty, 0 \leq x_4 < \infty
\end{aligned}
$$

where x_3 and x_4 are slack variables.
Let $A_B = (a_3 \;\; a_4) = I$, $A_N = (a_1 \;\; a_2)$, $A_{N_1} = (a_1 \;\; a_2)$, $A_{N_2} = \phi$ yielding the basic solution as $x_3 = -1,\ x_4 = -1$ with corresponding net evaluations of the non-basic variables as $Z_1 - c_1 = 0 - 2 = -2$, $Z_2 - c_2 = 0 - 4 = -4$. Theorem 1 implies that this solution is not dual feasible.
Consider the constraint

$$
\begin{aligned}
& \sum_{j \in N_1} (x_j - l_j) + \sum_{j \in N_2} (u_j - x_j) \leq M \\
& \Leftrightarrow (x_1 - 1) + (x_2 - 3) \leq M \\
& \Leftrightarrow (x_1 - 1) + (x_2 - 3) + x_o = M \\
& \Leftrightarrow x_1 + x_2 + x_o = M
\end{aligned} \tag{23}
$$

where $x_o \geq 0$ and $M > 0$ is a sufficiently large number.
Then the augmented problem $\widehat{P2}$ is

$$
\begin{aligned}
(\widehat{P2}) \qquad & \max\ Z = 2x_1 + 4x_2 \\
\text{subject to} \qquad & -x_1 + x_2 + x_3 = 1, \\
& -x_1 + 2x_2 + x_4 = 4, \\
& x_1 + x_2 + x_o = M \\
& 1 \leq x_1 < \infty,\ 3 \leq x_2 \leq 5, \\
& 0 \leq x_3 < \infty,\ 0 \leq x_4 < \infty,\ 0 \leq x_o < \infty
\end{aligned}
$$

Here $|c_k| = \max\{\ |c_j| : c_j > 0,\ j \in N_1;\ c_j < 0,\ j \in N_2\} = \max\{\ |2|,\ |4|\ \} = 4 = |c_2|$.
From (21), $x_2 = M - x_1 - x_o$.
Substituting this value of x_2 in $\widehat{P2}$, we get the equivalent augmented problem

Table 3: Initial dual feasible solution for $P2_{aug}$

		$c_j \rightarrow$	-4	-2	0	0	0
		Bounds	l_o	l_1	b	b	b
c_B	B	x_B	a_o	a_1	a_2	a_3	a_4
0	$a_{b_1} = a_3$	$3 - M$	-1	-2	0	1	0
0	$a_{b_2} = a_4$	$7 - 2M$	-2	3	0	0	1
0	$a_{b_3} = a_2$	$M - 1$	1	1	1	0	0
	$Z = 4M - 2$	$Z_j - c_j$	4	2	0	0	0

Table 4: Termination table for $P2_{aug}$

		$c_j \rightarrow$	-4	-2	0	0	0
		Bounds	l_o	b	u_2	b	b
c_B	B	x_B	a_o	a_1	a_2	a_3	a_4
0	$a_{b_1} = a_3$	$M - 9$	1	0	2	1	0
0	$a_{b_2} = a_1$	$M - 5$	1	1	1	0	0
0	$a_{b_3} = a_4$	$M - 11$	1	0	3	0	1
	$Z = 2M + 10$	$Z_j - c_j$	2	0	-2	0	0

$P2_{aug}$ as

$$
\begin{aligned}
(P2_{aug}) \qquad & \max\ Z = 4M - 2x_1 - 4x_o \\
\text{subject to} \qquad & -x_o - 2x_1 + x_3 = 1 - M \\
& -2x_o - 3x_1 + x_4 = 4 - 2M \\
& x_o + x_1 + x_2 = M \\
& 1 \leq x_1 < \infty,\ 3 \leq x_2 \leq 5, \\
& 0 \leq x_3 < \infty,\ 0 \leq x_4 < \infty,\ 0 \leq x_o < \infty
\end{aligned}
$$

Now, taking $\hat{A}_B = (a_3 \ \ a_4 \ \ a_2)$, $\hat{A}_{N_1} = (a_o \ \ a_1)$, $\hat{A}_{N_2} = \phi$, obtain the initial basic solution of $P2_{aug}$ which will be dual feasible as given in the simplex table (Table 3).

