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An Interior Multiobjective Linear Programming Algorithm

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This paper presents an interactive interior point algorithm for determining a best compromise solution to a multiple objective linear programming problem in situations with an implicitly defined utility function. We construct a sequence $X^0, X^1, ..., X^k, ..$ of a smaller and smaller polytopes which shrink towards the compromise solution . The algorithm generates a sequence of points $x^0, x^1, ..., x^k, ..$, such that x^k is in the interior of polytope X^k, by maximizing a linear function over an ellipsoid.

Key Words. Multiobjective linear programming, Interior point methods, Interactive methods, compromise solution.

1 Introduction

Multiobjective mathematical programming (MOMP) problem arise in differents fields, and researchers have devoted considerable attention to methods for solving them. The purpose of the solution methods is to optimize the decision maker's utility (peference) function. Depending on the assumption made on the utility function, the methods can be classified as utility theory, goal programming, and the progressive articulation and posterior articulation approaches. The progressive articulation approach, called the interactive method, is becoming increasingly popular and is considered a promising approach. The interactive methods are designed to find one "most preferred " solution for problem (MOMP) , that is dictated by an explicit or implicit utility function that belongs to the DM. For more details of each approach, refer to the textbooks by [Chankong & Haims,1983], and [R.E.Steuer,1986].

On the other hand, recent breakthroughs in linear programming (LP) have revolutionized the field. In particular, when Khachian and Karmarkar demonstrated that it is computationaly more efficient to work with an ellipsoid that approximate the polytope it self. With the algorithm introduced by N.Karmarkar in 1984 another computationnal approach is now available for solving linear programming problems. This original algorithm led to the development of other variants that are commonly referred to as interior point algorithms for linear programming.

Although researchers have developed numerous interactive solution methods, there are only a few interior point methods in the literature, [B.Aghezzaf and T.Ouaderhman,1998], [A.Arbel,1994], [A.Arbel and S.Oren,1996], [A.Arbel & P.Korhonen,1996] and [T.Trafalis et al,1990]. The solution approaches that are currently available use the affine scaling primal algorithm, or the primal-dual algorithm with various schemes for defining a single interior search direction. Two others approachs developed by T.Trafalis et al[T.Trafalis et al,1990] use the so-called method of centers to both linear and nonlinear multiple objective programming problem, where the decision space is a convex polyhedron defined by linear inequalities. The first algorithm is a generalization of Renegar's algorithm [Renegar J.,1986] to multiple objective setting, that finds an efficient point of the polytope. The second, is based on techniques of algebraic geometry related to the parametrization of algebraic varieties in n-dimentional spaces. It approximates a portion of the set of efficient faces by an algebraic surface using the nonnnegative orthant as the ordering cone to reduce the decision space . Since the cone depends on the inscribed ellipsoid, the best compromise solution may not be retained in the reduced space.

To remedy to this situation, we present an interactive interior point algorithm for determining a best compromise solution to a multiobjective linear programming problem (MOLP) in situations with an implicitly defined utility function which is illustrated by deriving the local trade-offs (the marginal rates of substitution) by interacting with the decision maker (DM) . The algorithm is based in principle on the same idea used by Vaidya[Vaidya,1990] in single objective algorithm. Employing this approach we construct a sequence $X^0, X^1, ..., X^k, ..$ of a smaller and smaller polytopes which shrink towards the compromise solution , the algorithm generates a sequence of points $x^0, x^1, ..., x^k, ..$, such that x^k is in the interior of polytope X^k, by maximizing a linear function over an ellipsoid. In this paper we do not need to add the trade-off cut at each iteration as is done in [B.Aghezzaf and T.Ouaderhman,1998].

The paper is arranged as follows. In section 2, the problem statement and some fundamental definitions are stated. Section 3 describes the algorithmic procedure of the method and the results of the computational studies with the method are included in section 4.

