AMO - Advanced Modeling and Optimization, Volume 2, Number 3, 2000

NUMERICAL EXPERIMENTS ON ATTRACTION BASIN

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ABSTRACT. In this article we shall study the attraction basins [2] of different iterative methods. There has been built a function that defines the attraction basins [3] of the method and the equation that is to be solved. The graphic representation of this function on a rectangular domain allows us to compare different iterative methods; this, from the point of view the convergence, which is determined by the choice of the initial iteration and the norm choice.

1. INTRODUCTION

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differential Fréchet application. For solving the equation $F(x) = 0$ an iterative method is used:

$$x^k = \Phi(x^{k-p}, x^{k-p+1}, \ldots, x^{k-1}) \quad k = \overline{p, km} \quad x^0, x^1, \ldots, x^{p-1} \; are\; given$$

Let $d_{F,\Phi,\|\cdot\|} : \mathbb{R}^{n\times p} \times \mathbb{R}_+ \times \mathbb{N} \to \mathbb{N}$ be the application which depends on F and Φ, defined as follows:

$$d_{F,\Phi,\|\cdot\|}(x^0, \ldots, x^{p-1}, \varepsilon, km) = \begin{cases} smallest\,\mathbf{k} & if\, \exists\, \Phi(x^{k-p}, \ldots, x^{k-1}) \,\wedge\, \left\|F(x^k)\right\| < \varepsilon \\ \mathbf{km} & if\, \overline{\exists}\, \Phi(x^{k-p}, \ldots, x^{k-1}) \,\vee\, \left\|F(x^k)\right\| \geq \varepsilon\, \forall\, k \end{cases}$$

from $k \quad p \leq k \leq km$.

This application counts the iterations of the Φ method for approximating the roots of the equation $F(x) = 0$. The result in an integer $\mathbf{k}$, $p \leq k \leq km$, if $\mathbf{k} = \mathbf{km}$ the method does not converge, if $\mathbf{k} \neq \mathbf{km}$ the method converges in $\mathbf{k}$ steps to a root that carries out the condition $\left\|F(x^k)\right\| < \varepsilon$.

The condition for $\Phi(x^{p-1}, x^{p-2}, \ldots, x^{p-s})$ to be defined, may be stated, if the case is concrete, through mathematical conditions. In order to exemplify the Newton method is presented. Let there be the application F and the iterative method:

$$N(x) = x - F'(x)^{-1} \cdot F(x)$$

$$x^k = N(x^{k-1}) \quad k = \overline{1, m} \quad x^0\, given$$

then

1991 *Mathematics Subject Classification.* AMS 65H05.

Key words and phrases. Nonlinear equation, Newton method, Steffensen method, Secant method, Broyden method, Fridman method, Conjugate Gradient method and attraction basin.

$$d_{F,N,\|\cdot\|}(x^0,\varepsilon,km)=\begin{cases} smallest\,\mathbf{k} \quad for\left|F'(x^{k-1})\right|\neq 0\wedge \left\|F(x^k)\right\|<\varepsilon\\ \mathbf{km} \quad if\left|F'(x^{k-1})\right|=0\vee \left\|F(x^k)\right\|\geq\varepsilon\,\forall\, k\end{cases}$$

from $k \quad p\leq k\leq km$.

The function $d_{F,\Phi,\|\cdot\|}$ depends on the chosen norm. A procedure that defines the application d, is now presented (the notation $d_{F,\Phi,\|\cdot\|}$ is dropped). The application is defined in Mathcad 2000, where the vector x, the precision ε, the value of km, and the application F may be defined, while the application F the **J**acobean matrix.

For $n=2$ in the rectangle $[a,b]\times[c,d]$ we consider the discrete points given by the formulas:

$$x^0=\begin{bmatrix}a+lh\\ c+jq\end{bmatrix}$$

where $l=\overline{0,m}$, $j=\overline{0,s}$, $h=\dfrac{b-a}{m}$ and $q=\dfrac{d-c}{s}$ if we equalize the (l,j) pair x^0 point, then we can define function d as follows:

$$d(l,j,\varepsilon,km)=\begin{cases} k \quad if\, the\, method\, converges\, in\, k\, steps,\, with\, an\, \varepsilon\, precision\\ km \quad if\, method\, does\, not\, converges\, km\, steps,\, with\, an\, \varepsilon\, precision\end{cases}$$

