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PRECONDITIONED KACZMARZ ALGORITHMS WITH RELAXATION PARAMETERS

ELENA PELICAN, CONSTANTIN POPA, "Ovidius" University, Faculty of Mathematics and Computer Science, Blvd. Mamaia 124, 8700 Constanta, Romania; email: epelican@univ-ovidius.ro, cpopa@univ-ovidius.ro

Abstract. In this paper we describe iterative algorithms for numerical solution of linear least-squares problems. They are based on combinations between an extension, with relaxation parameters for the classical Kaczmarz's projections method (obtained by one of the authors in a previous work) and approximate orthogonalization techniques due to Z. Kovarik. We prove that the new algorithms converge to any solution of an inconsistent and rank-defficient least-squares problem (with respect to the choice of the initial approximation), the convergence being much faster than for classical Kaczmarz - like methods. Numerical experiments on a first kind integral equation are described in the last section of the paper.

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1 Extended Kaczmarz and Kovarik methods

Let A be a real $m \times n$ matrix and $b \in I\!\!R^m$. We shall denote by $A^t, (A)_i, (A)^j$, $r(A), R(A), N(A), b_i$ the transpose, i-th row, j-th column, rank, range, null space of A and i-th component of b, respectively (all the vectors that appear being considered as column vectors). The notations $\rho(B), \sigma(B)$ will be used for the spectral radius and spectrum of a (square) matrix B and $\parallel A \parallel$ will be the spectral norm of A defined by $\parallel A \parallel^2 = \rho(A^tA) = \rho(AA^t)$. $P_S(x)$ will be the orthogonal projection of x onto the vector subspace S with respect to the Euclidean scalar product and the associated norm, denoted by $< \cdot, \cdot >$ and $\parallel \cdot \parallel$, respectively. We shall consider the linear least-squares problem : find $x^* \in I\!\!R^n$ such that

$$\parallel Ax^* - b \parallel = min! \tag{1}$$

It is well known (see e.g. [1]) that the set of all (least-squares) solutions of (1), denoted by $LSS(A;b)$ is a nonempty closed convex subset of $I\!R^n$ containing a unique solution with minimal norm, denoted by x_{LS}. More than that, we have

$$x^* \in LSS(A;b) \Leftrightarrow A^tAx^* = A^tb \tag{2}$$

and if $b_A = P_{R(A)}(b)$, then $LSS(A;b) = S(A;b_A)$, where by $S(A;b_A)$ we denoted the set of all (classical) solutions of the (consistent) system $Ax = b_A$. We shall also suppose that the rows and columns of A satisfy

$$(A)_i \neq 0,\ i = 1,\dots,m;\quad (A)^j \neq 0,\ j = 1,\dots,n. \tag{3}$$

Then, for $\alpha, \omega \neq 0$ two given real numbers we can define the linear applications (matrices)

$$f_i(\omega;A;b;x) = x - \omega\frac{< x,(A)_i > - b_i}{\parallel (A)_i \parallel^2}(A)_i, \tag{4}$$

$$\varphi_j(\alpha;A;y) = y - \alpha\frac{< y,(A)^j >}{\parallel (A)^j \parallel^2}(A)^j, \tag{5}$$

$$K(\omega;A;b;x) = (f_1 \circ \dots \circ f_m)(\omega;A;b;x), \tag{6}$$

$$\Phi(\alpha;A;y) = (\varphi_1 \circ \dots \circ \varphi_n)(\alpha;A;y), \tag{7}$$

$$P_i^\omega(x) = x - \omega\frac{< x,(A)_i >}{\parallel (A)_i \parallel^2}(A)_i,\quad Q^\omega = P_1^\omega \dots P_m^\omega \tag{8}$$

and R^ω the real $n \times m$ matrix of which i-th column $(R^\omega)^i$ is given by

$$(R^\omega)^i = \frac{1}{\parallel (A)_i \parallel^2}P_1^\omega P_2^\omega \dots P_{i-1}^\omega((A)_i), \tag{9}$$

with $P_0^\omega = I$ (the unit matrix). These applications have the following properties (for the proof see [6]).