2 Statement of the problem

Multi-objective linear programming (MOLP) problem are concerned with determining scalar values of n continuous decision variables in order to maximize $l (l \geq 2)$ objectives simultaneously. A mathematical formulation of the problem is :

$$(P) \left\{ \begin{array}{rrcl} \max & f(x) & = & \{f_1(x), ..., f_l(x)\} \\ \text{s.t} & x & \in & X = \{x \in I\!R^n |\ Ax \geq b\} \end{array} \right.$$

where $f_i(x) = c_i^T x, (1 \leq i \leq l)$. C and A are $l \times n$ and $m \times n$ matrices respectively, and $b \in I\!R^m$. Here the row $c_i^T, 1 \leq i \leq l$ of C are the coefficients of l linear objective functions f_i. In order to make the problem nontrivial, it will be assumed that the objectives are in conflict and incommensurable. Due to the conflicting nature of the objectives, an optimal solution that simultaneously maximizes all the objectives is usually not obtainable . Instead, there are several solutions, called efficient solutions (nondominated or pareto optimum), that have the property that no improvement in any one objective is possible without sacrificing on one or more of the other objectives. The set of all efficient solutions(in continuous case) is knwon as the efficient frontier.

Definition 2.1 *a solution $\bar{x} \in X$ is said to be efficient if there exists no point $x \in X$ such that :*

$$f(x) \geq f(\bar{x}) \quad and \quad f(x) \neq f(\bar{x})$$

With the above definition , the solution to a MOMP problems reduces to the problem of finding all efficient solutions. Often there are an infinite number and they are not comparable. Hence, it is assumed that the decision maker has a real valued utility function (value function or preference function) U, defined on the values of objectives, which has not been accessed and is not explicitly known, with this assumption, the MOMP problem can be redefined as follows :

$$(P_u) \begin{cases} \max & U(f_1(x), ..., f_l(x)) \\ \text{s.t} & x \in X \end{cases}$$

Within this mathematical framework, the primary objective of the interactive approaches is to find a best compromise solution through interaction with the DM.

Definition 2.2 *A best compromise solution (or optimal solution) to MOLP is an efficient solution that maximizes the decision maker's utility function. That is, a feasible solution $\bar{x}$ is the best compromise solution if and only if:*

$$U(f(\bar{x})) \geq U(f(x)), \qquad for\ all \quad x \in X$$

In this research, with (P_u) as problem under consideration, an interactive interior point algorithm that can be used to solve MOLP will be developed and tested . For the development of the method, it is assumed that X is bounded , and U is strictly increasing in each of its arguments, concave and differentiable on $I\!R^l$. As the polytope X is bounded we can assume that $m \geq n$, and the columns of A are linearly independent.

3 The proposed algorithm

To solve the multiobjective optimization problem we have to find a way to approximate the utility function at the current iterate and take a step in a direction that results in an improvement of the value of the objective function. The partial information on

the utility function of interest is the local trade-offs (marginal rates of substitution) between the objective functions which obtained by interacting with the decision maker. This concept is first proposed in multiple creteria decision making by Geoffrion, Dyer and Feinberg [Geoffrion & al,1972]. We assume that $f_1(x)$ is as a reference criterion .

Definition 3.1 *The tradeoff ratio between the objective* $f_1(x)$ *and* $f_i(x)$ *at point* x^k *is :*

$$\omega_{1i} = \frac{\frac{\partial U}{\partial f_i}(x^k)}{\frac{\partial U}{\partial f_1}(x^k)}$$

Consider the property of the utility function, the partial derivative of U at a point x^k is :

$$\begin{aligned} \nabla_{x^k} U &= \left[\frac{\partial U}{\partial x_1^k}, ..., \frac{\partial U}{\partial x_n^k}\right]. \\ &= (\frac{\partial U}{\partial f_1})\left[\sum_{i=1}^{l} \omega_{1i}^k \frac{\partial f_i}{\partial x_1^k}, ..., \sum_{i=1}^{l} \omega_{1i}^k \frac{\partial f_i}{\partial x_n^k}\right] \\ &= (\frac{\partial U}{\partial f_1})\dot{\nabla}_{x^k} U = (\frac{\partial U}{\partial f_1})\omega^k.C \end{aligned}$$

where: $\omega^k = [\omega_{11}^k, ..., \omega_{1l}^k]$ and $\dot{\nabla}_{x^k} U = \omega^k.C$
$\dot{\nabla}_{x^k} U$ is termed a positively scaled local gradient of U at x^k.