This two allow us to compare different iterative methods, considering the attraction basin inside the $[a,b]\times[c,d]$ rectangle. the graphic representation of the d function can supply us a synthetic image of the attraction basins. The regions colored in bright shad's represent the bigger values of function d (therefore the represent regions in which the number of iterations is large). If the value of the d function is km then, the method does not converge in km steps with a ε precision. The Newton, Steffensen, Secant, Broyden, Conjugated Gradient and Fridman methods [1] have been compared for nonlinear equations:

$$(1.1)\qquad \begin{cases} 3\cdot x_1^2-x_2^2=0\\ \dfrac{1}{2}\cdot cosh(\dfrac{5\cdot x_1}{3})+\dfrac{3}{5}\cdot x_1-x_2-\mathbf{\dfrac{3}{5}}=0\end{cases}$$

$$(1.2)\qquad \begin{cases} 3\cdot x_1^2-x_2^2=0\\ \dfrac{1}{2}\cdot cosh(\dfrac{5\cdot x_1}{3})+\dfrac{3}{5}\cdot x_1-x_2-\mathbf{\dfrac{1}{5}}=0\end{cases}$$

$$(1.3)\qquad \begin{cases} 3\cdot x_1^2-x_2^2=0\\ \dfrac{1}{2}\cdot cosh(\dfrac{5\cdot x_1}{3})+\dfrac{3}{5}\cdot x_1-x_2=0\end{cases}$$

$$(1.4)\qquad \begin{cases} 3\cdot x_1^2-x_2^2=0\\ \dfrac{1}{2}\cdot cosh(\dfrac{5\cdot x_1}{3})+\dfrac{3}{5}\cdot x_1-x_2+\mathbf{\dfrac{1.15}{5}}=0\end{cases}$$

The studied $[a,b]\times[c,d]$ rectangle is the $[-6,6]\times[-9,9]$ square and the number of discrete points equals $21901=121\times181$; imposed ε precision is of 0.001. The maximum number (km) of iterations is different, depending on the convergence order of the method. The convergence order of the Newton method is 2, obviously, if the method does not converge in 15 steps to a solution (with ε precision); it will not converge for a bigger km.

2. Newton and Steffensen method

The convergence of both Newton and Steffensen methods are not influenced by the chosen norm. In figures 1: B1N for 1.1, B2N for 1.2, B3N for 1.3 and B4N for 1.4 in Appendix 1 the attraction basins of Newton method are presented. In figures 2: B1St for 1.1, B2St for 1.2, B3St for 1.3 and B4St for 1.4 Appendix 1 the attraction basins of Steffensen method are presented.

$$f_1(x) := 3 \cdot x_1^2 - x_2^2$$
$$f_2(x) := \frac{1}{2} \cdot cosh\left(\frac{5 \cdot x_1}{2}\right) + \frac{3}{5} \cdot x_1 + x_2 - \frac{3}{5}$$

$$F(x) := \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

$$no(r, x) := (x_1^r + x_2^r)^{\frac{1}{r}}$$

$$J(x) := \begin{bmatrix} \frac{d}{dx_1} f_1(x) & \frac{d}{dx_2} f_1(x) \\ \frac{d}{dx_1} f_2(x) & \frac{d}{dx_2} f_2(x) \end{bmatrix}$$

$$\begin{aligned}
N(l, j, r, \varepsilon, km) :=\, & k \leftarrow 0 \\
& x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix} \\
& k \leftarrow km \ on \, error \ w \leftarrow F(x) \\
& k \leftarrow km \ on \, error \ \delta \leftarrow no(r, w) \\
& while \, (\delta \geq \varepsilon) \wedge (k < km) \\
& \quad k \leftarrow k + 1 \\
& \quad k \leftarrow km \ on \, error \ x \leftarrow x - J(x)^{-1} \cdot w \\
& \quad k \leftarrow km \ on \, error \ w \leftarrow F(x) \\
& \quad k \leftarrow km \ on \, error \ \delta \leftarrow no(r, w) \\
& return \ k
\end{aligned}$$

$$
\begin{aligned}
St(l,j,r,\varepsilon,km) :=\, & k \leftarrow 0 \\
& x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix} \\
& k \leftarrow km \; on\,error \; w \leftarrow F(x) \\
& k \leftarrow km \; on\,error \; \delta \leftarrow no(r,w) \\
& while\,(\delta \geq \varepsilon) \wedge (k < km) \\
& \quad k \leftarrow k + 1 \\
& \quad k \leftarrow km \; on\,error \; x \leftarrow x - JS(x,w)^{-1} \cdot w \\
& \quad k \leftarrow km \; on\,error \; w \leftarrow F(x) \\
& \quad k \leftarrow km \; on\,error \; \delta \leftarrow no(r,w) \\
& return \; k
\end{aligned}
$$

$$
JS(x,h) := \begin{bmatrix} \dfrac{f_1\left[x + \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] - f_1(x)}{h_1} & \dfrac{f_1\left[x + \begin{pmatrix} 0 \\ h_2 \end{pmatrix}\right] - f_1(x)}{h_2} \\ \dfrac{f_2\left[x + \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] - f_2(x)}{h_1} & \dfrac{f_2\left[x + \begin{pmatrix} 0 \\ h_2 \end{pmatrix}\right] - f_2(x)}{h_2} \end{bmatrix}
$$