Proposition 1 *(i) We have*

$$K(\omega;A;b;x) = Q^\omega x + R^\omega b,\ Q^\omega + R^\omega A = I,\ R^\omega y \in R(A^t),\ \forall\ y \in I\!R^m. \tag{10}$$

(ii) $N(A)$ *and* $R(A^t)$ *are invariant subspaces for* Q^ω *and*

$$Q^\omega = P_{N(A)} \oplus \tilde{Q}^\omega,\quad P_{N(A)}\tilde{Q}^\omega = \tilde{Q}^\omega P_{N(A)} = 0, \tag{11}$$

where $\tilde{Q}^\omega$ is the linear application defined by

$$\tilde{Q}^\omega = Q^\omega P_{R(A^t)}. \tag{12}$$

(iii) $N(A^t)$ and $R(A)$ are invariant subspaces for $\Phi^\alpha = \Phi(\alpha; A; \cdot)$ and

$$\Phi^\alpha = P_{N(A^t)} \oplus \tilde{\Phi}^\alpha, \quad P_{N(A^t)}\tilde{\Phi}^\alpha = \tilde{\Phi}^\alpha P_{N(A^t)} = 0, \tag{13}$$

where $\tilde{\Phi}^\alpha = \tilde{\Phi}(\alpha; A; \cdot)$ is the linear application defined by

$$\tilde{\Phi}^\alpha = \Phi^\alpha P_{R(A)}. \tag{14}$$

(iv) The applications $\tilde{Q}^\omega$ and $\tilde{\Phi}^\alpha$ satisfy

$$\| \tilde{Q}^\omega \| < 1, \quad \| \tilde{\Phi}^\alpha \| < 1. \tag{15}$$

The following algorithm (described by the author in [6]) is an extension of the classical Kaczmarz's projections method with relaxation parameters: let $x^0 \in I\!R^n, y^0 = b$; for $k = 0, 1, \ldots$ compute

$$y^{k+1} = \Phi(\alpha; A; y^k), \ \beta^{k+1} = b - y^{k+1}, \ x^{k+1} = K(\omega; A; \beta^{k+1}; x^k). \tag{16}$$

In [6] the following results were proved.

Proposition 2 *(i) Let G^ω be the $n \times m$ matrix defined by*

$$G^\omega = (I - \tilde{Q}^\omega)^{-1} R^\omega. \tag{17}$$

Then, for any $\alpha, \omega \in (0, 2)$, any matrix A satisfying (3), $b \in I\!R^m$ and $x^0 \in I\!R^n$ the sequence $(x^k)_{k \geq 0}$ generated with the algorithm (16) converges and

$$\lim_{k \to \infty} x^k = P_{N(A)}(x^0) + G^\omega b_A. \tag{18}$$

(ii) We have the equalities

$$LSS(A; b) = \{P_{N(A)}(x^0) + G^\omega b_A, \ x^0 \in I\!R^n\}; \quad x_{LS} = G^\omega b_A. \tag{19}$$

Remark 1 *The first and third steps from (16) are no more (succesive) orthogonal projections onto the hyperplanes generated by the columns and rows of A, as in [7] (for $\alpha = \omega = 1$). But, it can be easily proved that closer to 90° are the values of the angles between columns and rows of A, faster will be the convergence of the algorithm (16).*

With respect to the above Remark 1 we shall present in what follows two "approximate orthogonalization" algorithms proposed by Z. Kovarik in [4] for matrices with linearly independent rows. Let $(a_k)_{k\geq 0}$ be the sequence of positive real numbers

$$a_k = \frac{1}{2^{2k}} \frac{(2k)!}{(k!)^2}, \; k \geq 0 \tag{20}$$

and $(q_k)_{k\geq 1}$ a sequence of positive integers.

Algorithm (A) Start with $A_0 = A$ and recursively define the matrices H_k and A_{k+1} by

$$H_k = I - A_k A_k^t, \quad \Gamma_k = I + a_1 H_k + \ldots + a_{q_k} H_k^{q_k}, \tag{21}$$

$$A_{k+1} = \Gamma_k A_k, \quad k \geq 0. \tag{22}$$

Algorithm (B) Start with $A_0 = A$ and recursively define the matrices K_k and A_{k+1} by

$$K_k = 2(I + A_k A_k^t)^{-1} - I, \quad \Gamma_k = I + K_k \tag{23}$$

and A_{k+1} as in (22), with Γ_k from (23).