3.1 Generating a polytopes

Motivation for our approach arose from some difficulties encountered with the T.Trafalis et al approach [T.Trafalis et al,1990]. With the T.Trafalis algorithm, an approximation of a portion of the efficient frontier is given by constructing a sequence of algebraic surfaces $\{S_k\}$. The general surface S_k can be considered the loci of analytical centers of intersection of an ordering cone Λ and the polytope X, where the apex λ of Λ is moving on S_{k-1} by taking the initialising manifold S_0 to be a part of the inscribed ellipsoid of X. But the question which arises is how to construct the cone Λ that illustrate the DM preferences.
In using the cone of increasing direction $C^{\geq} = \{x \in I\!R^n / Cx \geq 0\}$ as the ordered cone, the portion of the efficient frontier obtained by T.Trafalis algorithm depends also on the geometry of the polytope X as it depends on Λ in the general case, since the algorithm is dependent on the inscribed ellipsoid of X. Consequently, the final surface S^* generated by the algorithm might possibly be a portion of the efficient frontier that is of no further interest, as will as one can never be completely sure about the existence of one λ^k which lets the desired solution in the restricted polytope and how to select it if that exist (see FIG.1b).

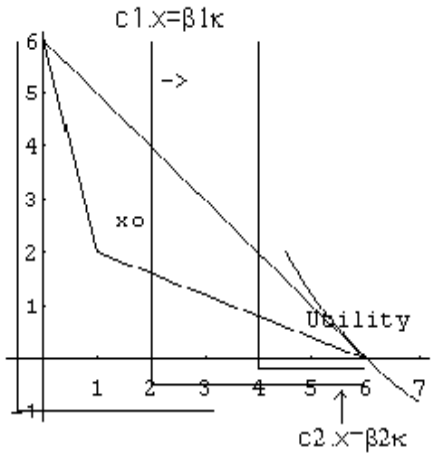

FIG.1a:

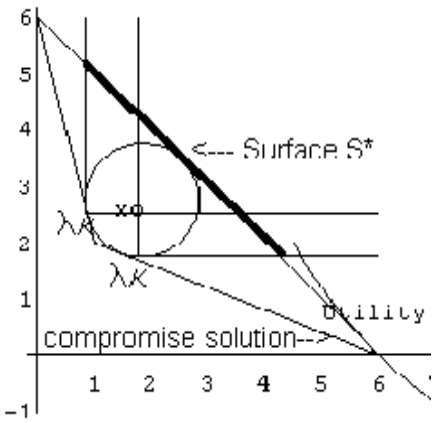

FIG.1b:

In developping our approach, the basis has been to overcome the difficulties encountered with the T.Trafalis algorithm mentioned before, and generating a single interior direction at current iterate that improves the utility function without combining it with other directions resulting from anchoring points as in [Arbel,1994]. This algorithm also follows the Renegar approach[Renegar,1986] , but there are three critical differences which enable us to solve the above difficulties.
First, the reduced space is obtained by using an updating formula to update the lower bounds of objective functions instead of the ordered cone. Second, the potential function is formulated in a manner to orient the iterates to desired solution. Third, the generate iterate is computed along a direction that could be different from the direction in Newton method. On the other hand, the problem is how much the lower bounds values of the objective functions can be changed in order to remain the compromise solution in the reduced space. We must find an appropriate way to update them.