3. Secant and Broyden method

The Secant method is 2 steps method; the first step being considered the $(0,0)$ point. The Broyden method needs an initial matrix H from which the approximation of the Jacobean begins. The convergence of these methods is not influenced by the chosen norm. In figures 3: B1Sec for 1.1, B2Sec for 1.2, B3Sec for 1.3 and B4Sec for 1.4 in Appendix 1 the attraction basins of Secant method are presented. In figures 4: B1B for 1.1, B2B for 1.2, B3B for 1.3 and B4B for 1.4 Appendix 1 the attraction basins of Broyden method are presented.

$$
\begin{aligned}
Sec(l,j,r,\varepsilon,km) :=\, & k \leftarrow 0 \\
& x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix} \\
& h \leftarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} - x
\end{aligned}
$$

$k \leftarrow km$ *on error* $w \leftarrow F(x)$
$k \leftarrow km$ *on error* $\delta \leftarrow no(r, w)$
$while\ (\delta \geq \varepsilon) \wedge (k < km)$
  $k \leftarrow k + 1$
  $k \leftarrow km$ *on error* $h \leftarrow JS(x, h)^{-1} \cdot w$
  $k \leftarrow km$ *on error* $x \leftarrow x - h$
  $k \leftarrow km$ *on error* $w \leftarrow F(x)$
  $k \leftarrow km$ *on error* $\delta \leftarrow no(r, w)$
$return\ k$

$B(l, j, r, \varepsilon, km) := k \leftarrow 0$
$x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix}$
$k \leftarrow km$ *on error* $w \leftarrow F(x)$
$k \leftarrow km$ *on error* $\delta \leftarrow no(r, w)$
$H \leftarrow \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$
$while\ (\delta \geq \varepsilon) \wedge (k < km)$
  $k \leftarrow k + 1$
  $k \leftarrow km$ *on error* $s \leftarrow -(H \cdot w)$
  $k \leftarrow km$ *on error* $y \leftarrow F(x + s) - w$
  $k \leftarrow km$ *on error* $v \leftarrow H \cdot y$
  $k \leftarrow km$ *on error* $H \leftarrow H + \dfrac{(s - v) \cdot s^T}{s \cdot v} \cdot H$
  $k \leftarrow km$ *on error* $x \leftarrow x + s$
  $k \leftarrow km$ *on error* $w \leftarrow F(x)$
  $k \leftarrow km$ *on error* $\delta \leftarrow no(r, w)$
$return\ k$

4. Conjugate Gradient and Fridman method

The convergence of the Conjugate Gradient and Fridman methods is influenced by the chosen norm. It is known that these methods best converge in the Euclidian norm $\|\cdot\|_2$.In figures 5: B1CG for 1.1, B2CG for 1.2, B3CG for 1.3 and B4CG for 1.4 in Appendix 1 the attraction basins of Gradient Conjugate method are presented. In figures 6: B1Fr for 1.1, B2Fr for 1.2, B3Fr for 1.3 and B4Fr for 1.4 Appendix 1 the attraction basins of Fridman method are presented.

$$
\begin{aligned}
GC(l,j,r,\varepsilon,km) :=\; & k \leftarrow 0 \\
& x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix} \\
& k \leftarrow km \; \mathit{on\,error} \; w \leftarrow F(x) \\
& k \leftarrow km \; \mathit{on\,error} \; \delta \leftarrow no(r,w) \\
& p \leftarrow w \\
& \mathit{while}\,(\delta \geq \varepsilon) \wedge (k < km) \\
& \quad k \leftarrow k + 1 \\
& \quad \beta \leftarrow \frac{1}{\delta^2} \\
& \quad k \leftarrow km \; \mathit{on\,error} \; y \leftarrow J(x)^T \cdot p \\
& \quad k \leftarrow km \; \mathit{on\,error} \; x \leftarrow x - \frac{w \cdot p}{no(r,y)^2} \cdot y \\
& \quad k \leftarrow km \; \mathit{on\,error} \; w \leftarrow F(x) \\
& \quad k \leftarrow km \; \mathit{on\,error} \; \delta \leftarrow no(r,w) \\
& \quad \beta \leftarrow \beta \cdot \delta^2 \\
& \quad p \leftarrow w \; \mathit{if} \; mod(k,2) = 0 \\
& \quad p \leftarrow w + \beta \cdot p \quad \mathit{otherwise} \\
& \mathit{return}\; k
\end{aligned}
$$

$$
\begin{aligned}
Fr(l,j,r,\varepsilon,km) :=\; & k \leftarrow 0 \\
& x \leftarrow \begin{pmatrix} a + l \cdot h \\ c + j \cdot q \end{pmatrix} \\
& k \leftarrow km \; \mathit{on\,error} \; w \leftarrow F(x) \\
& k \leftarrow km \; \mathit{on\,error} \; \delta \leftarrow no(r,w) \\
& \mathit{while}\,(\delta \geq \varepsilon) \wedge (k < km) \\
& \quad k \leftarrow k + 1 \\
& \quad k \leftarrow km \; \mathit{on\,error} \; y \leftarrow J(x)^T \cdot w \\
& \quad k \leftarrow km \; \mathit{on\,error} \; x \leftarrow x - \frac{no(r,w)^2}{no(r,y)^2} \cdot y \\
& \quad k \leftarrow km \; \mathit{on\,error} \; w \leftarrow F(x) \\
& \quad k \leftarrow km \; \mathit{on\,error} \; \delta \leftarrow no(r,w) \\
& \mathit{return}\; k
\end{aligned}
$$

