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Modelling Inter-temporal Capacity Transfer in an Aluminium Smelter*

by

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Abstract

In this paper, an existing model of a multi-period nonlinear bilevel model of an aluminium smelter which allows for the transfer of stocks of raw materials throughout the planning period is extended to allow for the inter-temporal transfer of *capacity*. The situation arises in reality where a part of the smelter is closed for scheduled maintenance and in order to adhere to the prime directive of the aluminium smelter Management, “don't stop production", it is necessary to transfer capacity from previous months to the month in question. Capacity *per se* cannot be transferred, however, the stocks of needed intermediate products (i.e., partly processed materials required for the next stage of production) can be, thus allowing the plant to continue operation. In incorporating the necessary infrastructure into the existing multi-period model, a number of substantial alterations were required. The most important change was the introduction of an additional variable to allow for the separation of the levels of activity of two main areas of the plant that produce the needed materials to allow smelting to take place. Numerical examples are given. Additional work on the already foreshadowed heuristic solution algorithm for the multi-period model is also highlighted.

Keywords: multi-period modelling, nonlinear bilevel programming, applied modelling.

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Introduction

The formulation of a simple model to represent the operations of an aluminium smelter (in Portland, Australia) in a single month has been developed and extensively documented [Nicholls and Hedditch, 1993] and [Nicholls, 1995]. The initial monthly model was developed in a stand-alone context with one of the main drivers of the smelter essentially specified by smelter management. The model did have the facility to “roll over" from month to month to approximate a multi-period model. However, this allowed sub-optimization to occur since all that happened after solution was that stocks were adjusted, deliveries added and then the next month “optimised”. The model was subsequently altered [Nicholls, 1995] to permit a more robust solution algorithm to operate. Currently, research is continuing (along the lines suggested in [Nicholls, 1996] on a heuristic solution algorithm that allows for the proper solution of the multi-period model. This research reduces the likelihood of sub-optimisation and lays the foundations for the development of robust sensitivity analysis together with the ability to plan for altered delivery schedules and plant shut-down due to maintenance. However, the solution algorithm is heuristic and thus a global optimal solution cannot be guaranteed.

The model developed in [Nicholls and Hedditch, 1993] and applied to the smelter had a subtle but significant effect on the operation of the smelter. It brought about a greater understanding of the interactions of various parts of the smelter and the importance of allowing areas responsible for certain activities to make their own operating decisions, given knowledge of their impact on others.

The model initially developed, despite its drawbacks, facilitated planning with respect to production and other levels of activity within the smelter for periods of up to a year ahead. One consequence resulting from the use of the model was that what had hitherto been thought of as a relatively unimportant area within the smelter (with respect to the overall levels of activity and cost) was seen correctly as a highly critical one. Additionally, the model gave management the ability to locate correctly bottle-necks in the production process. Indeed, it transpired that what was thought to be a bottle-neck within the smelter was not, subsequently halting planned capital expansion. In fact, the area concerned was able to be shut down for regular planned maintenance more frequently without affecting production. This in turn reduced unplanned and total maintenance costs and altered management's attitudes towards the use of “planning tools" *per se*. The model was also valuable as a means of assessing new production technologies, changes to the characteristics of raw materials and alternative capital investment options.

Other deficiencies of the model (as developed in [Nicholls and Hedditch, 1993] and [Nicholls, 1995]) were that it did not provide true multi-period solution capacity or allow the proper planning for important plant shut-downs in critical areas of the smelter. For the latter to occur, a build up of stocks of certain important materials (termed intermediate products) is required (a complex operation which this paper addresses). The lack of a true multi-period solution algorithm also prohibited the proper analysis of the impact of delivery schedules of alumina, pitch and other raw materials. The rectification of these deficiencies was required before it could be a completely satisfactory planning tool. While the model has been used extensively for many aspects of plant management without the incorporation of the above facilities, its usefulness is limited.

In order to address these problems, work was begun firstly to develop the multi-period optimisation capacity; and then to incorporate the inter-temporal capacity transfer mechanism (which would allow planned maintenance with minimum, if any, disruption to smelter operations).

As previously indicated, in this paper, the general monthly model is extended to represent the *inter-temporal* activities of the smelter [Nicholls, 1996] and [Nicholls, 1997a] associated with stocks and flows of certain raw materials between months. The multi-period mathematical model is a staircase one linked together by the opening and closing stocks of these raw materials. A solution heuristic has also been suggested for this formulation [Nicholls, 1997a]. The multi-period formulation is an important development since it allows for the first time the impact of inventory policies and shipping schedules to be evaluated in terms of the smelter's overall operations. It also facilitates a more realistic solution to the smelter, as the delivery of raw materials can never be a certain activity and problem areas should be identified *a priori*.

Allowing for the transfer of plant capacity from one period to another through the build-up and run down of specific stocks of intermediate products (i.e., materials produced within one area of the smelter *per se* which are then in turn an input into another subsequent area) is a significant and complex extension of the model. The capacity transfer activity requires the levels of real activity within specific areas of the smelter to be redefined in more detail, thereby increasing the number of variables and complicating the interactions. This necessitates some production and feed-back activities to be substantially reformulated. The intermediate products referred to for the smelter are specifically related to the anode production area.

In summary, this paper briefly outlines the previous formulation of the multi-period model and then discusses the extension of this to incorporate *capacity shifts between time periods* through the build-up and run-down of stocks of specific intermediate products.

A multi-period model of the smelter allowing inter-temporal transfer of raw materials (but no capacity transfer)

In this section, the model of the smelter that facilitates optimisation over any number of periods (normally a year) with the free transfer of raw materials (i.e., those materials supplied externally of the smelter) between periods is presented and a solution heuristic discussed. The solution heuristic is currently under development and discussed only briefly in this paper.

In the aluminium smelter the two variables are kilo-Amperes (kA) and the Setting Cycle (SC). The power input into the smelting process (kA) is proportionally related to the output of aluminium and thus maximizing kA maximises the output of aluminium. The Setting Cycle (SC) is the length of time that anodes are left in the reduction cells (in which the aluminium is smelted) before they are replaced. The anodes (allowing an electric current to flow in the reduction cells) are blocks of baked carbon made on site. The aluminium production *per se* is independent of the production of anodes for use in the reduction cells except for some specific technical conditions and the recycling of used anodes (new anodes *must* contain amounts of used anodes). In this formulation, the leader variable is kA and the follower variable, SC.

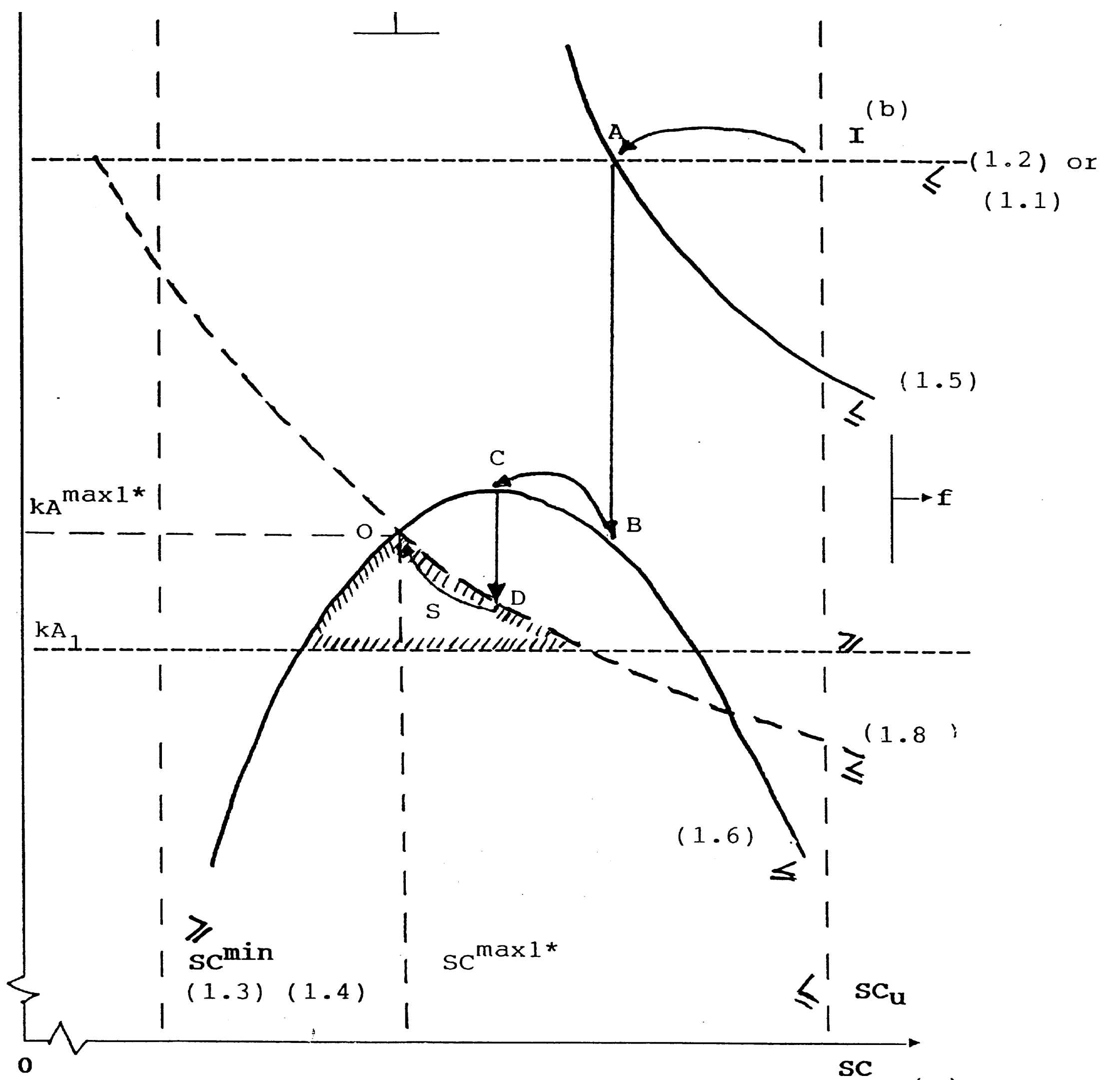
The model shown in (1) is linked together by the stocks of various raw materials in a staircase manner. The staircase model therefore involves many variables, i.e., kA and SC over each month in the planning period under consideration (kA_t and SC_t) as well as the opening and closing stocks. The solution algorithm developed for the monthly model [Brooke, Kendrick and Meeraus, 1988] is utilised as the engine in a broader heuristic solution algorithm to cope with the optimisation over all the months. The staircase mathematical model of the smelter for a typical month in the planning period is as follows:

P1: $\mathrm{Max}_{kA}\ F(kA_t,0) = kA_t \quad \forall\, t$ where SC_t solves;
P2: $\mathrm{Max}_{SC}\ f(0,SC_t) = SC_t \quad \forall\, t.$
s.t.,

$$\mathbf{a}_{\mathbf{1},t}\, kA_t \le \mathbf{b}_{1,t} \tag{1.1}$$
$$\mathbf{a}_{\mathbf{2},t}\, kA_t \le (\mathbf{CS}_{\mathbf{1},t-1} + \mathbf{DS}_{\mathbf{1},t}) \tag{1.2}$$
$$\mathbf{a}_{\mathbf{3},t}\, SC^{-1}_t \le \mathbf{b}_{2,t} \tag{1.3}$$
$$\mathbf{a}_{\mathbf{4},t}\, SC^{-1}_t \le (\mathbf{CS}_{\mathbf{2},t-1} + \mathbf{DS}_{\mathbf{2},t}) \tag{1.4}$$
$$- a_{5,t}\, kA^{-1}_t + a_{6,t}\, SC_t \le 0 \tag{1.5}$$
$$f_{1,t}\, kA_t + d_{1,t}\, SC^{-1}_t + e_{1,t}\, kA_t\, SC_t - g_{1,t} \le (CS_{c,t-1} + DS_{\mathbf{c},t}) \tag{1.6}$$
$$(f_{2,t}\, kA_t + d_{2,t}\, SC^{-1}_t + e_{2,t}\, kA_t\, SC_t - g_{2,t})(h_t\, SC^{-1}_t + j_t)^{-1} \ge b_{4,t} \tag{1.7}$$
$$(f_{2,t}\, kA_t + d_{2,t}\, SC^{-1}_t + e_{2,t}\, kA_t\, SC_t - g_{2,t})(h_t\, SC^{-1}_t + j_t)^{-1} \le b_{5,t} \tag{1.8}$$
$$\mathbf{CS}_{\mathbf{1},t-1} - \mathbf{a}_{\mathbf{2},t}\, kA_t + \mathbf{DS}_{\mathbf{1},t} - \mathbf{CS}_{\mathbf{1},t} = 0 \tag{1.9}$$
$$\mathbf{CS}_{\mathbf{2},t-1} - \mathbf{a}_{\mathbf{4},t}\, SC^{-1}_t + \mathbf{DS}_{\mathbf{2},t} - \mathbf{CS}_{\mathbf{2},t} = 0 \tag{1.10}$$
$$CS_{c,t-1} - (f_{1,t}\, kA_t + d_{1,t}\, SC^{-1}_t + e_{1,t}\, kA_t\, SC_t - g_{1,t}) + DS_{\mathbf{c},t} - CS_{\mathbf{c},t} = 0 \tag{1.11}$$
$$kA_t \ge kA_{l,t};\ SC_t \le SC_{u,t}$$
$$\text{all variables} \ge 0, \quad t = 1, \ldots, T \tag{1}$$

The problem represented by (1) is not bicriteria but strictly bilevel and nonlinear. The solution of multilevel problems is not necessarily easily undertaken and the interested reader is referred to [Bard, 1983] and [Wen and Hsu, 1991] for details. The higher level decision maker, in this case the areas of the smelter that is actually producing the aluminium and is therefore in control of the kilo-Amperes, aims to maximise the output of aluminium (i.e., maximise the kilo-Amperes (kA)). The lower level decision maker, in this case the area associated with the production of baked anodes used in the smelting process, aims to minimise the level of activity and therefore cost by maximising the setting cycle (SC). The higher and lower level decision makers each have control over a specific region of the feasible solution space and are assumed to play a two person Stackleberg game. The higher level decision maker *initially* maximises F (i.e., problem 1 [P1]) and then the lower level decision maker maximises f (i.e., problem 2 [P2]) bearing in mind the value of kA chosen in P1. Since there are regions within the feasible solution space where *both* decision makers coexist, the final solution will be arrived at after appropriate interaction, i.e., the solution for any given month will lie within a "reaction set". The solution heuristic developed in [Nicholls, 1997a] makes use of the special nature of the Portland problem and involves the solution of the monthly problems using a relatively simple heuristic.

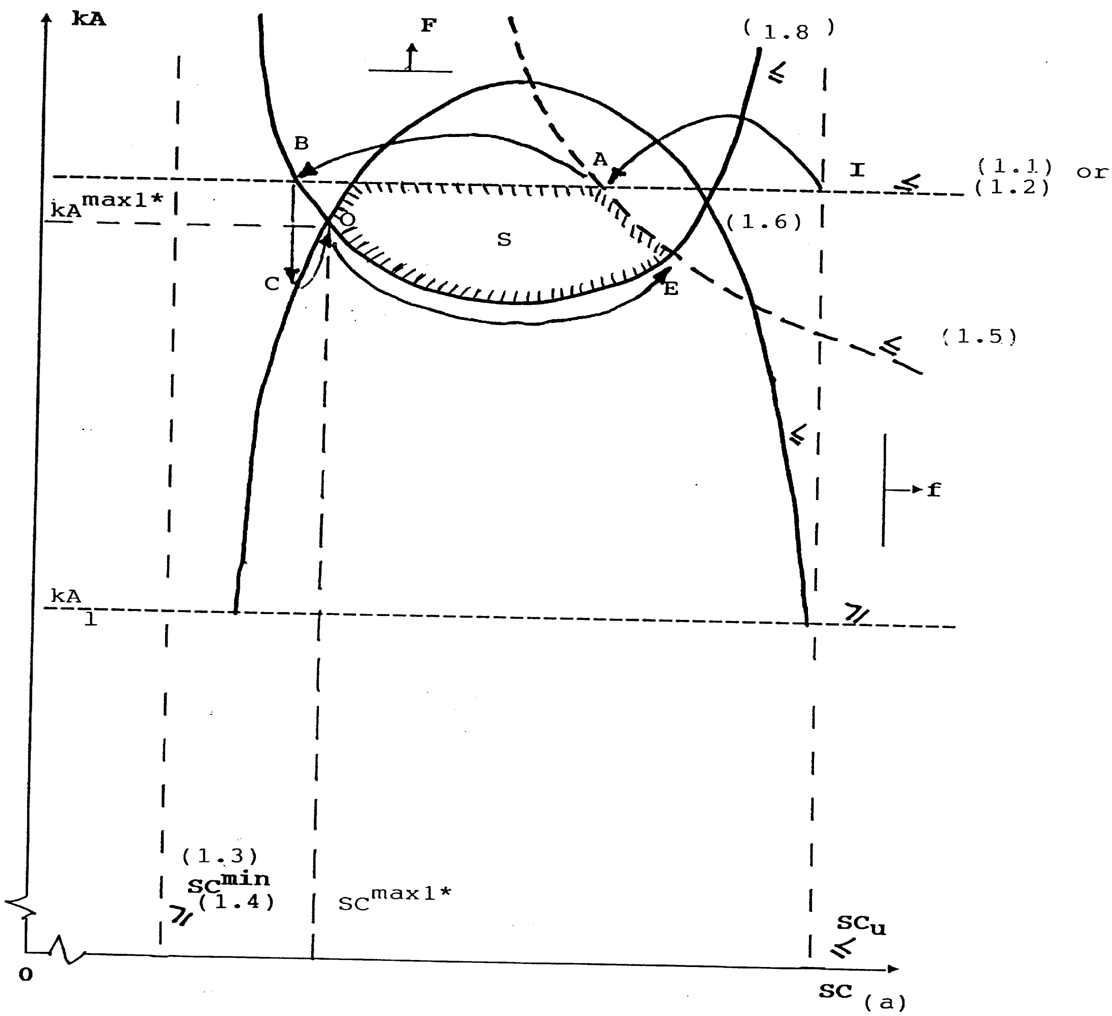
Figure 1 and **Figure 2** below illustrate the feasible solution spaces that can be representative of certain smelter conditions and are also illustrative of the way in which the solution heuristic operates for the monthly model. It is interesting to note the way in which the global solutions are found from the exterior initial point working eventually *into* the feasible solution space, thus forming the "exterior point" nature of the solution algorithm.



(a) Note that the time subscripts have been omitted for simplicity.

(b) I = Initial starting point of solution algorithm.

Figure 1 *Representative Solution Space for Model (1) (Including Path of Solution Algoritm)*



(a) Note time subscript has been omitted for simplicity.