Let x^k be an interior point of X and consider in the following the updating formula used to update the objectives lower levels , that gives in iteration k the added set C^k of constraintes :

$$C^k = \{x \in I\!R^n \quad | \quad Cx \geq \beta(x^k)\}$$

where :

$$\begin{cases} \beta(x^k) & = & [\beta_1^{k+1}, ..., \beta_l^{k+1}]^T. \\ \beta_i^{k+1} & = & \beta_i^k + \alpha_i^k(c_i^T x^k - \beta_i^k), \qquad 1 \leq i \leq l \quad \text{and} \quad 0 \leq \alpha_i^k < 1. \\ \beta_i^0 & \leq & \min\limits_{x \in X} c_i^T.x, \qquad\qquad 1 \leq i \leq l. \end{cases}$$

where the coefficients $\{\alpha_i^k\}$ will be changed, in general, from one iteration to another. By choising convenably the values of the step-sizes α_i^k $(0 \leq \alpha_i^k < 1)$ at each iteration in the updating formulas, any point in the efficient frontier can be obtained, which is not availible with the suggested translation process of $C^{\geq}$ in T.Trafalis et al algorithm

(see Figure 1.a).
Thus, the original decision space is further reduced by adding the above constraints to the existing set of constraints X. Let X^k denote the reduced decision space at the kth iteration, then:

$$X^k = \{x \in I\!R^n |\ A.x \geq b\} \cap C^k.$$

By using the following equality :

$$\begin{array}{rcl} \sum_{i=1}^{i=l}(\beta_i^{k+1} - \beta_i^k) & = & \sum_{i=1}^{i=l} \alpha_i^k (c_i^T x^k - \beta_i^k) \\ & = & \sum_{i=1}^{i=l} \alpha_i^k (c_i^T x^k - z_i) + \sum_{i=1}^{i=l} \alpha_i^k (z_i - \beta_i^k) \end{array}$$

where $z_i = c_i^T.x, x \in X$, and if the lower bounds satisfy the aspirations of the decision maker, he will preferred not to update them, hence: $\sum_{i=1}^{i=l}(\beta_i^{k+1} - \beta_i^k) = 0$. The coefficients $\{\alpha_i^k\}$ that yields us $z = \beta(x^k)$ will satisfy $\sum_{i=1}^{i=l} \alpha_i^k (c_i^T x^k - z_i) = 0$, which could be guaranteed by $\alpha_i^k = \rho\omega_{1i}^k \quad$ or $\quad c_i^T x^k = z_i$.
So, we propose that the values of the step-sizes α_i^k depends on the DM preferences and given by:

$$\beta_i^{k+1} = \beta_i^k + \omega_{1i}^k / (\sqrt{m+l} \sum_{i=1}^{i=l} \omega_{1i}^k)(c_i^T x^k - \beta_i^k). \quad 1 \leq i \leq l$$

3.2 Locating an improving direction of U in the reduced space

The purpose of this step is to locate a direction of increase for (MOLP) problem in the reduced decision space. The scaled local gradient of U is calculated at the current interior point x^k, that is completly specified once the local tradeoffs are obtained from the DM, and used to locate a improving direction of U in the reduced space X^k.
Consider the following problem

$$(P^k) \begin{cases} \max & \dot{\nabla}_{x^k} U.x \\ \text{s.t} & x \in X^k \end{cases}$$

The associated potential function would be :

$$F^k(x) = \sum_{i=1}^{m} \ln(A_i.x - b_i) + \sum_{s=1}^{l} \ln(c_s^T.x - \beta_s^{k+1}) + (m+l+1)\ln(\dot{\nabla}_{x^k} U.x - \lambda^k)$$

where :

$$\lambda^k = \omega^k.\beta(x^k) + \alpha(\dot{\nabla}_{x^k} U.x^k - \omega^k.\beta(x^k)), \quad \text{and} \quad 0 < \alpha < 1$$

Since the columns of A(and $m \geq n$) are linearly independent, $F^k(x)$ is a strictly concave function over the interior of X^k, and the analytic center of X^k given by maximizing