References

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5. Appendix 1

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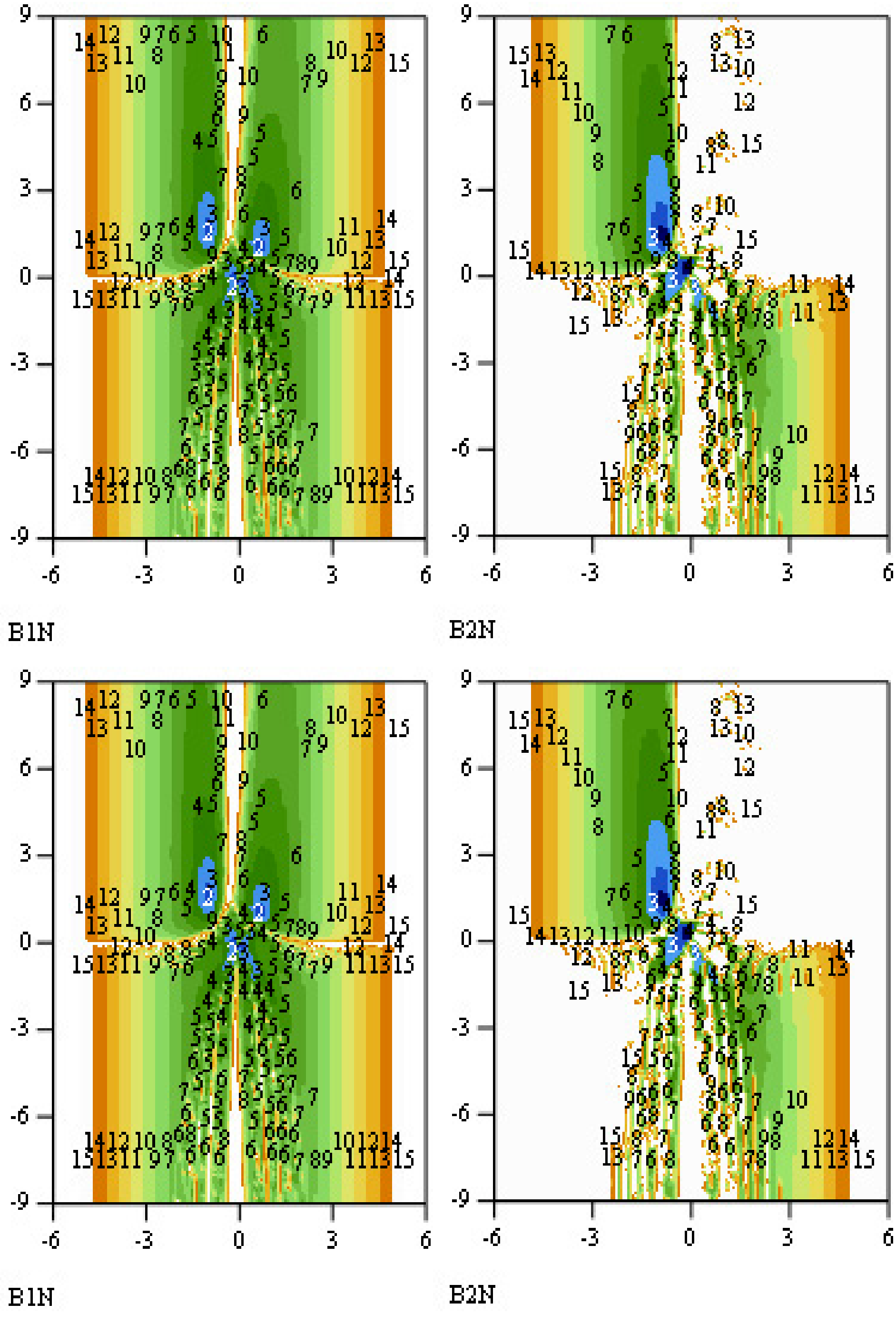