Figure 2 *Representative Solution Space for Smelters Model (2) (Including Path of Solution Algoritm)*

In model (1), the first constraint group (1.1) contains the activities associated with the consumption of resources and plant capacities (represented by the vector ($\mathbf{b}_{1,t}$)) which are solely associated with the reduction cells (i.e., the smelting process *per se*) and are linearly consumed with respect to kA. Note that these resources and plant capacities *cannot* be transferred from period to period. In the case of Portland this group consists of only one *resource* (power) but may also contain various plant capacities. Here the $\mathbf{a_1}$ is a vector of reduction cell related coefficients which have been considerably simplified as in reality they comprise in many instances complex algebraic expressions with up to forty parameters. Constraint group (1.2) is similarly concerned with the resources (raw materials) solely associated with smelting *per se.* However, these raw materials *may* be stockpiled and transferred from period to period**.** In constraint group (1.2) the closing stocks from the previous period (represented by the vector $\mathbf{CS_{1}}_{,t\text{-}1}$) plus the deliveries scheduled for the current period (represented by the vector $\mathbf{DS_{1,t}}$) constitute the availability of resources. Constraint groups (1.1) and (1.2) belong to the higher level decision maker and in the solution heuristic set the upper bound for kA (kA^{max1}). In the case of Portland there are approximately twenty to thirty such constraints in (1.1) and (1.2). The smallest value that kA may take on is set by technical aspects associated with the smelting process and is an explicit bound (kA_l).

The third constraint group (1.3) represents the consumption of plant and other capacities as well as resources that cannot be transferred from one period to another (represented by the vector $\mathbf{b}_{2,t}$). These resources and capacities are solely associated with the production of the baked anodes (excluding recycled quantities of anodes from the reduction cells) and their consumption and usage is reciprocally related to SC. Here $\mathbf{a_3}$ is a vector of production coefficients. Constraint group (1.4) is similar to (1.3). However, these resources (excluding baked anodes which must be consumed in the same month of manufacture in this model formulation) *may* be transferred from period to period. The resource availabilities are represented by the closing stock from the previous period plus the scheduled deliveries for the current month (i.e., the vectors $\mathbf{CS_{2}}_{,t-1}$ and $\mathbf{DS_{2}}_{,t}$ respectively). These two constraint groups (i.e., (1.3) and (1.4)) belong to the lower level decision maker and within the solution algorithm are used to establish the lower bound for the SC (SC^{min}).

Note that in the formulation of constraint groups (1.2) and (1.4), care has been taken to leave them in the very simplest form so that the constraints involve as few variables as possible. This is done so that the solution heuristic can use them to establish the upper and lower bounds for kA and SC respectively. The determination of the closing stocks for the current period for both these resource groupings ($\mathbf{CS_{1}}_{,t-1}$, $\mathbf{CS_{2}}_{,t-1}$) is handled separately and simply by constraint groups (1.9) and (1.10) in accordance with the requirements of the solution heuristic. In the constraint groups (1.1), (1.2), (1.3) and (1.4) a total of approximately one hundred constraints can be involved.

The fifth constraint establishes an upper limit to the setting cycle **given** the value of the kA currently in use and is a reciprocal constraint with respect to kA. The origin of this constraint lies in the technical requirement that a minimum proportion of the anode must remain at the time of replacement and it represents an area of interaction between the higher and lower level decision makers (thus involving kA *and* SC). Essentially, the constraint is a more conveniently rearranged form of the constraint:

$$SC_t \le (a_{5,t}/a_{6,t})\, kA^{-1}{}_t \qquad (1.5A)$$

and is an interface between the higher and lower level decision makers.

The sixth constraint represents the consumption of petroleum coke in the production of anodes. This consumption is determined by the amount of recycled anodes available and the fixed proportion of pitch (the other ingredient required to make the anodes). Consequently, coke is a filler raw material influenced by both kA *and* SC. The constraint is based on the following representation:

Total materials required for baked anodes - Recycled anodes - a fixed proportion of pitch $\leq$ *coke available.*

Again, the coke may be passed from one period to another and its availability is represented by the opening stock ($CS_{\mathbf{c},t-1}$) plus the scheduled deliveries for the month ($DS_{\mathbf{c},t}$). Similarly, as for constraint groups (1.2) and (1.4), the closing stock constraint is specifically established in (1.11).

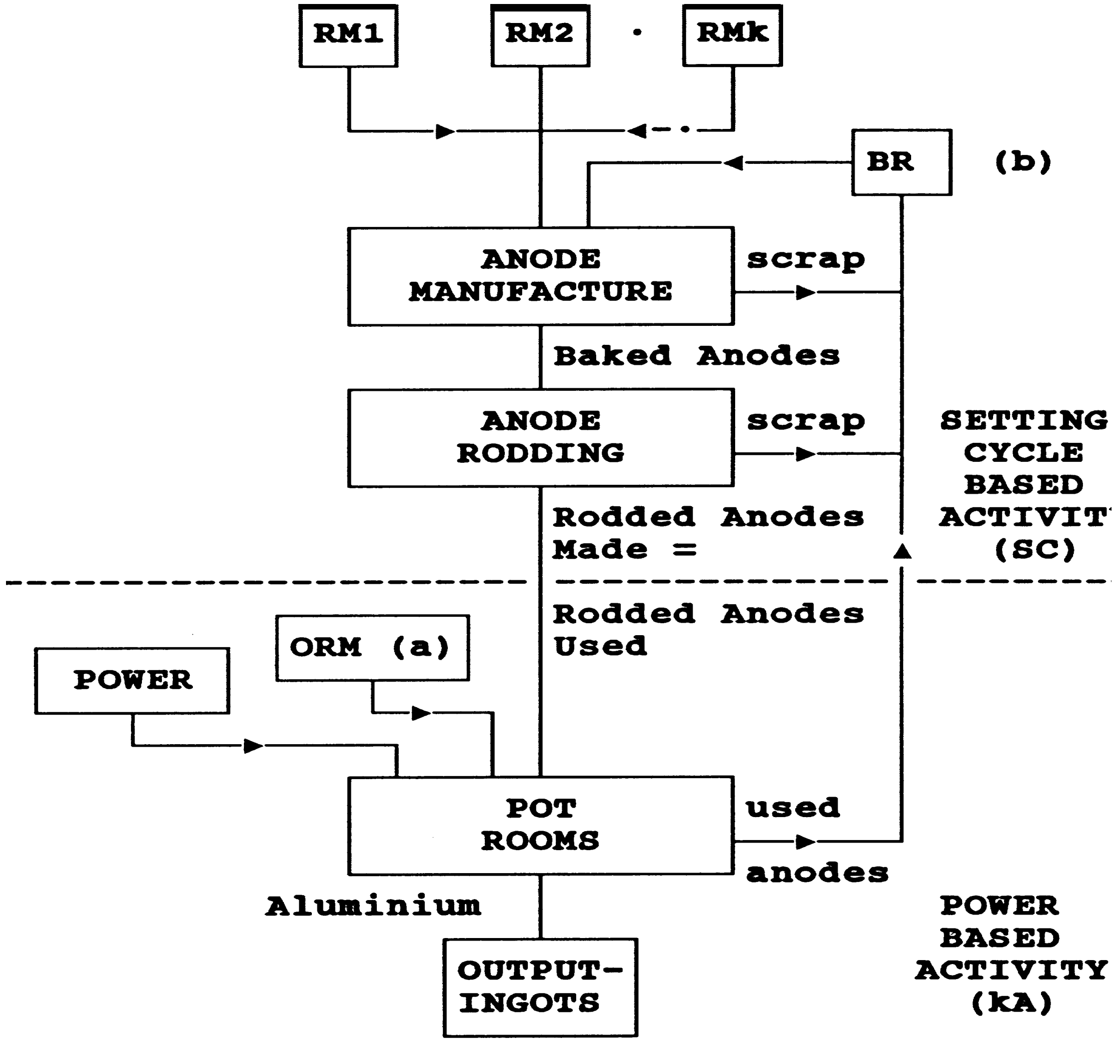
Note that in constraint (1.6), if the kA is raised with the SC remaining unchanged, the consumption of anodes increases. Hence for a given period of time there will be less anode left to recycle. The longer the SC, the less anode is left at the time of replacement and therefore the more coke is needed. However, offsetting this is the fact that longer (larger) the setting cycle, the less anodes are required per month. This constraint may be *treated* as a concave quadratic (with respect to SC) within the feasible solution space defined for this problem.

The seventh and eighth constraints represent the minimum and maximum proportions respectively of recycled anode material permitted in new (manufactured) anodes. The eventual proportion of recycled anode material is influenced by both the SC and kA variables in a reciprocal and linear way respectively. As with constraint (1.5), these two constraints represent an area of interaction between the higher and lower decision makers. Both these constraints are convex quadratic with respect to SC. The constraints are based on the general form:

$b_{4,t} \leq$ (*Recycled Anodes/Total Anode Material*) $\leq b_{5,t}$ (1.8A)

In (1), the ninth and tenth constraint groups (as already indicated) establish the closing stocks of various raw materials associated with the reduction cell area and the anode production process respectively ($\mathbf{CS_1}_{,t}$, $\mathbf{CS_2}_{,t}$). The consumption of raw materials in constraint group nine is determined by kA while in constraint group ten, the consumption is determined by SC. In constraint eleven, the closing stock of coke is established ($CS_{c,t}$). The usage of coke is determined by both kA and S. The *initial* opening stocks for t=1 are hard coded as bounds but for t>1 the previous period's closing stocks are used. These constraints link the various monthly models together and form the staircase model for the planning period.

The overall flow of materials and product in the formulation above is illustrated in **Figure 3**. It is important to understand these flows as when capacity transfer is permitted they become more complicated.



(a) Includes Alumina and **O**ther **R**equired raw **M**aterials

(b) Butts Returned i.e., the residual of the anodes from the potrooms

Figure 3 *Production Activity and Process Flow for the Aluminium Smelter*

The heuristic solution algorithm for the multi-period model of the smelter with no capacity transfer

The cascade type solution algorithm outlined in [Nicholls, 1997a] and based on the approaches outlined in [Jayakumar and Ramasesh, 1994] is based on two phases with the first phase devoted to finding a feasible solution to the multi-period problem and the second phase determining the global or local optimal solution. As has been pointed out, the solution algorithm is a heuristic and only in some well defined circumstances can the solution be *known* to be globally optimal. For simplicity, the assumption is made that all production coefficients in the constraints remain unchanged over all the periods in the planning horizon.