F^k is indeed a unique point. Hence, the problem is converted into a new form by maximizing $F^k(x)$ over the interior of X^k.
One more interesting concept is to introduce the inscribed ellipsoid of X^k, denoted by $E^k(1)$, and maximizing F^k over it. The ellipsoid $E^k(1)$ is given by:

$$E^k(1) = \{x \in I\!R^n |\ (x - x^k)^T.H^k.(x - x^k) \leq 1\}$$

where: $H^k = \sum_{i=1}^{m} \frac{A_i.A_i^T}{(A_i.x - b_i)^2} + \sum_{s=1}^{l} \frac{c_s.c_s^T}{(c_s^T.x - \beta_s^{k+1})^2} + (m + l + 1)\frac{(\dot{\nabla}_{x^k}U)^T.(\dot{\nabla}_{x^k}U)}{(\dot{\nabla}_{x^k}U.x - \lambda^k)^2}$.
At iteration k, the next iterate is obtained by maximizing the linear approximation of $F^k(x)$ over $E^k(1)$ and we have to solve:

$$\begin{cases} \max & \nabla F(x^k).x \\ \text{s.t} & x \in E^k(1) \end{cases}$$

Let z^k be the solution point of this problem. We shall now describe how to find the next iterate x^{k+1} from x^k. From the theory of convex optimization $(z^k - x^k)$ satisfies the system of linear equations, given by the (K.K.T) conditions:

$$H^k(z^k - x^k) = t\nabla F(x^k).$$

For some scalar $t > 0$. We first compute ξ^k, a vector in the direction of $(z^k - x^k)$ by solving the system of linear equations:

$$H^k.\xi^k = \nabla F(x^k).$$

Next, we find a scalar $t^k > 0$, such that :

$$(t^k)^2(\xi^k)^T.H^k.\xi^k \leq 1.$$

Once the direction ξ^k and t^k are available, we take the next interior step according to the updating formula given by :

$$x^{k+1} = x^k + \rho t^k \xi^k. \qquad 0 < \rho \leq 1$$

Line search : the line search is executed along the direction $t^k\xi^k$ by varying the ρ value

3.3 Algorithm

In this section, the major steps of the algorithm are presented :

- step1: $k = 0$,
  initialization: x^0 , $A.x^0 > b$ and select a finite $\beta_i^0 \leq \min_{x \in X} c^T.x, \quad 1 \leq i \leq l$

- step2: iteration k :
  a) Determine the tradeoffs vector ω^k at x^k.
  b) evaluate :

$$\beta_i^{k+1} = \beta_i^k + \frac{\omega_{1i}^k}{(\sqrt{m+l})\sum_{i=l}^{i=1}\omega_{1i}^k}(c_i^T x^k - \beta_i^k). \quad 1 \leq i \leq l$$

$$\lambda^k = \omega^k.\beta(x^k) + \alpha(\dot{\nabla}_{x^k} U.x^k - \omega^k.\beta(x^k)), \qquad 0 \leq \alpha < 1.$$

  c) Solve the linear system:

$$H^k.\xi^k = \nabla F(x^k).$$

  d) Determine t^k: $(t^k)^2(\xi^k)^T.H^k.\xi^k \leq 1.$
  e) Find the new iterate x^{k+1} from : $x^{k+1} = x^k + \rho t^k \xi^k.$ $0 < \rho \leq 1$

- step3 : If the DM accept x^{k+1} as the compromise solution , stop.
  Otherwize increment the iteration counter: $k := k + 1$, goto step2

4 Illustrative example

The algorithm proposed in this paper will be illustrated using the following numerical example[Arbel,1994]. We have chosen this example because the approach will yield a final interior point better than that found in A.Arbel approach :

$$\begin{cases} \max & x_1 & & \\ \max & & x_2 & \\ \text{s.t} & x_1 + 5x_2 & \leq & 41 \\ & 2x_1 + 3x_2 & \leq & 33 \\ & 4x_1 + x_2 & \leq & 41 \\ & x_1 - 2x_2 & \leq & 8 \\ & x_1 + x_2 & \geq & 2 \\ & -4x_1 + x_2 & \leq & 4 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \end{cases}$$

For this example, an initial point and the initialization of β_i^k are available through :

$$x^0 = [2, 2], \qquad \beta_1^0 = \beta_2^0 = -1$$

To simulate the DM's choice behavior, we will assume the following implicit preference function in order to test our proposed algorithm :

$$U(x) = (x_1 + 4)(x_2 + 1)$$

The vector optimization problem has a unique solution given through :

$$x^\star = [7.00, 6.33] \quad \text{and} \quad U^\star = 80.67$$

Following the steps outlined above,we generate a sequence of steps samurized in table.1 and illustrated by the figure 2 .