FIGURE 1. Attraction Basins for Newton Method

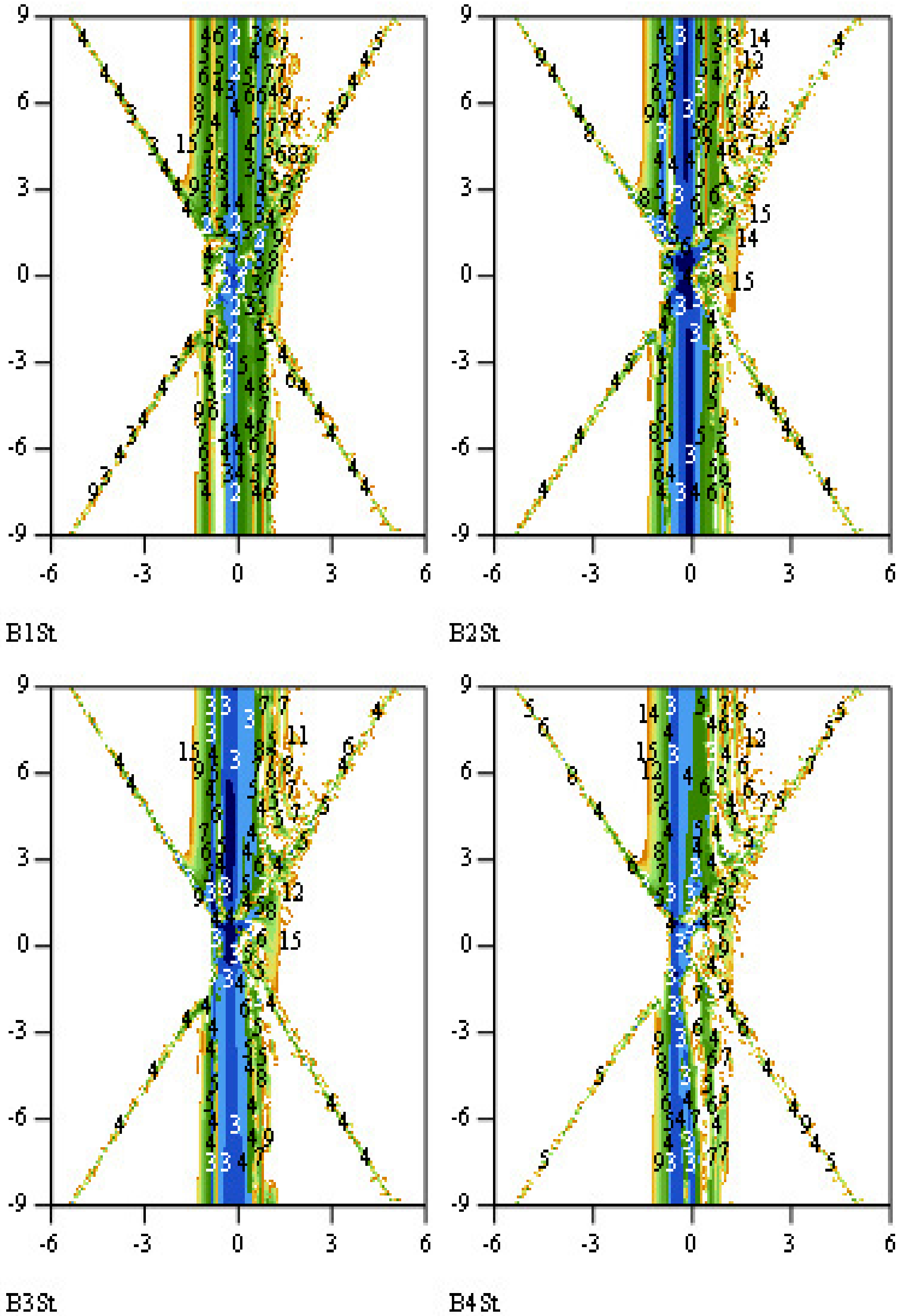

FIGURE 2. Attraction Basins for Steffensen Method