If this assumption is relaxed, the solution algorithm is not invalidated, but requires a small amount of expansion.

In the Phase 1 solution procedure each of the monthly models is solved in turn using the solution algorithm developed for the monthly model, where the raw materials not used in the current month are passed on to the next month (as opening stock) and then supplemented by subsequent deliveries. With the surplus stock of the raw materials passed on to the next period for use (if needed), the constraints associated with the raw materials in the current month now *appear* to be active. However, the true indication of a constraint being active (with respect to raw materials) is when the closing stock for the current period is zero.

Should the *first* monthly solution t=1 be infeasible, then no feasible solution exists for the multi-period model as there is no opportunity to reduce the consumption of appropriate resources in previous months and pass them on to the month where they are needed. Should an infeasible solution occur for a month where t>1, then an infeasibility resolution heuristic (IRH) is invoked. If there exists no feasible solution for this month, then there will be no feasible solution for the multi-period model and the solution procedure is terminated.

Upon having successfully obtained feasible (possibly sub-optimal in terms of the multi-period context) solutions for each of the individual months, an examination of the closing stocks is required to determine whether this is the global optimum solution. If all the closing stocks of raw materials (excluding power availability [mega-Watts] and capacities since these cannot be stockpiled and transferred between periods) over the full planning period are non-zero, then no resource has limited the production of aluminium or the activities of the anode and rodding areas. The other *technical constraints* representing say maximum or minimum ratios etc., may be limiting but they cannot be altered with respect to their position within the solution space and thus do not play a role in this section of the solution heuristic.

Given that there are no *raw material* constraints active within the multi-period model then the solution obtained from the individual monthly solutions is a global optimal for the multi-period model and Phase 2 is not required. It can be seen that in certain circumstances, the heuristic *will* produce a globally optimal solution.

If some of the raw material closing stocks are equal to zero for some t (t=1, . ,T) then Phase 2 will be required. Supposed that for at least one t (t^*), the following conditions are met:

$$b_{1,1,t^*} > kA_{t^*}\, a_{1,1,t^*} \qquad (2.1)$$

$$- a_{5,t^*}\, kA^{-1}{}_{t^*} + a_{6,t^*}\, SC_t < 0 \qquad (2.2)$$

$$(f_{2,t^*}\, kA_{t^*} + d_{2,t^*}\, SC^{-1}{}_{t^*} + e_{2,t^*}\, kA_{t^*}\, SC_{t^*} - g_{2,t^*})(h_{t^*}\, SC^{-1}{}_{t^*} + j_{t^*})^{-1} > b_{4,t^*} \qquad (2.3)$$

$$(f_{2,t^*}\, kA_{t^*} + d_{2,t^*}\, SC^{-1}{}_{t^*} + e_{2,t^*}\, kA_{t^*}\, SC_{t^*} - g_{2,t^*})(h_{t^*}\, SC^{-1}{}_{t^*} + j_{t^*})^{-1} < b_{5,t^*} \qquad (2.4)$$

(where $b_{1,1,t^*}$ is the power availability for period t^* and the **only** element in $\mathbf{b_{1,t}}$(for Portland), $t^* \in T$ and $t^*>1$). (2)

Since (from (2)) power is not limiting kA_{t*} for that month **nor** are any of the technical constraints (minimum and maximum ratio together with the setting cycle maximum given the kA_{t*}) then:

$$\sum_{t=1}^{T} kA^n_t \geq \sum_{t=1}^{T} kA^o_t \qquad (3)$$

where kA^n_t and kA^o_t are the kA_t values post Phase 2 and Phase 1 respectively. A Phase 2 heuristic solution algorithm is then employed to find the ΣkA^n_t (Phase 2.1).

Note that (2) implies:

$$\mathbf{a_4}_{,t}\, SC^{-1}_t = (\mathbf{CS_2}_{,t-1} + \mathbf{DS_2}_{,t}) \qquad (4.1)$$

and/or $$f_{1,t}\, kA_t + d_{1,t}\, SC^{-1}_t + e_{1,t}\, kA_t\, SC_t - g_{1,t} = (CS_{c,t-1} + DS_{\mathbf{c},t}) \qquad (4.2)$$

for at least one element of $\mathbf{a_4}_{,t}$ when $t=t^*$) (4)

are active (i.e., (1.4) and/or (1.6)). This is since (1.2) pertains to power (which is not limiting) and (1.3) is associated with inter-temporally *non-transferable* raw materials. If this were not the case then only kA based resources (1.1) and (1.2) would be limiting and since these are proportionally related to kA no possibility exists for a net kA gain. Rather, a maintenance of the *status quo* ($\Sigma kA^n_t = \Sigma kA^o_t$) i.e., a reduction in kA in a prior month can only be matched by a proportional increase (at least) in kA in t^*. **Figure 4** below illustrates the type of situation that Phase 2.1 is attempting to uncover and then exploit. Note **Figure 4** represents two successive time periods and their inter-connection.

Figure 4 illustrates that by decreasing the amount of coke available in period 1 (t=1) and transferring it to period 2 (t=2), the old optimal kA ($kA^{max1*}_{1,1}$) for period 1 falls to $kA^{max1*}_{2,1}$ and the old optimal setting cycle ($SC^{max1*}_{1,1}$) increases to the new optimal value of $SC^{max1*}_{2,1}$. For the next period (t=2), the consequences of the deliberate reduction in coke usage in period 1 can be seen with the old optimal kA ($kA^{max1*}_{1,2}$) increasing by more than the decrease in the previous period (t=1) to the new optimal value $kA^{max1*}_{2,2}$. The setting cycle thus decreases to $SC^{max1*}_{2,2}$. This net increase in kA is occasioned by the gradient change encountered in constraint (1.5). In the situation depicted in **Figure 4,** the parametric shift in constraint (1.6) (the coke constraint) caused by the transference of coke from the previous period, results in an intersection with constraint (1.5) in a steeper section.