Let A_∞ be the $m \times n$ matrix defined by

$$A_\infty = [(AA^t)^{\frac{1}{2}}]^+ A, \tag{24}$$

where by B^+ we denoted the Moore-Penrose pseudoinverse of the matrix B (see e.g. [1]). We shall suppose that

$$\| AA^t \| = \| A^t A \| = \rho(AA^t) < 1. \tag{25}$$

In [11] and [8] we applied the above algorithms to an arbitrary rectangular matrix A. The following result was proved.

Proposition 3 *If (25) holds, then the sequence of matrices $(A_k)_{k\geq 0}$ generated with one of the algorithms (21) - (22) or (23), converges to A_∞.*

Remark 2 *If A has linearly independent rows then we can replace the pseudoinverse $[(AA^t)^{\frac{1}{2}}]^+$ in (24) by the classical inverse $[(AA^t)^{\frac{1}{2}}]^{-1}$ and it can be proved that A_∞ has, in this case, mutually orthogonal rows (see [4]). This is no longer true for a general rectangular A, but an improvement is obtained concerning the angles between rows of A_∞ by comparing them with the angles between the rows of the initial matrix A; (see e.g. the numerical experiments from [11] and [8]).*

Remark 3 *The condition (25) is not restrictive. It can be fullfiled by an appropriate scalling of the elements of the matrix A, e.g. of the form ρA, where $\rho \in I\!R \setminus \{0\}$. This kind of scalling does not affect the set $LSS(A; b)$.*

2 Preconditioned Kaczmarz algorithm with relaxation parameters

Unfortunately, it is well known (both, from theoretical and experimental considerations; see e.g. [1] and references therein) that projections methods as (16) have slow convergence. From this view point and taking into account the above Remarks 1 and 2, we can think at a "combination" between (16) and one of the Kovarik's algorithms **(A)** and **(B)**. In such a combination, Kovarik's methods would have to action as a "preconditioner" for the extended Kaczmarz one (16). This can be done (at least) in one of the following two ways.

Algorithm A1 - Preprocessing preconditioning

Part I. Set $A_0 = A, b^0 = b$, $H_0 = I - A_0 A_0^t$ or $K_0 = 2(I + A_0 A_0^t)^{-1} - I$ and let $NKO \geq 1$ be an integer. For $k = 0, 1, \ldots, NKO - 1$ apply the transformations

$$A_{k+1} = \Gamma_k A_k, \quad b^{k+1} = \Gamma_k b^k, \tag{26}$$

with Γ_k from (21) or (23), respectively and H_k or K_k updated as in (32), and obtain A_{NKO} and b^{NKO}.

Part II. Perform NEK sweeps with the Kaczmarz-extended algorithm (16) ($NKE \geq 1$) with respect to A_{NKO} and b^{NKO}.

Algorithm A2 - Dynamic preconditioning

Let $x^0 \in I\!R^n$, $A_0 = A$, $b^0 = b$ and

$$H_0 = I - A_0 A_0^t \ \ \textit{or} \ \ K_0 = 2(I + A_0 A_0^t)^{-1} - I. \tag{27}$$

Step 1. Compute A_{k+1} and b^{k+1} by

$$A_{k+1} = \Gamma_k A_k, \quad b^{k+1} = \Gamma_k b^k, \tag{28}$$

with Γ_k from (21) or Γ_k from (23), respectively.

Step 2. Compute y^{k+1} and β^{k+1} by

$$y^{k+1} = \Phi^{k+1}(\alpha; A_{k+1}; b^{k+1}), \tag{29}$$

$$\beta^{k+1} = b^{k+1} - y^{k+1}. \tag{30}$$

Step 3. Compute the next approximation x^{k+1} by

$$x^{k+1} = K(\omega; A_{k+1}; \beta^{k+1}; x^k) \tag{31}$$

and update H_k or K_k to H_{k+1} or K_{k+1}, respectively by

$$H_{k+1} = I - A_{k+1}A_{k+1}^t \quad or \quad K_{k+1} = 2(I + A_{k+1}A_{k+1}^t)^{-1} - I. \tag{32}$$