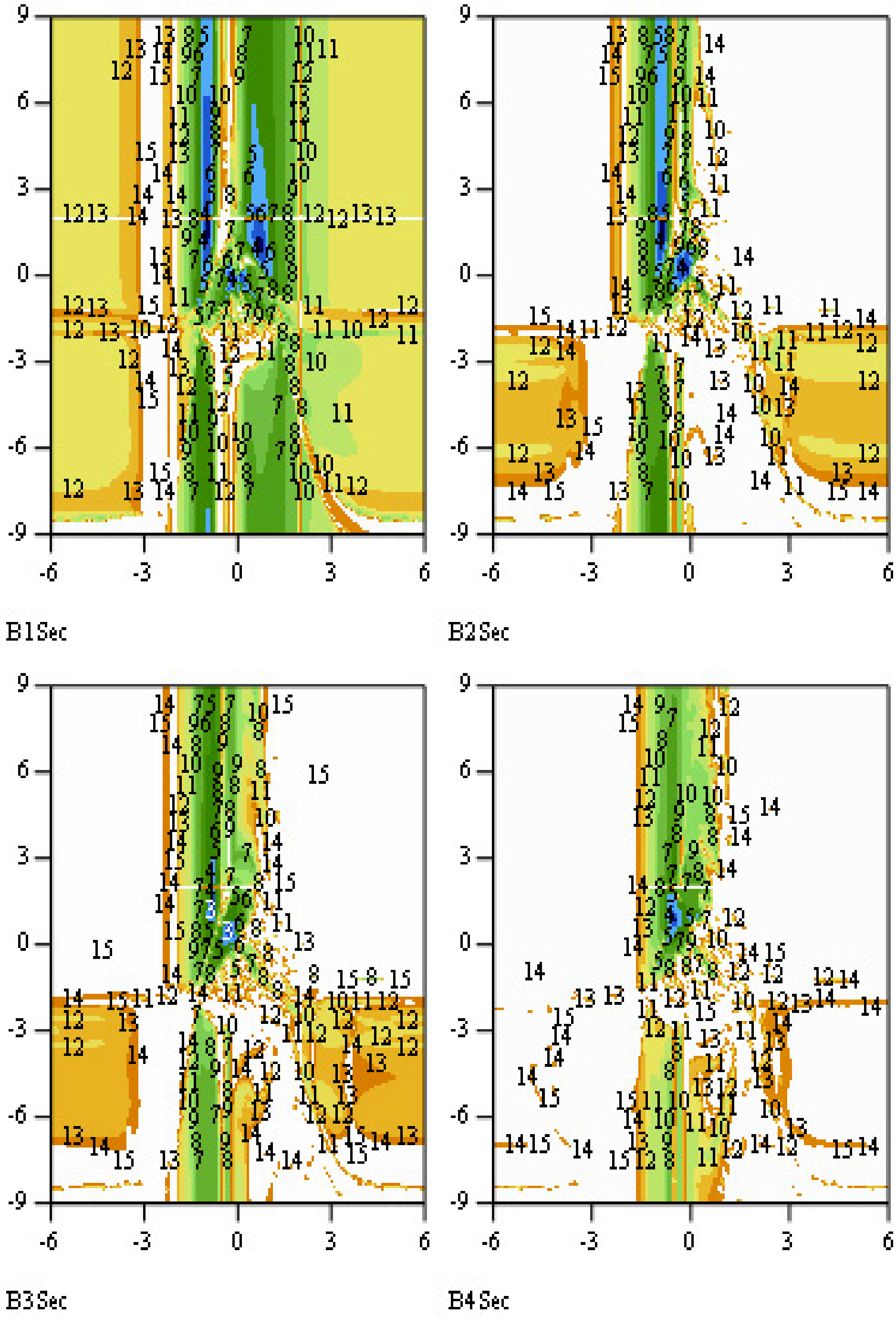

FIGURE 3. Attraction Basins for Secant Method

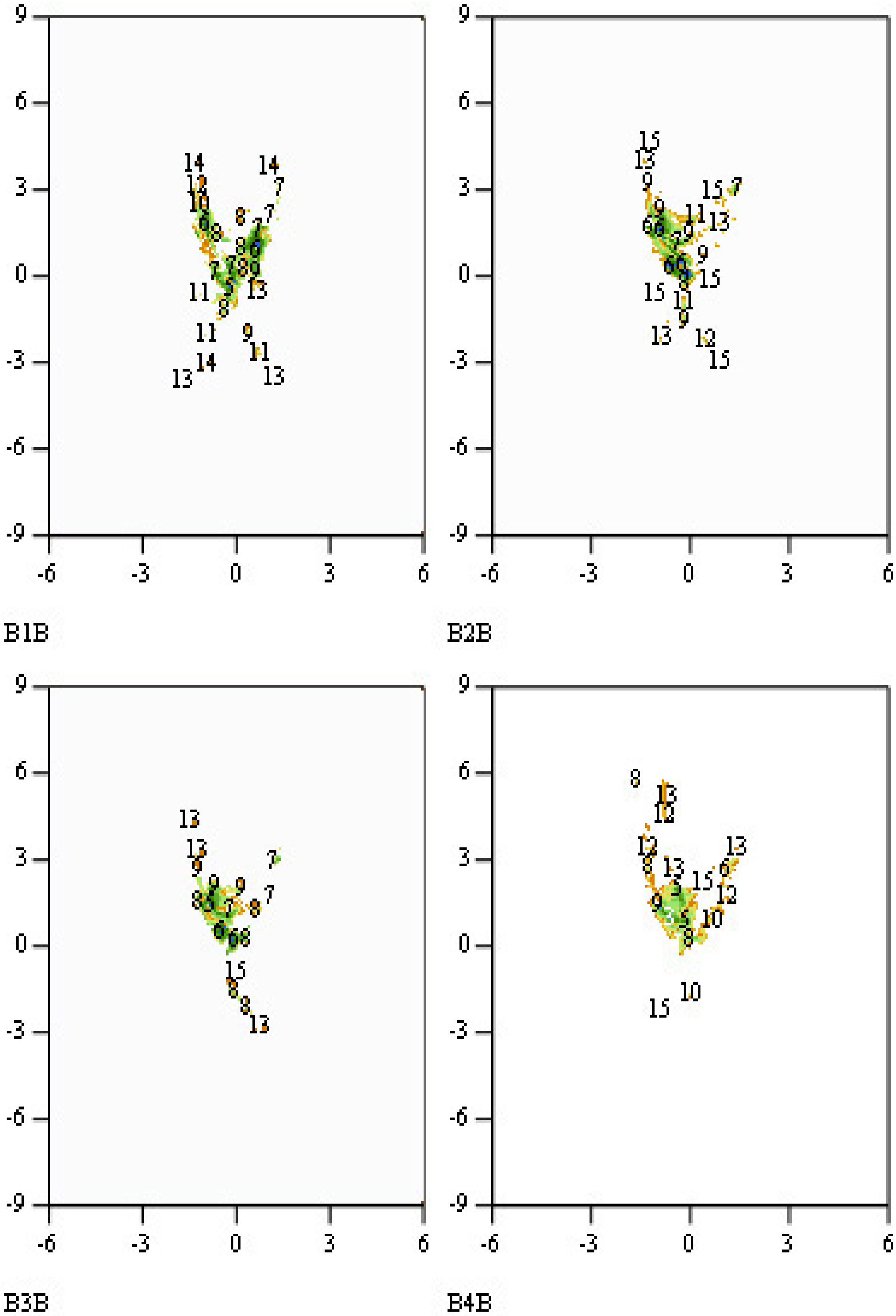