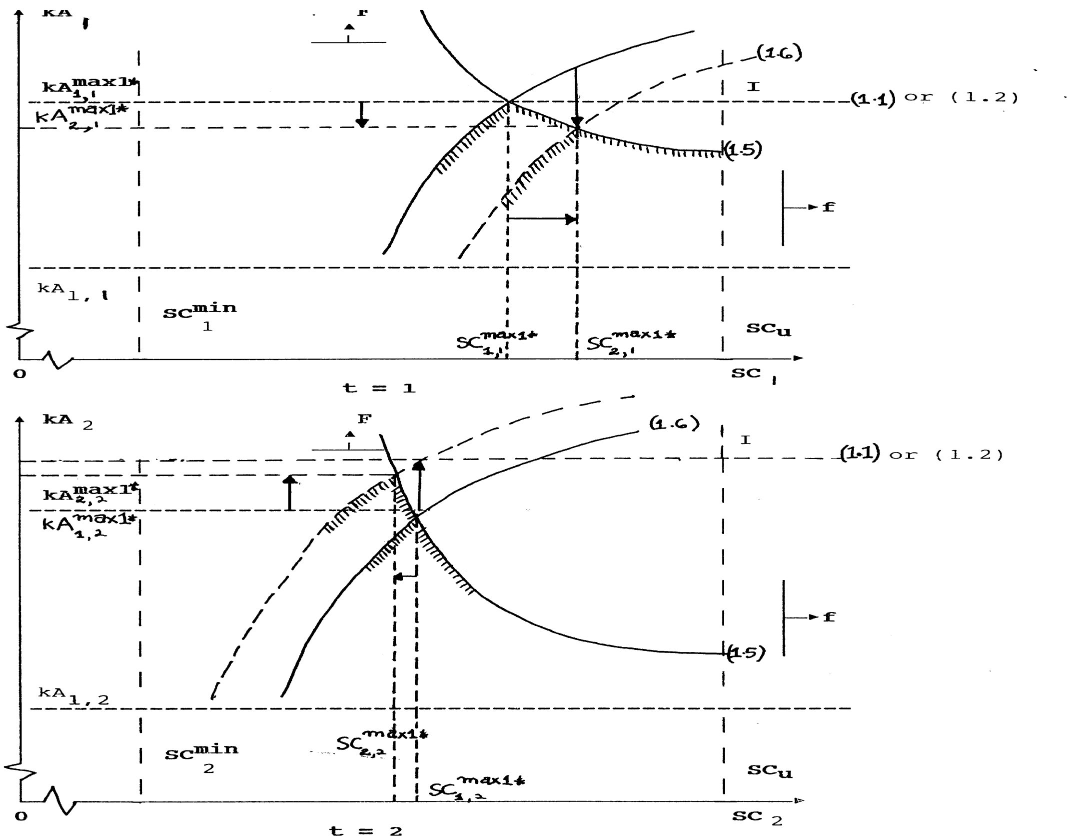


Figure 4 *Typical Phase 2.1 Scenario*

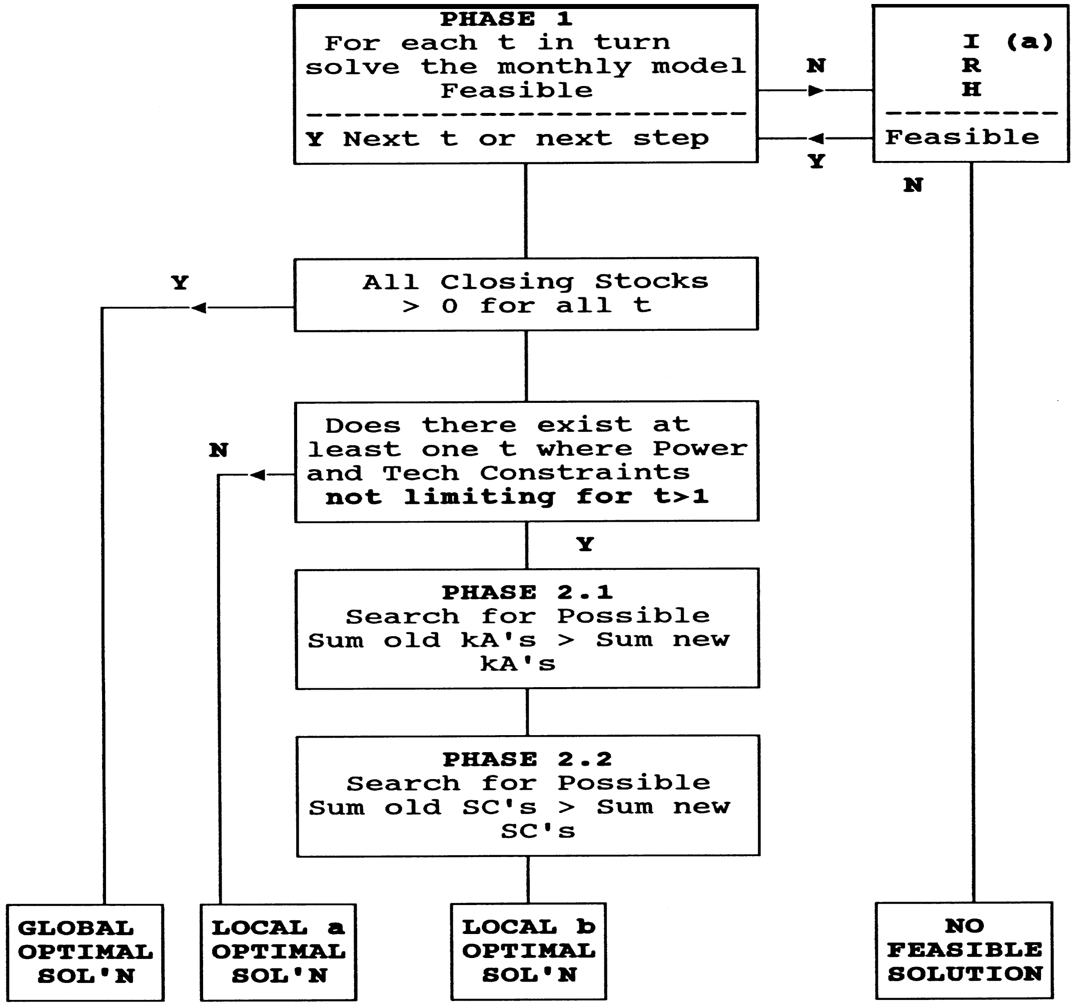
When Phase 2.1 has been completed, an examination of the SC_t values is performed to determine if there can be any re-allocation of resources between the months such that:

$$\sum_{t=1}^{T} SC^n_t > \sum_{t=1}^{T} SC^o_t \qquad (5)$$

The solution obtained will be *at least* equal to that obtained in Phase 1 as kA cannot be increased any further and certainly will not be decreased. This solution heuristic is designated Phase 2.2.

If the conditions in (2) are not met then the kA^o_t and SC^o_t values obtained from Phase 1 constitute the global optimum. **Figure 5** summarises the solution process.

One important aspect not yet covered is the operation of the Infeasibility Resolution Heuristic. When solving the single monthly model with the previously developed heuristic, it has been observed over a large number of instances that infeasibilities occur frequently. However, with the multi-period model these infeasibilities can often be overcome with the transfer of appropriate raw materials from previous months to the month where they are required. Depending on the type of infeasibility encountered, an appropriate resolution heuristic can be implemented in an attempt to allow a feasible and possibly global optimal solution to be obtained. This may result in a reduced overall kA (i.e., summed over the planning horizon).



(a) If t=1 then the **IRH** is not required since there exists no feasible solution.

Note: GLOBAL OPTIMAL SOL'N $\geq$ LOCAL a OPTIMAL SOL'N $\geq$ LOCAL b OPTIMAL SOL'N

Figure 5 *The Structure of the Cascade Solution Algorithm*

It is important to understand how the infeasibilities arise and how they might be overcome. Additionally, when capacity transfer *per se* (i.e., build-up and run-down of intermediate products) is permitted in the model, it is triggered by the occurrence of a particular type of infeasibility. When capacity transfer is triggered, a new mathematical model must be invoked, one that reduces to (1) when capacity transfer is not permitted.

There are four main circumstances where an infeasible solution might occur during the solution of the monthly model (a part of the staircase model (1)) and depending on what circumstances are present, some specific courses of action are required in the IRH. Details associated with the IRH (as with Phases 1, 2.1 and 2.2) will not be dealt with in this paper and are all currently under research.

As is illustrated in **Figure 6(a) and 6(c)** when capacity transfer (in the form of the transfer of intermediate products, such as baked anodes) is allowed in this model, the impact will be felt through the reduction in the SC^{min}. This is because the increase in capacity in the anode production area (where the only capacity transfer is permitted) will possibly allow the largest

SC, the limiting one, to take on a lower value, reflecting an increased level of activity. With the SC^{min} lower the possibility of attaining feasibility is enhanced. Note that SC^{min} is calculated thus;

$$SC^{min}_t = \underset{i}{\mathrm{Max}}\,(SC_{t,i}) =$$

where $SC_{t,i} = a_{3,i,t}/b_{2,i,t}$ $(i=1, . . ,I)$. (6)

Should circumstances such as those depicted in Figure 6(a) and 6(c) occur and not able to be resolved by the IRH, the use of a different formulation of the smelter operations will be required. As previously indicated, this formulation will reduce to (1) when capacity transfer is not required. This is dealt with below where numerical examples of the solution of model (1) and the alternative formulation of model (1) will also be given.

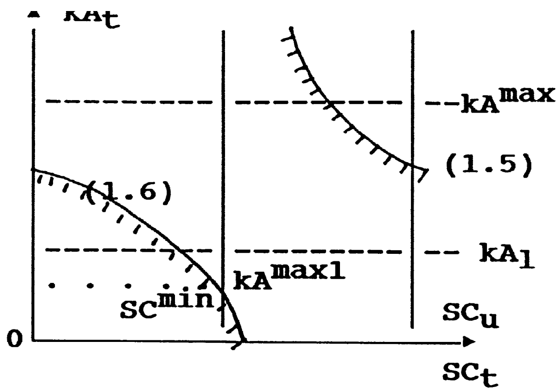


Figure 6a Infeasible
Solution: $kA^{max1} < kA_l$
Implying: $SC^{max1} < Sc^{min}$
Remedy: inc. Coke, dec. SC^{min}

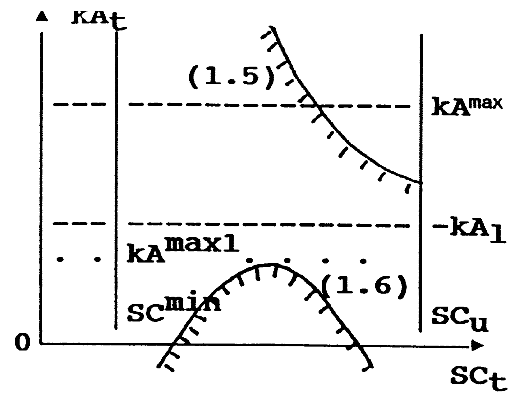


Figure 6b Infeasible
Solution: $kA^{max1} < kA_l$
Remedy: inc. Coke only

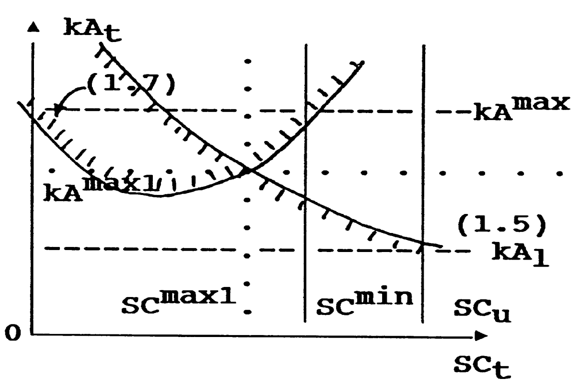


Figure 6c Infeasible
Solution: $SC^{max1} < Sc^{min}$
Remedy: dec. SC^{min} only

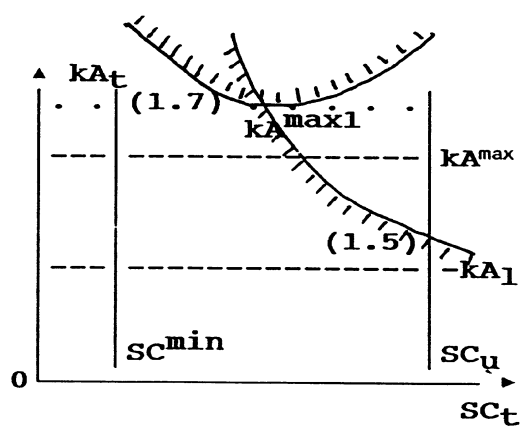


Figure 6d Infeasible
Solution: $kA^{max1} > kA^{max}$
Remedy: inc. kA^{max} only

Note: $\mathbf{SC^{min}}$ is the smallest SC permitted by Constraints (1.3), (1.4)
$\mathbf{SC_u}$ is the largest SC permitted technically
$\mathbf{kA_l}$ is the lowest kA permitted technically
$\mathbf{kA^{max}}$ is the highest kA permitted by Constraints (1.1), (1.2)
$\mathbf{kA^{max1}}$ is the kA from the Solution Algorithm
$\mathbf{SC^{max1}}$ is the SC from the Solution Algorithm
dec. is decrease and **inc.** is increase.

