AMO - Advanced Modeling and Optimization, Volume 19, Number 2, 2017 *

Multi-choice linear programming for multi-objective bimatrix game

Chandan Bikash Das

Department of Mathematics

Tamralipta Mahavidyalaya, Tamluk, Purba Midnapore-721636, West Bengal, India

E-mail: cdasmathematics@gmail.com

Abstract

The aim of this paper is to develop a multi-choice model for multiobjective bimatrix game problem where payoff matrices are multi-choice in nature. Then transfer it to a standard mathematical programming problem such a way that the multiple number of alternatives, out of which one multiobjective payoff matrix of each player is to be selected. The selection of alternatives should be such a manner that the combination of choices provide an optimal solution to the multiobjective bimatrix game. There may be more than one combination which will provide an optimal solution. This paper proposed a technique to formulate mixed integer programming model. Using standard soft ware, the proposed model can be solved. Finally numerical example is presented to illustrate the proposed model and solution procedure.

Keywords: Mixed integer programming, Multi-choice programming, Multiobjective bimatrix game, Transformation technique.

*AMO-Advanced Modeling and Optimization. ISSN: 1841 - 4311**

1 Introduction

Game Theory has a remarkable importance in both Operations Research and Systems Engineering due to its great applicability. Many real conflict problems can be modeled as games. Two-person zero-sum game models are accurate when stakes are small monetary amounts. But in reality sense, when the stakes are more complicated, as often in economic situations, it is not generally true that the interests of the two players are exactly opposed. Such type of game models are non-cooperative game models. In other words such situations give rise to two-person non-zero sum games, called bimatrix games. A bimatrix game can be considered as a natural extension of the matrix game in which the outcome of a decision dictate the verdict that what one player will strive for an outcome which provides him with the lowest possible loss. However, the encountered conflict problems in economical, military and political fields become more and more complex and uncertain due to the existence of diversified factors. This situation will bring some difficulties in application of classical game theory. To remove this difficulties, we have employed multi-choice options to bimatrix game.

In this paper, few references are presented including their work. The vector payoffs were first considered by Backwell [3] and later by Contini. Both writers formalized the problem with its full stochastic information. Fernandez, Puerto and Monroy [11] considered to solve the two-person multicriteria zero-sum games. As they have considered a multicriteria game, the solution concept is based on Pareto optimality and finally they obtained the Pareto efficient solution for their proposed games. Fernandez and Puerto [10], developed a methodology to get the whole set of Pareto-optimal security strategies which are based on solving a multiple criteria linear program. This approach shows the parallelism between these strategies in multicriteria games and minimax strategies in scalar zero-sum matrix games. This notion of security is based on expected payoffs. For this reason, only when the game is played many times these strategies provide us a real sense of security. In the contrary, if the game is played only once; as in one shot games, a better analysis should consider not only the payoffs but also the probability to get them. Ghose and Prasad [13], have been proposed as a solution concept based on Pareto-optimal security strategies for these games. They also introduced the concept based on the similarity with security levels determined by the saddle points in scalar matrix games. This concept is independent of the notion of equilibrium so that the opponent is only taken into account to establish the security levels for one's own payoffs. When it is used to select strategies, the concept of security levels has important property that the payoff obtained by these strategies cannot be diminished by the opponent's deviation in strategy. Borm, Vermeulen and Vorneveld [27] analyzed the structure of a set of equilibrium for the two-person multicriteria game. It turns out that the classical result for the set of equilibrium for bimatrix games is valid for multicriteria game if one of players has two pure strategies. In another paper [4] they generalized some axioms of the Nash equilibrium and it was shown that there exited no consistent refinement of Nash equilibrium concept that satisfy individual rationality and non emptiness on a reasonably large class of games. Nishizaki and Sakawa ([18],[19],[20]) proposed the resolution approach which can be regarded as a paradigm for bimatrix multi-objective non-cooperative game. This multi-objective non-cooperative game is a originally constructed by rigid mathematical theory and proofs. Roy [25], has presented the study of two different solution procedures for the two-person bimatrix game. The first solution procedure is applied to the game on getting the probability to achieve some specified goals along the player's strategy. The second specified goals along with the player's strategy by defining the fuzzy membership function defined on the pay-off matrix of the bimatrix game. Das and Roy ([8],[9],[26]) have presented some two-persons zero sum game under entropy environment. They also presented a solution concept through fuzzy programming via Genetic Algorithm ([7],[17],[14]).