Remark 4 *The step (29) means succesive application of* $\Phi(\alpha; A_{k+1}; \cdot)$ $(k+1)$ *- times to the initial vector* b^{k+1}*, i.e.*

$$\Phi^{k+1}(\alpha; A_{k+1}; b^{k+1}) = (\Phi(\alpha; A_{k+1}; \cdot) \circ \ldots \circ \Phi(\alpha; A_{k+1}; \cdot))(b^{k+1}). \tag{33}$$

Remark 5 *Because the matrices* Γ_k *and* B_k *are* (SPD)*, we can easily obtain that, if (25) holds and* $(A_k)_{k\geq 0}$ *is the sequence defined with the above algorithms* **(A)** *or* **(B)***, then*

$$N(A_k) = N(A), \quad N(A_k^t) = N(A^t). \tag{34}$$

and

$$LSS(A_k; b^k) = LSS(A; b), \ \forall\ k \geq 0. \tag{35}$$

(see also [3])

In what follows we shall prove the convergence of the above defined preconditoned algorithms **A1** and **A2**. Concerning the first one, the following convergence result is true.

Theorem 1 *Let* $\alpha, \omega \in (0, 2)$*,* ε *an arbitrary positive real number and* $x^0 \in \mathbb{R}^n$ *an arbitrary initial approximation. Then, for any positive integer* NKO *it exists a positive integer* NKE *(depending on* $\alpha, \omega, \varepsilon$ *and* NKO*) and a least-squares solution of (1),* $x^* \in \mathbb{R}^n$ *(depending on* ω, x^0 *and* NKO*) such that, if* $(x^k)_{k\geq 0}$ *is the sequence generated in the second part of* **A1** *(with respect to* x^0*,* A_{NKO} *and* b^{NKO}*) then*

$$\| x^k - x^* \| \leq \varepsilon, \ \forall\ k \geq NKE. \tag{36}$$

Proof. Let $\alpha, \omega, \varepsilon$, x^0 and NKO be arbitrary fixed, $b^{NKO}_{A_{NKO}}$, $G^{NKO,\omega}$ as in Section 1 (17) and

(17), respectively, but with respect to A_{NKO} and b^{NKO}, instead of A and b and $x^* \in I\!R^n$ defined by

$$x^* = P_{N(A_{NKO})}(x^0) + G^{NKO,\omega} b^{NKO}_{A_{NKO}}. \tag{37}$$

In [6] we proved that $x^* \in LSS(A_{NKO}; b^{NKO})$ and $(x^k)_{k\geq 0}$ converges to x^*. From this we obtain (36) and together with the equalities (35) we complete the proof.

Remark 6 *We may expect that for bigger values of NKO we shall get smaller values for NKE (considered as the smallest integer for which (36) holds).*

Concerning the algorithm **A2** in [3] we proved its convergence for the particular case $\alpha = \omega = 1$ and in [9] in the case of consistent problems as (1) (in this case, the algorithm (16) has the simplfied form

$$x^{k+1} = K(\omega; A; b; x^k),$$

$\forall k \geq 0$). Using the ideas from these results, we shall prove convergence in the general inconsistent case for (1) and for arbitrary values of $\alpha, \omega \in (0, 2)$. For this, let $Q^\omega_k, \tilde{Q}^\omega_k, \Phi^\alpha_k, \tilde{\Phi}^\alpha_k, R^\omega_k$ and G^ω_k be the matrices defined as in (8), (12), (7), (14), (9) and (17), respectively, but with A_k from (28) instead of A, b^k as in (28) and $b^k_{A_k}$ defined by

$$b^k_{A_k} = P_{R(A_k)}(b^k). \tag{38}$$

Then, as in [3] and using the relations (10) - (19) we obtain the following result.

Proposition 4 *Let $\alpha, \omega \in (0, 2)$ be arbitray fixed and $\tilde{Q}^\omega_\infty, \tilde{\Phi}^\alpha_\infty$ and R^ω_∞ are the matrices defined as in (12), (14) and (9), respectively, but with A_∞ from (24) instead of A. Then*

$$\lim_{k\to\infty} \tilde{Q}^\omega_k = \tilde{Q}^\omega_\infty, \quad \lim_{k\to\infty} \tilde{\Phi}^\alpha_k = \tilde{\Phi}^\alpha_\infty, \quad \lim_{k\to\infty} R^\omega_k = R^\omega_\infty. \tag{39}$$

Moreover the sequence $(b^k_{A_k})_{k\geq 0}$ is bounded.