Figure 6 *Causes of Infeasibility in Solving the Monthly Models Constituting the Multiperiod Model-Phase 1*

A multi-period model of the smelter allowing inter-temporal transfer of capacity (i.e., intermediate products)

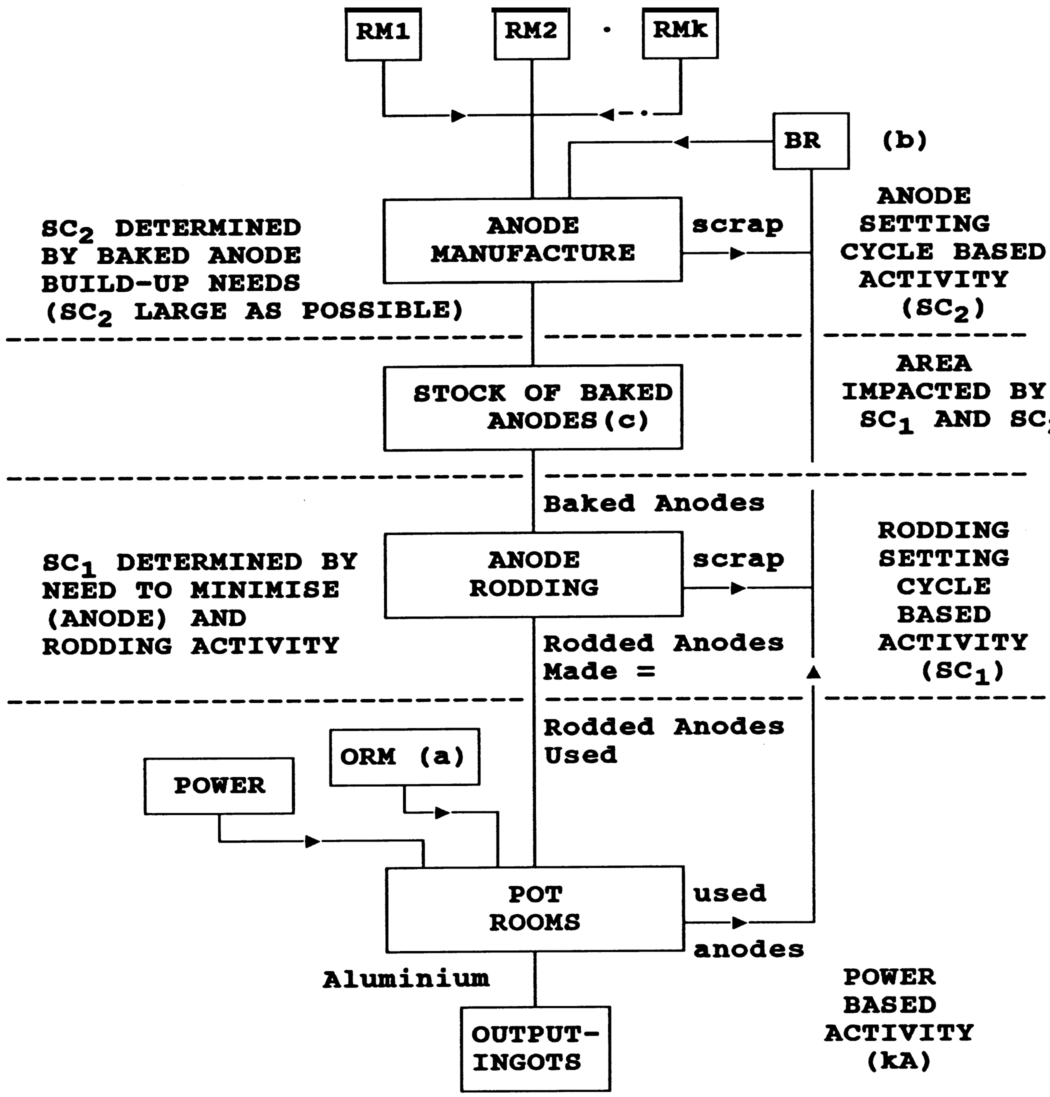
Transfer of capacity is required when part of the plant is shut down, or operating at substantially lower levels than normal, for maintenance in a period and a build up of intermediate product stocks normally produced by that part is needed to allow the rest of the plant to continue in normal (or near normal) operation. The extension discussed in this section foreshadowed in [Nicholls, 1997b] allows for the stockpiling of baked anodes prior to maintenance work on the carbon bake furnaces. This stockpiling requires that two setting cycles are defined. The model of the plant thus needs to be extended. The periods in which the stockpiling (and extension of the model) are required are determined within the multi-period model by the Infeasibility Resolution Heuristic (IRH) which then solves the problem for these periods by a different and more conventional approach. The rest of the problem is handled as before. The remainder of this section deals in turn with the physical background, the role and the *modus operandi* of the Infeasibility Resolution Heuristic when triggered by the need to transfer, the extended model and discussion.

The complication introduced by capacity transfer, in the form of stock build up of a specified intermediate product, is that the anode and rodding areas now require separate setting cycles (one for each sub-area). Consequently, the number of variables is now critically increased and additional operational complexities are introduced in representing the operations of the anode and rodding areas separately in the mathematical model. **Figure 7** below shows the production activity and process flow for the reconfigured smelter operations. It should be noted also, that a new mathematical formulation will only be used when the transfer of capacity is required. The model as represented by (1) will otherwise be used. The staircase model of the smelter over a given time period where capacity transfer is permitted will therefore consist of a combination of models (1) and an additional model developed below and defined in (14).

As **Figure 7** illustrates, with the need to build-up the level of stocks of baked anodes, the anode plant will need to work at a higher level of activity than the rodding plant. The rodding plant will determine its setting cycle (SC_1) based on the fact that it desires to minimise its level of activity, thus also minimising some operating costs. However, while the anode plant has the same objective, it must adjust its level of activity (as represented by its setting cycle SC_2) taking into account whether it is building up or reducing stock levels of baked anodes. When normal operations are present i.e., no build-up or run-down of baked anode stocks then $SC_1 = SC_2$. If there is a stock run-down, then the anode area's SC_2 will be larger than the SC_1 of the rodding area. The SC_1 is driven by normal activities while the SC_2 is driven by the stock build-up and run-down considerations. The stock is normally run down within the single period t^+ and does not continue past this period. The reformulation of the model to include two setting cycles complicates the carbon feed-back loop (i.e., the determination of the quantity of recycled carbon returned from the potrooms) which is now determined by SC_1, SC_2 and the kA. The coke demanded will also be determined by SC_1, SC_2 and kA.

The need for this new formulation is determined by the Infeasibility Resolution Heuristic. The new formulation will then be inserted into the staircase model for the current month (where the trigger occurred) and for a specific number of prior consecutive months. The solution process for Phase 1 is required to solve an overriding sub-problem comprising month t^+ and a number of consecutive months prior to t^+ determined by the solution procedure *per se*.

The sub-problem and its solution covered by this new sub-model will normally be left *in situ* for the remainder of Phase 1 and Phase 2 since it allows a feasible solution to the original problem (through an alternative formulation and solution procedure) to be obtained. Consequently, it remains a critical part of the solution. This alternative representation of the operations of the smelter (the overriding sub-problem) is developed below.



Note:
(a) Includes Alumina and **O**ther **R**equired raw **M**aterials
(b) Butts Returned i.e., the residual of the anodes from the potrooms determined by kA, SC_1 and SC_2.
(c) Stock of baked anodes effected by production (SC_1) and use (SC_2)
$SC_1 > SC_2$ represents a reduction in the stock of baked anodes
$SC_1 = SC_2$ represents the "normal" situation when a single SC used
$SC_1 < SC_2$ represents a build-up in the stock of baked anodes

Figure 7 *Production Activity and Process Flow for the Aluminium Smelter Allowing for Raw Material and Capacity Transfer*

The Infeasibility Resolution Heuristic when inter-temporal capacity transfer is allowed

The need to deal with capacity transfer inflicted by a specific set on conditions in this problem formulation, requires the application of a new formulation of model (1). This will result in an overriding sub-problem which may be solved using a mathematical programming approach (i.e., analytical rather than heuristic) since the complexity of the multilevel nature of the problem vanishes even though the number of variables critically increases. The increase occurs to the point where a *simple* heuristic as employed in the solution to the multi-period model with no capacity transfer facilities is inapplicable.

The solution of the sub-problem will however, have an additional complication inasmuch as it will require successive solutions of an expanding model to be obtained until feasibility, and thus potentially optimality is found. The discussion below establishes the exact conditions which necessitate the new formulation, what the new model is and establishes a methodology (and technique) for its solution.

The exact trigger for the inclusion of the new model formulation (and thus a different solution algorithm) occurs when there exists an infeasibility as depicted in **Figure 6a** or **Figure 6c.** This is specifically caused by the inadequate capacity of the carbon bakes for that period ($\mathbf{b_{2,1,t}}$). The exact conditions required for the new model formulation are as shown in (7) and (8) below.

Let the month in which the trigger occurs be designated t^+ and the months prior to this period requiring the new model be ($t = t^o, . . ,t^+-1$) where $t^o<t^+$. The solution heuristic within the IRH recursively determines the value of t^o which is finally required.

The specific infeasibility condition which triggers the need for the new model, incorporating capacity transfer in the form of stockpiling and then use of baked anodes, based on the model as depicted in (1) is as follows:

If for t:

$SC^{max1}_t < SC^{min}_t$

and $a_{3,1,t}\, SC^{-1}_t > b_{2,1,t}$ (7)

and $\max_i (SC_{t,i}) = SC_{t,1}$

where $SC_{t,i} = a_{3,i,t}/b_{2,i,t}$ (i=1, . . ,I) (8)

and $SC^{IRH}_t < SC^{min}_t$

where SC_t^{IRH} is the setting cycle obtained from the application of the IRH,

then $t = t^+$.

Note that $b_{2,1,t}$ is the capacity of the carbon bakes in month **t.** This is measured in terms of the number baked anodes that can be produced in that month.