It is clear from Theorem 1 that the above solution is dual feasible (primal optimal) but not primal feasible. The termination simplex table obtained by applying dual simplex method to problem $P2_{aug}$ is as given in Table 4.

The corresponding basic solution is an optimal feasible solution of $P2_{aug}$ but x_o is not a basic variable in it. Also Z is a function of M, therefore, in view of Subsection 3.1.3, the problem $P2_{aug}$ has an unbounded solution. Consequently, the original problem $P2$ also has an unbounded solution.

Example 3. Consider the problem having all the variables with both lower

Table 5: Initial dual feasible solution

		$c_j \rightarrow$	3	5	3	0	0
		Bounds	u_1	u_2	u_3	b	b
c_B	B	x_B	a_1	a_2	a_3	a_4	a_5
0	$a_{b_1} = a_4$	-1	1	2	1	1	0
0	$a_{b_2} = a_2$	-8	2	4	3	0	1
	$Z = 53$	$Z_j - c_j$	-3	-5	-3	0	0

and upper bounds finite.

$$\begin{aligned} (P3) \quad & \max\ Z = 3x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 19 \\ & 2x_1 + 4x_2 + 3x_3 \leq 33 \\ & 1 \leq x_1 \leq 5,\ 2 \leq x_2 \leq 7,\ 0 \leq x_3 \leq 1 \end{aligned} \tag{24}$$

For solving this problem by dual simplex method, we need a starting dual feasible solution, which in this case can be easily obtained as explained in Subsection 3.2.

The above problem is equivalent to the canonical form

$$\begin{aligned} & \max Z = 3x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 = 19 \\ & 2x_1 + 4x_2 + 3x_3 + x_5 = 33 \\ & 1 \leq x_1 \leq 5,\ 2 \leq x_2 \leq 7,\ 0 \leq x_3 \leq 1, \\ & 0 \leq x_4 < \infty,\ 0 \leq x_5 < \infty \end{aligned}$$

where x_4 and x_5 are slack variables. Let $A_B = (a_4 \ \ a_5) = I$ be the basis matrix. Let $A_N = (a_1 \ \ a_2 \ \ a_3)$, $A_{N_1} = (a_3)$, $A_{N_2} = (a_1 \ \ a_2)$.

This gives the basic solution $x_4 = 0,\ x_5 = -5$ and corresponding to this basic solution, $Z_1 - c_1 = 0 - 3 = -3$, $Z_2 - c_2 = 0 - 5 = -5$, $Z_3 - c_3 = 0 - 3 = -3$.

Since $Z_3 - c_3 < 0$ and $a_3 \in A_{N_2}$, therefore this solution is not dual feasible (Theorem 1). Therefore, flip the bound of x_3, i.e. set x_3 at its upper bound so that the solution becomes dual feasible.

Take $A_B = (a_4 \ \ a_5) = I$, $A_N = (a_1 \ \ a_2 \ \ a_3)$, $A_{N_1} = \phi$, $A_{N_2} = (a_1 \ \ a_2 \ \ a_3)$ and obtain the starting dual feasible solution as given in the following simplex table (Table 5).

It can be seen that this solution is dual feasible but primal infeasible, so dual simplex method can be applied directly to the given problem to obtain the optimal feasible solution as

$$x_1 = 5,\ x_2 = \frac{23}{4},\ x_3 = 0 \text{ and } Z = \frac{175}{4}$$

5 Concluding Remarks

1. In this paper, we have developed a systematic technique to solve a bounded variable linear program using dual simplex approach for the case when a

starting dual feasible solution is not readily available.

2. A particular case of linear programming problem with bounded variables arises when all the variables have finite lower and upper bounds. For finding a dual feasible solution for such a problem, a technique based on the concept of flip of bounds is also described in the paper. However, flipping the bounds fails for the problems where one or more variables do not possess finite bounds.

3. For any given problem, the constraint (8) can also be modified by considering only those non-basic variables where optimality is hampered.

4. This situation can be explored further for linear fractional programming problems with bounded variables.

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