The following result proves the convergence of the algorithm **A2**.

Theorem 2 *For any $\alpha, \omega \in (0,2)$ and any initial approximation $x^0 \in I\!R^n$, the sequence $(x^k)_{k \geq 0}$ generated with the algorithm (27)-(32) converges and*

$$\lim_{k \to \infty} x^k = P_{N(A)}(x^0) + G^\omega b_A. \tag{40}$$

Moreover, x_{LS} is obtained as limit in (40) if and only if $x^0 \in R(A^t)$.

Proof. Let $k \geq 0$ be arbitrary fixed and $b_*^k \in I\!R^m$ defined by

$$b_*^k = P_{N(A_k^t)}(b^k). \tag{41}$$

Then, from (38) and [7] we obtain

$$b^k = b_{A_k}^k \oplus b_*^k \tag{42}$$

and

$$LSS(A_k; b^k) = \{P_{N(A_k)}(x^0) + G_k^\omega b_{A_k}^k, x^0 \in I\!R^n\}, x_{LS} = G_k^\omega b_{A_k}^k = G^\omega b_A. \tag{43}$$

Let now $x^0 \in I\!R^n$ be arbitrary fixed. Following the proof of Lemma 3 from [10] it results

$$P_{N(A_k)}(x^k) = P_{N(A)}(x^k) = P_{N(A)}(x^0), \ \forall \ k \geq 0. \tag{44}$$

and using this together with (10), (31), (11), (**??**), (30), (42), (29), (13) and (41) we get the equalities

$$x^{k+1} - (P_{N(A)}(x^0) + G^\omega b_A) =$$

$$\tilde{Q}_{k+1}^\omega [x^k - (P_{N(A)}(x^0) + G^\omega b_A)] + R_{k+1}^\omega (\beta^{k+1} - b_{A_{k+1}}^{k+1}). \tag{45}$$

and

$$\beta^{k+1} - b_{A_{k+1}}^{k+1} = -(\tilde{\Phi^\alpha}_{k+1})^{k+1}(b_{A_{k+1}}^{k+1}). \tag{46}$$

Now, if $x^* \in I\!R^n$ is defined by (see (40)) $x^* = P_{N(A)}(x^0) + G^\omega b_A$, then from (45) and (46) we obtain

$$x^{k+1} - x^* = \tilde{Q}_{k+1}^\omega (x^k - x^*) - R_{k+1}^\omega (\tilde{\Phi^\alpha}_{k+1})^{k+1}(b_{A_{k+1}}^{k+1}), \ \ \forall \ \ k \geq 0. \tag{47}$$

By iterating this equality and taking norms we obtain

$$\| x^{k+1} - x^* \| \ \ \leq \ \ \| \tilde{Q}_{k+1}^\omega \| \dots \| \tilde{Q}_1^\omega \| \| \| x^0 - x^* \| +$$

$$\sum_{j=1}^{k}(\| \tilde{Q}^{\omega}_{k+1} \| \dots \| \tilde{Q}^{\omega}_{j+1} \| \| \| \tilde{\Phi}^{\alpha}_{j} \|^{j} \| R^{\omega}_{j} \| \| \| b^{j}_{A_j} \|)+$$

$$\| R^{\omega}_{k+1} \| \| \| \tilde{\Phi}^{\alpha}_{k+1} \|^{k+1} \| b^{k+1}_{A_{k+1}} \| . \quad (48)$$

From now one, the proof follows exactly the same way as in [3], also using the relations (see (15))

$$\| \tilde{Q}^{\omega}_{k} \| < 1, \ \| \tilde{\Phi}^{\alpha}_{k} \| < 1, \ \forall\ k \geq 0, \ \| \tilde{Q}^{\omega}_{\infty} \| < 1, \ \| \tilde{\Phi}^{\alpha}_{\infty} \| < 1. \quad (49)$$

We now may conclude our investigations with the following consequence of the above Theorems 1 and 2.