Multi-choice linear programming problems exist in many managerial decision making problems. Hiller and Lieberman [16] and Ravindran al. [22] have considered a mathematical model in which an appropriate constraint is to be chosen using binary variables required for a constraint is same as the total number of choices for the constraint.

Chang [5] has proposed a new idea for modeling the multi-choice goal programming problem. He used multiplicative terms of binary variables to handle the multiple aspiration levels. In his another paper [6], he replaces multiplicative terms of the binary variables by a continuous variable. Depending on this idea Panda al. [21] solve multiple pay off game problem. Biswal and Acharya [2] has presented a transformation model for multi-choice linear programming in requirement vector. The present multi-choice linear programming problem can not be solved by traditional LPP techniques. In order to solve the present problem, this paper proposes a new methodology to solve matrix game problem.

2 Mathematical model

A bimatrix game can be considered as a natural extension of the matrix game. A two-person non zero-sum game can be expressed by a bimatrix game, comprised of two $m \times n$ dimensional matrices, namely A and B, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

If player PI adopts the strategy "row i" and player PII adopts the strategy "column j" then a_{ij} denotes the expected payoff for player PI and b_{ij} denotes the expected payoff for player PII.

The multiple pair of $m \times n$ payoff matrices can be considered for two-person multi-criteria non zero-sum game as follows:

$$A(l) = \begin{bmatrix} a(l)_{11} & a(l)_{12} & \dots & a(l)_{1n} \\ a(l)_{21} & a(l)_{22} & \dots & a(l)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a(l)_{m1} & a(l)_{m2} & \dots & a(l)_{mn} \end{bmatrix}, l = 1, \dots, n_1$$

$$B(r) = \begin{bmatrix} b(r)_{11} & b(r)_{12} & \dots & b(r)_{1n} \\ b(r)_{21} & b(r)_{22} & \dots & b(r)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b(r)_{m1} & b(r)_{m2} & \dots & b(r)_{mn} \end{bmatrix}, r = 1, \dots, n_2$$

where the player PI and the player PII have respectively n_1 and n_2 numbers of objectives. We consider $n_1 = n_2 = s$ number of objectives.
The mixed strategy of the bimatrix game (1.1.1) for player PI and PII are defined as follows:

$$X = \{\ x \in R^m;\ \sum_{i=1}^{m} x_i = 1;\ x_i \geq 0, \quad i = 1, 2, \dots, m\ \} \tag{1}$$

$$Y = \{\ y \in R^n;\ \sum_{j=1}^{n} y_j = 1;\ y_j \geq 0, \quad j = 1, 2, \dots, n\ \} \tag{2}$$

We remark that the pure strategies for both players are the extreme points of X and Y.

Definition 7.3:(Expected Payoffs of Multi-criteria Bimatrix Game)

For the bimatrix game $(A(l), B(l))$, if the player PI chooses the mixed strategy $x \in X$ and the player PII chooses the mixed strategy $y \in Y$, the l^{th} expected payoff for the player PI is represented by $v_1(l) = x^t A(l) y, \quad l = 1, \dots, s$
and that of the l^{th} payoff for the player PII is represented by
$v_2(l) = x^t B(l) y, \quad l = 1, \dots, s.$

Here the player PI chooses a mixed strategy y and the player PII chooses a mixed strategy z in a multi-criteria bimatrix game $(A(l), B(l)), l = 1, \dots, s$.
Therefore, the two-person multi-objective bimatrix game with mixed strategies can be

formulated as follows:

$$\max_{x\in X}\{v_1(l) = x^t A(l)y\}, \quad l = 1,\ldots,s \tag{3}$$

$$\text{and} \quad \max_{y\in Y}\{v_2(l) = x^{*t} B(l)y\}, \quad l = 1,\ldots,s \tag{4}$$

It is very well known that multi-objective bimatrix game is equivalent to following quadratic programming problem.