k	x1	x2	ω^k(normalised)	u(x)
0	2.	2.	0.333333, 0.666667	18.
1	2.39496	2.43155	0.349213, 0.650787	21.9446
2	2.85379	2.8523	0.359824, 0.640176	26.4029
3	3.35306	3.26127	0.366898, 0.633102	31.3334
4	3.87801	3.66109	0.371724, 0.628276	36.7201
5	4.41687	4.0551	0.375231, 0.624769	42.5481
6	4.95869	4.44592	0.378068, 0.621932	48.7883
7	5.4921	4.83419	0.380665, 0.619335	55.3787
8	6.00419	5.21624	0.383235, 0.616765	62.1884
9	6.47812	5.57937	0.385717, 0.614283	68.9394
10	6.88057	5.89527	0.387901, 0.612099	75.0244
11	7.08763	6.14337	0.391825, 0.608175	79.2031

Table.1 :Solution results with $\alpha = 0.25, (\rho = 1)$.

We mentioned earlier that the value of the scalar α has some effect on the procedure. this effect on the progress of the iterative process when varying, α, is illustrated in table.2. The utility values at the current iterate are shown in figure.3 below.

α	kmax	x1	x2	u(x)
0.1	9	7.08004	6.046961	78.0806
0.2	10	7.06568	6.06217	78.1477
0.3	12	7.07822	6.17734	79.5121
0.4	15	7.03648	6.24678	79.979
0.5	21	7.08417	6.2228	80.1164

Table.2:Solution results when varying α.

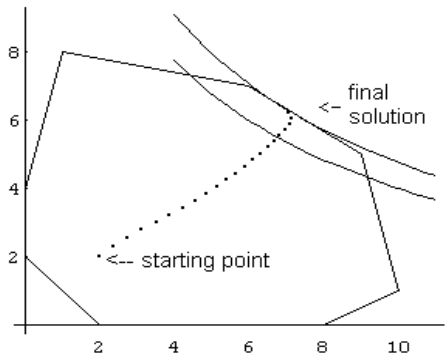

FIG.2: Solution trajectory ($\alpha = 0.5$)

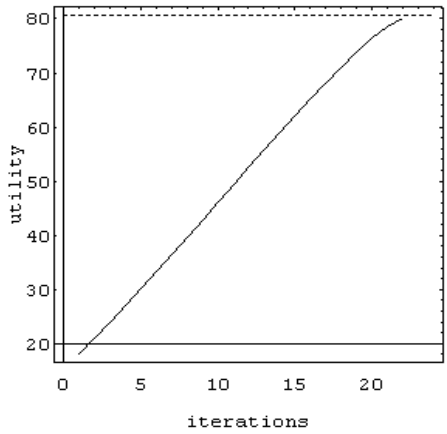

FIG.3: Utility values at current iterate ($\alpha = 0.5$)

In addition to the illustrative example, the problems presented by [Arbel,1994], [Arbel & al,1993], [Arbel & al,1996] and [Sherali,1987] were solved for comparison purposes. the results are given in table.3:

Example	kmax	x1	x2	u(x)
[Arbel and al,1996]	9	10.1135	9.7024	98.1291
[Arbel,1994]	6	5.11818	4.78668	24.4991
[Arbel and al,1993]	9	4.78413	5.08377	24.3214
[Sherali,1987]	7	3.72265	4.42778	-13.307

Table.3:Solution results by other testing examples ($\alpha = 0.1$).

5 Conclusion

To summarise: we have presented in this paper an interactive algorithm for solving multiobjective linear programming problems. The algorithm is based on modifying the method of centers to MOLP problems, this modification is based on finding an approximation to the gradient of the implicitly-known utility function by interacting with the DM. The marginal rates of substitution is obtained and used in deriving the approximation gradient and to update the lower bounds of the objectives.
Future research in this new area of interior MOLP algorithms should explore the construction of an ordered cone that illustrate the DM preferences. Additional areas should include exploring the type of ellipsoid that will be more reasonable.

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