Corollary 1 *In the corresponding hypothesis, for any $x^0 \in I\!R^n$ the sequence $(x^k)_{k\geq 0}$ generated with one of the algorithms* **A1** *or* **A2** *converges to a solution of the problem (1). Moreover, it converges to the minimal norm solution x_{LS} if and only if $x^0 \in R(A^t)$.*

3 Numerical experiments

We considered in our numerical experiments, as model problem, the following first kind Fredholm integral equation (see e.g. [2] and [12]): find $x^*(t) \in L^2([0,1])$ such that

$$(Tx)(s) = \int_0^1 k(s,t)\ x(t)dt = y(s), \ s \in [0,1], \quad (50)$$

where

$$k(s,t) = \frac{1}{\sqrt{1+(2s-1)^2(2t-1)^2}}, \ y(s) = 2s - 1 + w(s), \ t, s \in [0,1], \quad (51)$$

$$w(s) = \begin{cases} \frac{ln((2s-1)+\sqrt{1+(2s-1)^2})}{2s-1}, & s \neq 0.5 \\ 1, & s = 0.5 \end{cases} \quad (52)$$

and $L^2([0,1])$ is the vector space of Lebesgue square integrable functions $f : [0,1] \longrightarrow I\!R$ equiped with the norm

$$\| f \|_{L^2([0,1])} = \sqrt{\int_0^1 (f(t))^2 dt}. \quad (53)$$

If, by $R(T), R(T)^{\perp}$ we denote the range of T and its orthogonal complement in $L^2([0,1])$, respectively, then the function y from (51)-(52) (which is differentiable on $(0,1)$) satisfies

$$y \in R(T)^{\perp} \oplus R(T) = D(T^{+}), \tag{54}$$

$(w(s) = (Tx_0)(s)$, with $x_0 : [0,1] \longrightarrow [0,1]$ defined by $x_0(t) = 1, \forall t \in [0,1])$, where $D(T^{+})$ is the domain of the Moore-Penrose pseudoinverse of T. Then, we reformulated the problem (50) in the least-squares sense as follows (see [2]): find $x \in L^2([0,1])$ the minimal $L^2([0,1])$ - norm solution of

$$\parallel Tx - y \parallel_{L^2([0,1])} = \min! \tag{55}$$

We discretized the problem (55) using the least-squares collocation algorithm described in [2], i.e. for two given integers $m, n (m, n \geq 2)$ we considered the points $s_i, \sigma_j \in [0,1]$, $i = 1, \ldots, m,\ j = 1, \ldots, n$

$$s_i = \frac{i-1}{m-1},\ \sigma_j = \frac{j-1}{n-1}, \tag{56}$$

the $m \times n$ real matrix $\hat{A}$,

$$(\hat{A})_{ij} = \int_0^1 k(s_i, t)\ k(\sigma_j, t)\ dt \tag{57}$$

and the vector $b \in I\!R^m$

$$(b)_i = y(s_i). \tag{58}$$

But, because the integrals in (57) can not be exactly computed, we approximated them using the rectangles quadrature formula ("midle point") with 16 equally spaced points in $[0,1]$. If we denote by A the $m \times n$ matrix obtained by these approximations, then, instead of the "exact" discrete problem associated to (55), i.e. find $x^{app} \in I\!R^n$ the minimal Euclidean norm solution of

$$\parallel \hat{A}x^{app} - b \parallel = \min!, \tag{59}$$

we considered the "approximate" discrete problem (of the form (1)) : find $x^c \in I\!R^n$ the (computed) minimal Euclidean norm solution of

$$\parallel Ax^c - b \parallel = \min!. \tag{60}$$

Remark 7 *If by x_j^{app}, $j = 1, \ldots, n$ we denote the components of the minimal norm solution x^{app} from (59) and $X^{app} \in L^2([0,1])$ is defined by*

$$X^{app}(t) = \sum_{j=1}^{n} x_j^{app} k(\sigma_j, t), \ t \in [0,1], \tag{61}$$

then the following approximation result holds (see [2] and [5]):

$$\lim_{m,n \to \infty} \| x - X^{app} \|_{L^2([0,1])} = 0 \tag{62}$$

(where x is the minimal norm solution from (55)). Thus, bigger are m and n in (56), better will approximate X^{app} the exact solution x from (55).