The objective of the new sub-problem is very specific; to build up sufficient stocks of baked anodes (SBA_t) prior to t^+ to allow a feasible solution to be obtained in that period. The stock build-up must be exactly the required amount and the greatest build-up should occur as close as possible to t^+. This is due to the wealth and health issues surrounding storage of baked anodes.

Consequently, a simple objective function that assures this requirement is:

$$\text{Min } Z^o = \sum_{t^a=t^o}^{t^+} (t^+ - t^a)\, SBA_{t^a} \tag{9}$$

Thus the biggest build-up of stock to achieve the required value (for feasibility) of SBA_{t^+}, will occur in the closest period to the need (t^+). Here t^o is recursively found by assuming initially that the sub-problem consists of two periods, t^+ and t^+-1. If the stock build-up cannot be established within that one period, then the model will comprise three months, t^+, t^+-1 and t^+-2 and so on until the required stock (SBA_{t^+}) has been achieved and t^o is then known (the last value of t added to the model). The solution to this sub-model will remain set at the values obtained through solution of this sub-problem in any additional Phase 1, 2.1 or 2.2 operations, as it is the only solution (kA_t, $SC_{1,t}$ and $SC_{2,t}$) that meets the stock build-up objective and attains feasibility. Also, it will almost certainly transpire that with the setting cycles set at specific values to achieve the required stock build-up of baked anodes, the kA will be reduced in order to meet resource and technical constraints. Hence, in a way the SC and kA values are being determined by the stock build-up. In effect, the objective of stock build-up has completely overridden the old multilevel objectives of maximizing kA and then SC, as without the reformulation and new objective the problem is infeasible. Consequently, the sub-problem for this period of time ($t = t^o, . . , t^+$) i.e., ($t \in T^+$) is not multilevel, has an increased number of variables and can be solved using a more conventional Langrangian approach (using packages such as GAMS [Brooke, Kendrick and Meeraus, 1988] or LANCELOT [Conn, Gould and Toint, 1992]. For an excellent survey of the approaches and packages available for solving this sub-problem the reader is referred to [Murray, 1994].

While the problem is a complication *per se*, its solution is simpler than if it were a bilevel problem requiring a more complicated solution heuristic than that developed for the solution of the multi-period model discussed above.

The mathematical model representing this new formulation requires the addition of a number of new constraints, namely a constraint to represent the build-up of baked anodes for ($t \in T^+$) and a constraint that specifies the target for the stock build-up (i.e., SBA_t).

From (8), the required *target* stock of baked anodes for t^+ ($RSBA_{t^+}$) is determined prior to solving the new sub-problem by evaluating the *first* of the constraints in constraint block (1.3) which represents the anode baking furnace capacity, via:

$$a_{3,1,t^+} (SC_{t^+}^{IRH})^{-1} - b_{2,1,t^+} - RSBA_{t^+} = 0 \qquad (10)$$

The stock of baked anodes on hand for t^+-1 (SBA_{t^+-1}) must be equal to this amount, i.e.,

$$SBA_{t^+-1} = RSBA_{t^+} \qquad (11)$$

The SBA_{t^+-1} in (11) is cumulative across ($t \in T^+$) and is the closing stock when $t = t^+-1$ which is contained in the stock constraint:

$$SBA_{t-1} - a_{3,1,t}\, SC^{-1}{}_{1,t} + a_{3,1,t}\, SC^{-1}{}_{2,t} - SBA_t = 0 \qquad (12)$$

($t \in T^+$)

and;

$$SBA_{t^o-1} = 0,\ SBA_{t^+} = 0 \qquad (13)$$

In (12), $a_{3,1,t}\ SC^{-1}{}_{1,t}$ represents the stock of baked anodes consumed during t while $a_{3,1,t}\ SC^{-1}{}_{2,t}$ represents the number of anodes produced during t.

The new mathematical model based on (1) for use in periods ($t = t^o, . . ,t^+$) where $t^o > 1$, is as follows:

$$\text{Min } Z^o = \sum_{t^a=t^o}^{t^+} (t^+ - t^a)\, SBA_{t^a}$$

s.t.,

$$\mathbf{a}_{\mathbf{1},t}\, kA_t \le \mathbf{b}_{1,t} \qquad (14.1)$$

$$\mathbf{a}_{\mathbf{2},t}\, kA_t \le (\mathbf{CS}_{\mathbf{1},t-1} + \mathbf{DS}_{\mathbf{1},t}) \qquad (14.2)$$

$$\mathbf{a}_{\mathbf{3},t}\, SC^{-1}{}_{1,t} \le \mathbf{b}_{2,t} \qquad (14.3)$$

$$\mathbf{a}_{\mathbf{4},t}\, SC^{-1}{}_{1,t} \le (\mathbf{CS}_{\mathbf{2},t-1} + \mathbf{DS}_{\mathbf{2},t}) \qquad (14.4)$$

$$\mathbf{a}_{\mathbf{5},t}\, SC^{-1}{}_{2,t} \le \mathbf{b}_{3,t} \qquad (14.5)$$

$$\mathbf{a}_{\mathbf{6},t}\, SC^{-1}{}_{2,t} \le (\mathbf{CS}_{\mathbf{3},t-1} + \mathbf{DS}_{\mathbf{3},t}) \qquad (14.6)$$

$$- a_{7,t}\, kA^{-1}{}_t + a_{8,t}\, SC_{1,t} \leq 0 \tag{14.7}$$

$$f_{1,t}\, kA_t + d_{1,t}\, SC^{-1}{}_{2,t} + d_{2,t}\, SC^{-1}{}_{1,t} + e_{1,t}\, kA_t\, SC_{1,t} - g_{1,t} \leq (CS_{c,t-1} + DS_{\mathbf{c},t}) \tag{14.8}$$

$$(f_{2,t}\, kA_t + d_{3,t}\, SC^{-1}{}_{2,t} + d_{4,t}\, SC^{-1}{}_{1,t} + e_{2,t}\, kA_t\, SC_{1,t} - g_{2,t})(h_t\, SC^{-1}{}_{2,t} + j_t)^{-1} \geq b_{4,t} \tag{14.9}$$

$$(f_{2,t}\, kA_t + d_{3,t}\, SC^{-1}{}_{2,t} + d_{4,t}\, SC^{-1}{}_{1,t} + e_{2,t}\, kA_t\, SC_{1,t} - g_{2,t})(h_t\, SC^{-1}{}_{2,t} + j_t)^{-1} \leq b_{5,t} \tag{14.10}$$

$$SBA_{t-1} - a_{3,1,t}\, SC^{-1}{}_{1,t} + a_{3,1,t}\, SC^{-1}{}_{2,t} - SBA_t = 0 \tag{14.11}$$

$$SBA_{t^+-1} = RSBA_{t^+} \tag{14.12}$$

$$SBA_{t^o-1} = 0 \tag{14.13}$$

$$SBA_{t^+} = 0 \tag{14.14}$$

$$\mathbf{CS_1}_{,t-1} - \mathbf{a_2}_{,t}\, kA_t + \mathbf{DS_1}_{,t} - \mathbf{CS_1}_{,t} = 0 \tag{14.15}$$

$$\mathbf{CS_2}_{,t-1} - \mathbf{a_4}_{,t}\, SC^{-1}{}_{1,t} + \mathbf{DS_2}_{,t} - \mathbf{CS_2}_{,t} = 0 \tag{14.16}$$

$$\mathbf{CS_3}_{,t-1} - \mathbf{a_6}_{,t}\, SC^{-1}{}_{2,t} + \mathbf{DS_3}_{,t} - \mathbf{CS_3}_{,t} = 0 \tag{14.17}$$

$$CS_{c,t-1} - (f_{1,t}\, kA_t + d_{1,t}\, SC^{-1}{}_{2,t} + d_{2,t}\, SC^{-1}{}_{1,t} + e_{1,t}\, kA_t\, SC_{1,t} - g_{1,t}) + DS_{\mathbf{c},t} - CS_{\mathbf{c},t} = 0 \tag{14.18}$$

$$kA_t \geq kA_{l,t};\ SC_t \leq SC_{u,t}$$

$$\text{all variables} \geq 0,\ (t = t^o, \ldots, t^+) \tag{14}$$

Model (14) will be solved using a conventional Langrangian approach (as discussed previously) initially with two periods (t^+ and t^+-1) and then with successive further periods if necessary (prior to t^+) until a feasible, and therefore optimal, solution is obtained. At this point t^o will have been defined as, prior to this, t^o was only a *pro tem* value. The solution to (14), if a feasible one exists, will also see the $SBA_{t^+} = 0$ since the build-up of anodes was targeted at providing the exact shortfall in period t^+. This also implies that the $SC_{1,t}$, $SC_{2,t}$ and kA_t obtained using the new formulation in ($t \in T^+$) will remain the same since resource shortages have occurred prior to this and the $RSBA_{t^+}$ must be met. Effectively, the switch in the objective function of the model together with the implied objective of attaining feasibility has conspired to remove ($t \in T^+$) from the Phase 1 and Phase 2 solution heuristics.

When a resolution to this sub-problem is obtained and $t^+<T$, Phase 1 continues with the (t^++1) month's individual model then worked on until Phase 1 has been completed. However, as mentioned, the model formulation and solution for all ($t \in T^+$) will remain unchanged for

any successive Phase 1, 2.1 or 2.2 operations. An exception to this occurs if after having entered the IRH and solved this sub-problem, another infeasibility as defined by (7) and (8) arises and cannot be resolved without encroaching on the time frame of the first sub-problem. Then and only then, will the results of this original sub-problem be subject to alteration, with the new sub-problem now encompassing both periods where infeasibilities arise that trigger the new model formulation. However, the mechanism is present to allow this joint sub-problem to be treated in exactly the same way as a single one with the exception being that two infeasibilities with respect to furnace capacity occur and therefore two stock targets of baked anodes must be met within the time frame of the sub-problem.