FIGURE 4. Attraction Basins for Broyden Method

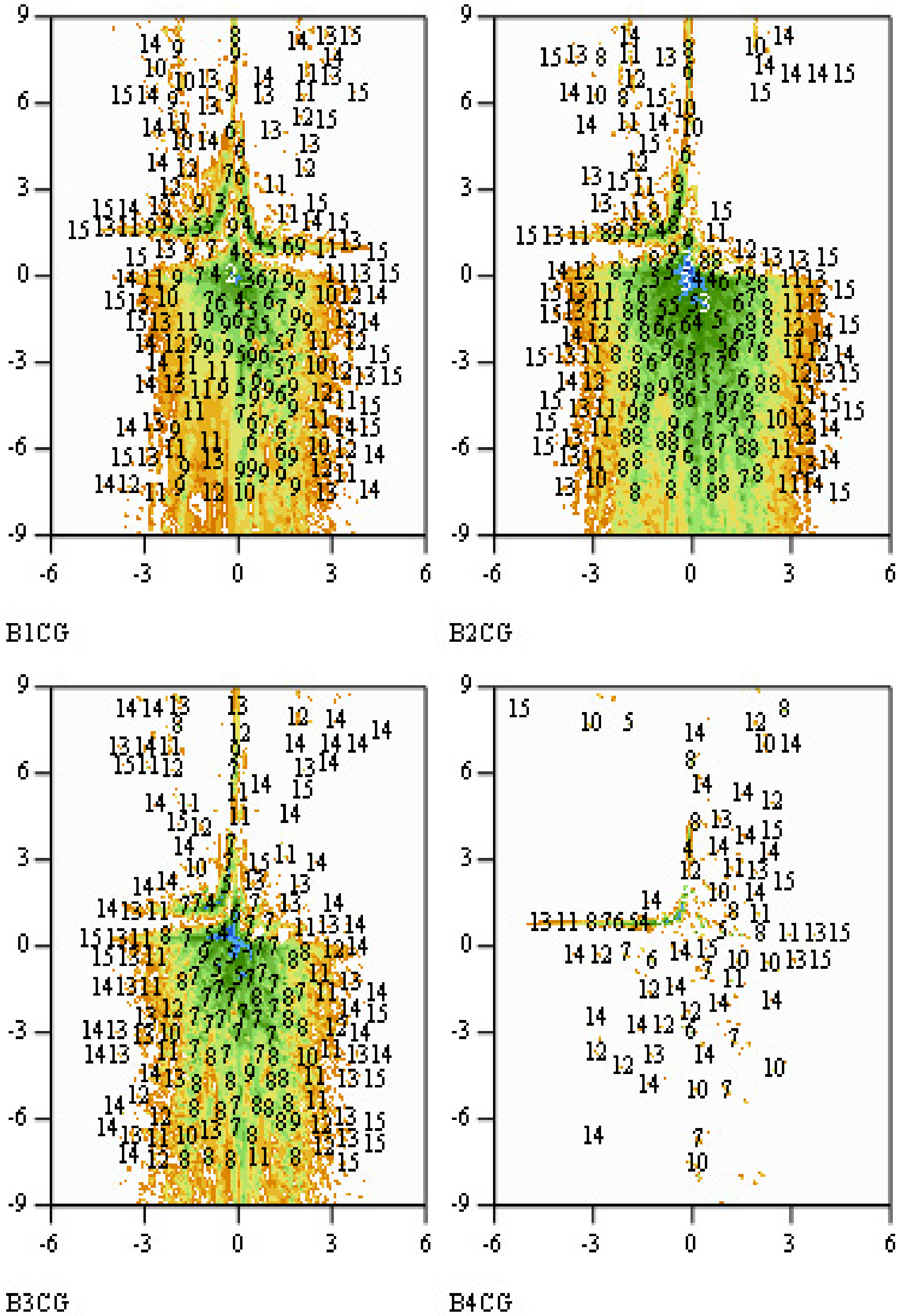

Figure 5. Attraction