Remark 8 *The approximation made by using x^c from (60) instead of x^{app} from (59) is directly related to the numerical integration used in (57). In our case, because the kernel $k(s,t)$ in (51) is smooth enough, we observed that the above mentioned rectangle quadrature formula (with 16 equally spaced points in [0,1]) gave an enough good approximation.*

The problem (60) is (very) ill-conditioned. For its numerical solution we considered the (standard) Tikhonov regularization technique (see e.g. [2]): find $x_{reg}^c \in I\!R^n$ such that

$$\| Ax_{reg}^c - b \| + \gamma \| x_{reg}^c \| = \min! \tag{63}$$

where γ is the regularization parameter. We used in our experiments only the algorithm **A1** (see also Remark 9 below) and the following stopping tests:
- for **Part I**

$$\| A_{k+1} - A_k \|_\infty \leq 10^{-6} \tag{64}$$

(which gave us a "good" value for NKO (see Remark 6));
- for **Part II**

$$\frac{\| x^{k+1} - x^k \|_\infty}{\| x^{k+1} \|} \leq \epsilon \tag{65}$$

(for different values of ϵ - see the comments below). This gave us the smallest NKE (see again Remark 6). We used this kind of "relative error" stopping test because the values of the components of the exact (and computed) solutions were (very) big numbers in comparison with the entries of the matrices arising during the iterative procedure.

In our tests we used only Kovarik's **Algorithm (B)** (for tests using **Algorithm (A)** see [3], [9], [10], [13]). The optimal value of the regularizing parameter γ in (63) was determined by numerical tests and found equal to 10^{-14}. For this value, the relative ($exrer$) and absolute ($exaer$) errors, computed with the formulas

$$exrer = \frac{\| x^c - x^c_{reg} \|_\infty}{\| x^c \|}, \quad exaer = \| x^c - x^c_{reg} \|_\infty \tag{66}$$

had the values from Table 1 (we must take into account the fact that, for the "square" case ($m = n$), the components of the "regularized" exact solution x^c_{reg} from (60) were of order 10^{13}, and in the "rectangular" case ($n = 9, m > n$) these values were of order $10^5 - 10^6$). In the 5-th column of Table 1 we present the minimal numbers of iterations for the stopping test (64). In the columns 6 and 7 of Table 1 we described the minimal numbers of iterations to fullfil the stopping test (65) with **Part II** of the algorithm **A1** for $\alpha = 1$ and the two indicated values for ω. We observe a small improvement with respect to these two different choices.

Remark 9 *Unfortunately, only the algorithm* **A1** *could be used for the (good) value $\alpha = 10^{-14}$ of the regularizing parameter. The algorithm* **A2** *was totally unstable for such small values of γ. The last (minimal) value for which it could be used was $\gamma = 10^{-8}$ (see also the numerical results from [3]).*

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Elena Pelican graduated Faculty of Mathematics and Computer Science at the "Ovidius" University of Constanta, Romania, where she is teaching as assistant professor since 1999. In the same time she is a Ph.D student at Institute of Applied Mathematics of the Romanian Academy of Sciences, Bucharest. Her research interests are concerned with efficient numerical solvers for integral equations of the first kind.

Constantin Popa graduated Faculty of Mathematics and obtained the Ph.D at the University of Bucharest, Romania. Since 1990 he is teaching at the "Ovidius" University of Constanta, having now a full professor position. He had two DAAD research grants in Germany (Kiel and Augsburg) and a one year postdoc grant (1996-1997) at Department of Mathematics, Weizmann Institute, Israel. His research interests are centered on preconditioning techniques, iterative solutions of least - squares formulations and regularization techniques for inverse problems.

Table 1 Numerical experiments with algorithm **A1**

m	n	$exrer$	$exaer$	$iter.$ $(refeq27)$	$\omega = 1$	$\omega = 0.8$
9	9	0.002	6600	53	1	1
17	17	0.004	574	54	7	6
33	33	0.008	71	55	13	9
65	65	0.02	53	56	17	11
129	129	0.33	2384	57	19	13
257	257	0.49	5712	58	22	16
513	513	0.55	10^4	59	24	19
17	9	10^{-7}	0.02	53	7	5
33	9	10^{-7}	0.3	54	15	10
65	9	10^{-7}	0.08	54	29	20
129	9	10^{-7}	0.002	55	58	39
257	9	10^{-7}	0.004	55	116	77
513	9	10^{-6}	0.03	56	230	150