Numerical Examples of the Solution of Model 1 and Model 14

In this section some very simple examples are given to illustrate the solution process associated with model 1 and model 14. The same problem is used for both with the appropriate infeasibility occurring in period 3 to trigger the use of model 14. The simplifications made to the constraints and their limited number in no way reduce the usefulness of the examples. It should be pointed out that for such a simple case of Model 14 generated below, an easily applied heuristic can be used rather than a Langrangian approach.

The Solution of Model 1 problem using the solution heuristic suggested in Figure 5.

The problem used here is for three months and is specific as follows:

P1: $\text{Max}_{kA}\ F(kA_t,0) = kA_t \quad \forall\ t$ where SC_t solves;
P2: $\text{Max}_{SC}\ f(0,SC_t) = SC_t \quad \forall\ t.$

s.t.,

$kA_t \le \mathbf{b}_{1,t}$ (1.1)

$396.2\ 10^3\ SC^{-1}_t \le \mathbf{b}_{2,t}$ (1.3)

$20\ 10^3\ SC^{-1}_t \le (\mathbf{CS_2}_{,t-1} + \mathbf{DS_2}_{,t})$ (1.4)

$-kA^{-1}_t + .000122\ SC_t \le 0$ (1.5)

$3.63\ kA_t + 116.945\ SC^{-1}_t + 0.33\ kA_t\ SC_t - 8130 \le (CS_{c,t-1} + DS_{\mathbf{c},t})$ (1.6)

$(-.0024\ kA_t + 10\ SC^{-1}_t + 0.00024\ kA_t\ SC_t - 0.001)(SC^{-1}_t + 0.1)^{-1} \ge 10.5$ (1.7)

$(-.0024\ kA_t + 10\ SC^{-1}_t + 0.00024\ kA_t\ SC_t - 0.001)(SC^{-1}_t + 0.1)^{-1} \le 12.0$ (1.8)

$\mathbf{CS_1}_{,t-1} - \mathbf{a_2}_{,t}\ kA_t + \mathbf{DS_1}_{,t} - \mathbf{CS_1}_{,t} = 0$ (1.9)

$\mathbf{CS_2}_{,t-1} - \mathbf{a_4}_{,t}\ SC^{-1}_t + \mathbf{DS_2}_{,t} - \mathbf{CS_2}_{,t} = 0$ (1.10)

$CS_{c,t-1} - (f_{1,t}\ kA_t + d_{1,t}\ SC^{-1}_t + e_{1,t}\ kA_t\ SC_t - g_{1,t}) + DS_{\mathbf{c},t} - CS_{\mathbf{c},t} = 0$ (1.11)

$kA_t \ge 290;\ SC_t \le 32$

all variables ≥ 0, $\quad t = 1, \ldots, 3$ (1B)

In (1B) above,$b_{2,1,t}$ = 302, 300 and 295 kA (000's) for period 1, 2 and 3 respectively;

$b_{2,1,t}$ = 16,900, 16,500 and 16,500 baked anodes for period 1, 2 and 3 respectively;

$CS_{2,1,0} = 0$ and $DS_{2,1,t}$ = 800, 750 and 750 for t =1, 2 and 3 respectively;

$CS_{c,0} = 0$ and $DS_{c,t} = 8130$ for each period.

The solution to this using the heuristic outlined in **Figure 5** and which requires a Phase 1 solution procedure only, is as follows;

Table 1

Solution for Model 1B

	SC^{max1*}_t	SC^{min}_t	$SC_{u,t}$	kA^{max1*}	$kA_{1,t}$
t=1	27.1415	25.00	32	302	290
t=2	27.3224	24.60	32	300	290
t=3	27.7855	24.06	32	295	290

Note that in this simple case the SC^{min}_t is defined by the Pitch constraint (1.4) for each period. The solution above is also within the upper and lower limits for the percentage of Butts in the green mix (constraints (1.7) and (1.8)). Since no resource is limiting the solution (only technical considerations such as upper limit on SC given a kA and the limit on kA *per se*), a global optimal solution is obtained.

Table 2

Capacities and Stocks

	Furnace Capacity		Coke Stocks		Pitch Stocks	
	Used	Available	Open	Cls	Open	Cls
t=1	14,598	15,900	0	20.10	0	63.12
t=2	14,501	16,500	20.1	76.02	63.12	81.12
t=3	14,259	16,500	76.02	221.40	81.12	111.32

Note that *Open* and *Cls* represent Opening and Closing Stocks.

Solution time for such a trivial problem is irrelevant, but the algorithm is efficient.

The Solution of Model 14 problem using a solution heuristic.

As mentioned, the simplicity of the problem for solution meant that a simple heuristic could be invoked to obtain a solution. This negated the need for a Langrangian approach.

In this situation, the problem is the same as for (1B) above, however, due to furnace repairs, the capacity in period 3 has dropped from 16,500 to 13,500. This means that the solution algorithm as summarised in **Figure 5** finds an infeasibility in period 3 since the minimum setting cycle is 29.34 in order to cope with the reduced furnace capacity. The SC in period 3 can only go to a maximum of 28.2 given the lower limit on kA of 290. Consequently, the IRH is entered and SC_t+^{IRH} is seen to be 28.2 with kA_t^{IRH} = 290 (the lower limit). This situation means that the number of baked anodes required has dropped to 14,050 (from (10), (11) and (12) indicating that in order to obtain feasibility with these values of SC and kA (the closest to feasible values as possible), 550 baked anodes must be manufactured in *prior* period(s). These will become the opening stock for period 3. In other words, the alternative model (14) is required.

In this very simple case, model (14) contracts to the following for (t^+-1):

$\text{Min } Z^o = (t^+-1)\, SBA_2$

s.t.,

$$kA_t \leq 300 \tag{14.1}$$

$$396.3\ 10^3\ SC^{-1}{}_{2,t} \leq 16{,}500 \tag{14.5}$$

$$20\ 10^3\ SC^{-1}{}_{2,t} \leq (63.12 + 750) \tag{14.6}$$

$$-kA^{-1}{}_t + 0.000122\ SC_{1,t} \leq 0 \tag{14.7}$$

$$3.63\ kA_t + 100.5\ SC^{-1}{}_{2,t} + 16.445\ SC^{-1}{}_{1,t} + 0.33\ kA_t\ SC_{1,t} - 8130 \leq 8{,}150.1 \tag{14.8}$$

$$(-0.0024\ kA_t + 9\ SC^{-1}{}_{2,t} + 1\ SC^{-1}{}_{1,t} + 0.00024\ kA_t\ SC_{1,t} - 0.001)(SC^{-1}{}_{2,t} + 0.1)^{-1} \geq 10.5 \tag{14.9}$$

$$(-0.0024\ kA_t + 9\ SC^{-1}{}_{2,t} + 1\ SC^{-1}{}_{1,t} + 0.00024\ kA_t\ SC_{1,t} - 0.001)(SC^{-1}{}_{2,t} + 0.1)^{-1} \leq 12.0 \tag{14.10}$$

$$SBA_{t-1} - 396.3\ 10^3\ SC^{-1}{}_{1,t} + 396.3\ 10^3\ SC^{-1}{}_{2,t} - SBA_t = 0 \tag{14.11}$$

$$SBA_2 = 550 \tag{14.12}$$

$$SBA_1 = 0 \tag{14.13}$$

$$SBA_{t+} = 0 \tag{14.14}$$

$$kA_t \geq 290;\ SC_t \leq 32 \text{ all variables} \geq 0,\ (t=2) \tag{14B}$$

All the stocks and deliveries are as specified for Model (1B).

The single period model above in (14B) is easily solved using a simple heuristic. Note that the required stock of baked anodes can be easily built up in period 2 therefore there is no need for the alternative model to go back another period and become a two period model. The solution for the three periods is as summarised in **Table 3**

Table 3
Solution for Model 1B

	sc^{max1*}_t	$sc_1^{max1*}{}_t$	$sc_2^{max1}{}_t$	$sc^{IRH}{}_t$	$kA^{max1*}{}_t$	$kA^{IRH}{}_t$
t=1	27.1415	na	Na	na	302	na
t=2	na	27.332	26.32	na	300	na
t=3	na	na	Na	28.2	na	290

All the raw materials and other stocks were not limiting, only the technical constraints and power.

The implications of the infeasibility which triggered the need for the alternative model, is that a lower setting cycle than was previously the case for period 2, was needed for the *anode production* area. This means that with increased activity there will therefore be increased costs etc. The same setting cycle as before for the *usage* of anodes in period 2 has no effect.

The examples above show how easily the solutions for the initial problem and then the alternative problem can be obtained.

Concluding Remarks

With the development of the initial multi-period model of the smelter, the facility to take into account the deliveries of raw materials as well as the existing stocks of raw materials and their usage was allowed. This meant that a model of the smelter based on a time period of around a year was possible. The solution of such a model allows for the first time alterations to be suggested to the shipping schedules that will facilitate more demonstrably efficient operation of the smelter. Also, the evaluation of the impact of various inventory policies on the smelter can be undertaken. However, the need for capacity transfer in order to obtain feasible solutions to some month's operations, where a part of the plant has been closed for maintenance, was still required. This paper has seen the foundations laid for the development of a solution heuristic that takes this latter aspect into account. The capacity transfer is affected by the stock build-up and then run down of specific intermediate products in the anode production process. The need to maintain as small as possible stock of baked anodes (in this case) was also taken into account within the suggested framework for a solution heuristic.

With the ability to allow for the transfer of raw materials as well as capacity now incorporated into the model of the smelter, alterations to the smelter operations can be affected to allow for shipping schedules and planned plant shut-down for maintenance over whatever planning period is appropriate. This is the first time that such a tool has been available to do this for the smelter.

The development of the proper multi-period formulation of the smelter and its solution together with the incorporation of the inter-temporal capacity transfer facility was undertaken at the suggestion of management in order to deliver a planning tool that will have greater applicability and usefulness in the years to come. Management await with anticipation the finalisation of this research.

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