Basins for Conjugate Gradient Method

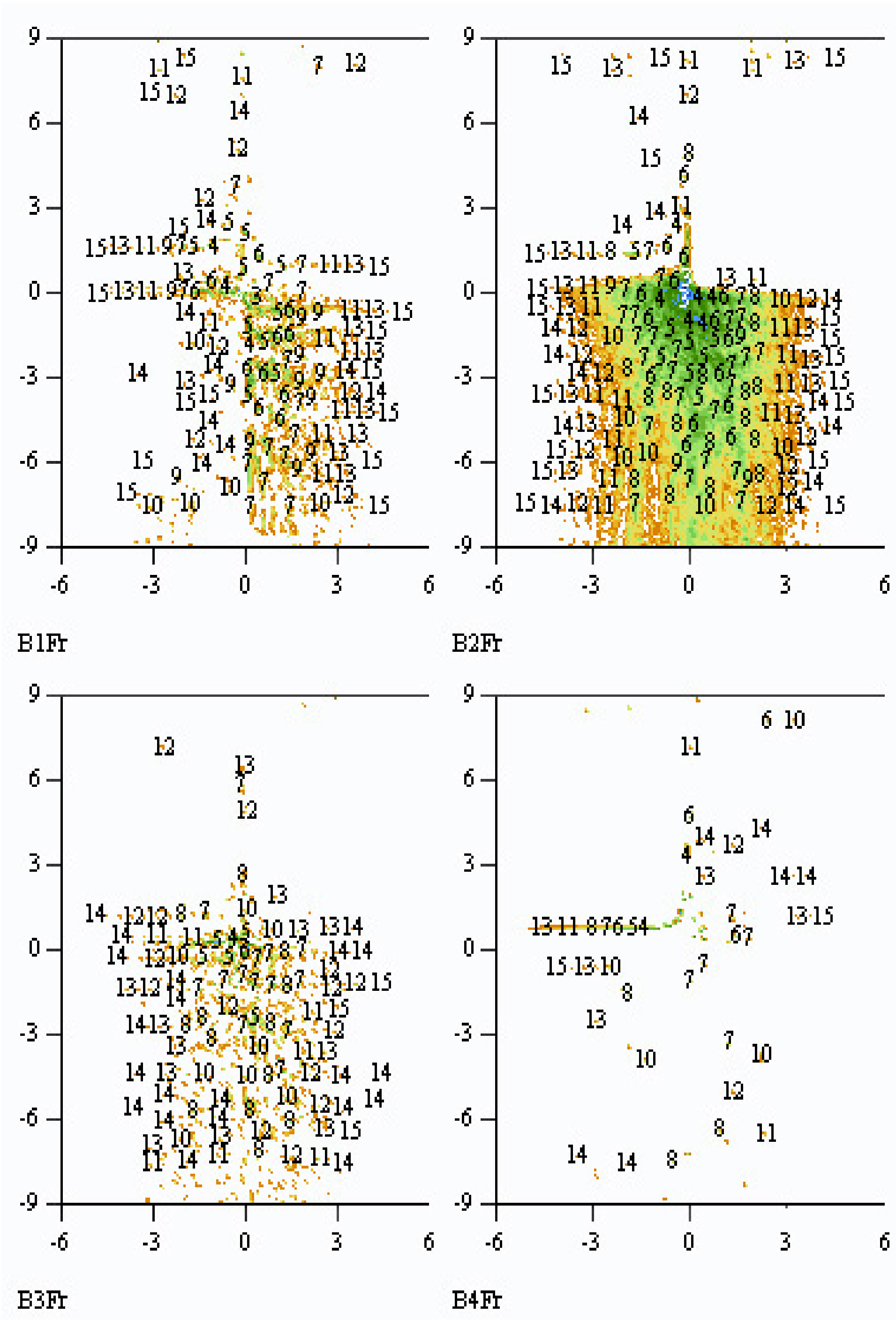

FIGURE 6. Attraction Basins for Fridman Method