Model 1

$$\begin{aligned} \max \quad &: \quad v_1(l) \quad l = 1,\ldots,s \\ \max \quad &: \quad v_2(l) \quad l = 1,\ldots,s \\ \text{subject to} \quad v_1(l) \quad &= \quad x^T A(l)y \quad l = 1,\ldots,s \qquad (5) \\ v_2(l) \quad &= \quad x^T B(l)y \quad l = 1,\ldots,s \qquad (6) \\ x \in X, \; y \quad &\in \quad Y \end{aligned}$$

solving the above model we get the optimal solutions.

Definition 7.2:(Nash Equilibrium Solution)

A pair $(x^* \in X, y^* \in Y)$ is said to constitute a Nash equilibrium solution to a bimatrix game $(A(l), B(l))$ in mixed strategies, if the following inequalities are satisfied for all $x \in X$ and $y \in Y$ and t denotes the transpose of a matrix:

$$x^{*t}A(l)y^* \;\geq\; x^t A(l)y^*, \quad l = 1,\ldots,s \tag{7}$$

$$x^{*t}B(l)y^* \;\geq\; x^{*t}B(l)y, \quad k = 1,\ldots,s \tag{8}$$

Here, the pair $(x^{*t}A(l)y^*, x^{*t}B(l)y^*)$, $l = 1,\ldots,s$ is known as a Nash equilibrium outcome of the bimatrix game in mixed strategies.

2.1 Multi-choice Non-linearProgramming for Multi-objective Bimatrix Game(MCNLMBG)

The mathematical model of a (MCNLMBG) is presented for player PI as:

Find $x \in X$ so as to **Model 2**

Model 2

$$\begin{aligned}
\max &: \quad v_1(l) \;\; l = 1, \dots, s \\
\max &: \quad v_2(l) \;\; l = 1, \dots, s \\
\text{subject to} \quad v_1(l) &= x^T(A^{(1)}(l), A^{(2)}(l) \dots, A^{(k)}(l))y \;\; l = 1, \dots, s \qquad (9) \\
v_2(l) &= x^T(B^{(1)}(l), B^{(2)}(l) \dots, B^{(k)}(l))y \;\; l = 1, \dots, s \qquad (10) \\
x \in X, \; y &\in Y
\end{aligned}$$

Right hand side (RHS)of each constraints (9) and (10) have k(pair wise) number of alternatives where only one(pair) is to be selected. To solve **Model 2** it is necessary to transform the problem to standard mathematical programming problem.

3 Transformation of equivalent models

The proposed model accommodates a maximum of eight alternatives in LHS. Seven cases are presented bellow for $k = 2, \dots, 8$

Case(i) $k = 2$

Setting $k = 2$, we have (9) and (10) as:

$v_1(l) = x^T(A^{(1)}(l), A^{(2)(l)})y \;\; l = 1, \dots, s$

and $v_2(l) = x^T(B^{(1)}, B^{(2)})y \;\; l = 1, \dots, s$

For each $l = 1, \dots, s$ RHS of the constraint has two(pair wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$ and $(A^{(2)}, B^{(2)})$, out of which one is to be selected. Since total number of elements of the set is 2, one binary variable $z^{(1)}$ is required. Taking this binary variable, the model is formulated as:

Model 3

$$\begin{aligned}
\max &: \quad v_1(l) \;\; l = 1, \dots, s \\
\max &: \quad v_2(l) \;\; l = 1, \dots, s \\
\text{subject to} \quad v_1(l) &= x^T\{z^{(1)}A^{(1)}(l) + (1 - z^{(1)})A^{(2)}(l)\}y \;\; l = 1, \dots, s \qquad (11) \\
v_2(l) &= x^T\{z^{(1)}B^{(1)}(l) + (1 - z^{(1)})B^{(2)}(l)\}y \;\; l = 1, \dots, s \qquad (12)
\end{aligned}$$

$$
\begin{aligned}
z^{(1)} &= 0/1 \qquad (13)\\
x &\in X\\
y &\in Y
\end{aligned}
$$

Case(ii) $k = 3$

Setting $k = 3$, we have (9) and (10) as:
$v_1(l) = x^T(A^{(1)}(l), A^{(2)}(l), A^{(3)}(l))\ \ l = 1, \ldots, s$
$v_2(l) = x^T(B^{(1)}(l), B^{(2)}(l), B^{(3)}(l))\ \ l = 1, \ldots, s$
For each $l = 1, \ldots, s$ RHS of the constraint has three(pair wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$, $(A^{(2)}(l), B^{(2)}(l))$ and $(A^{(3)}(l), B^{(3)}(l))$,, out of which one(pair) is to be selected. Since total number of elements of the set is 3 and $2^1 < 3 < 2^2$, two binary variable $z^{(1)}, z^{(2)}$ is required. Express 3 as $(2C_2 + 2C_1)$ or $(2C_1 + 2C_0)$. Hence we have to give restriction to remaining one(4-3) term by introducing additional constraint. In case two models are formulated. Taking the binary variables and introducing additional constraint, two models are formulated as:

Model 4.a

$$
\begin{aligned}
\max &: \quad v_1(l)\ \ l = 1, \ldots, s\\
\max &: \quad v_2(l)\ \ l = 1, \ldots, s\\
\text{subject to}\ \ v_1(l) &= x^T\{(1 - z^{(1)})(1 - z^{(2)})A^{(1)}(l)\\
&+ (1 - z^{(1)})z^{(2)}A^{(2)}(l) + z^{(1)}(1 - z^{(2)})A^{(3)}(l)\}y\ \ l = 1, \ldots, s\\
v_2(l) &= x^T\{(1 - z^{(1)})(1 - z^{(2)})B^{(1)}(l)\\
&+ (1 - z^{(1)})z^{(2)}B^{(2)}(l) + z^{(1)}(1 - z^{(2)})B^{(3)}(l)\}y\ \ l = 1, \ldots, s\\
z^{(1)} + z^{(2)} &\leq 1\\
z^{(1)}, z^{(2)} &= 0/1\\
x &\in X\\
y &\in Y
\end{aligned}
$$

Model 4.b

$$\begin{aligned}
\max &: \quad v_1(l) \;\; l = 1, \ldots, s \\
\max &: \quad v_2(l) \;\; l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) &= x^T\{(1 - z^{(1)})z^{(2)}A^{(1)}(l) \\
&+ z^{(1)}(1 - z^{(2)})A^{(2)(l)} + z^{(1)}z^{(2)}A^{(3)}(l)\}y \;\; l = 1, \ldots, s \\
v_2(l) &= x^T\{(1 - z^{(1)})z^{(2)}B^{(1)(l)} \\
&+ z^{(1)}(1 - z^{(2)})B^{(2)}(l) + z^{(1)}z^{(2)}B^{(3)}(l)\}y \;\; l = 1, \ldots, s \\
z^{(1)} + z^{(2)} &\geq 1 \\
z^{(1)}, z^{(2)} &= 0/1 \\
x &\in X \\
y &\in Y
\end{aligned}$$

Case(iii) $k = 4$

Setting $k = 4$, we have (9) and (10) as:

$v_1(l) = x^T(A^{(1)}(l), A^{(2)}(l), A^{(3)}(l), A^{(4)}(l))$ l=1,. . . ,s

$v_2(l) = x^T(B^{(1)}(l), B^{(2)}(l), B^{(3)}(l), B^{(4)}(l))$ l=1,. . . ,s

For each $l = 1, \ldots, s$ RHS of the constraint has four(pair-wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l)), (A^{(2)}(l), B^{(2)}(l)), (A^{(3)}(l), B^{(3)}(l)), (A^{(4)}(l), B^{(4)}(l))$, out of which one(pair) is to be selected. Since total number of elements of the set is 4 , two binary variable $z^{(1)}, z^{(2)}$ is required. Taking the binary variables, model is formulated as:

Model 5

$$\begin{aligned}
\max &: \quad v_1(l) \;\; l = 1, \ldots, s \\
\max &: \quad v_2(l) \;\; l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) &= x^T\{z^{(1)}z^{(2)}A^{(1)}(l) + z^{(1)}(1 - z^{(2)})A^{(2)}(l) \\
&+(1 - z^{(1)})z^{(2)}A^{(3)}(l) + (1 - z^{(1)})(1 - z^{(2)})A^{(4)}(l)\}y \;\; l = 1, \ldots, s \\
v_2(l) &= x^T\{z^{(1)}z^{(2)}B^{(1)}(l) + z^{(1)}(1 - z^{(2)})B^{(2)}(l) \\
&+(1 - z^{(1)})z^{(2)}B^{(3)}(l) + (1 - z^{(1)})(1 - z^{(2)})B^{(4)}(l)\}y \;\; l = 1, \ldots, s
\end{aligned}$$

$$\begin{aligned} z^{(1)}, z^{(2)} &= 0/1 \\ x &\in X, \quad y \in Y \end{aligned}$$

Case(iv) $k = 5$

Setting $k = 5$, we have (9) and (10) as:

$v_1(l) = x^T(A^{(1)}, A^{(2)}, A^{(3)}, , A^{(4)}, A^{(5)}) \quad l = 1, \ldots, s$

$v_2(l) = x^T(B^{(1)}, B^{(2)}, B^{(3)}, , B^{(4)}, B^{(5)}) \quad l = 1, \ldots, s$

For each $l = 1, \ldots, s$ RHS of the constraint has five(pair-wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$, $(A^{(2)}(l), B^{(2)}(l))$, $(A^{(3)}(l), B^{(3)}(l))$, $(A^{(4)}(l), B^{(4)}(l))$, $(A^{(5)}(l), B^{(5)}(l))$, out of which one(pair) is to be selected. Since total number of elements of the set is 5 and $2^2 < 5 < 2^3$, two binary variable $z^{(1)}, z^{(2)}, z^{(3)}$ is required. Express 3 as $(3C_1{+}3C_2{-}1)$. Hence we have to give restriction to remaining three(8-5) term by introducing additional constraint. In case two models are formulated. Taking the binary variables and introducing additional constraints, three models are formulated as:

Model 6.a

$$\begin{aligned} \max &: \quad v_1(l) \quad l = 1, \ldots, s \\ \max &: \quad v_2(l) \quad l = 1, \ldots, s \\ \text{subject to} \quad v_1(l) &= x^T\{z^{(1)}(1 - z^{(2)})(1 - z^{(3)})A^{(1)}(l) \\ &+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})A^{(2)}(l) \\ &+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}A^{(3)}(l) \\ &+ z^{(1)}z^{(2)}(1 - z^{(3)})A^{(4)}(l) \\ &+ (1 - z^{(1)})z^{(2)}z^{(3)}A^{(5)}(l)\}y \quad l = 1, \ldots, s \\ v_2(l) &= x^T\{z^{(1)}(1 - z^{(2)})(1 - z^{(3)})B^{(1)}(l) \\ &+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})B^{(2)}(l) \\ &+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}B^{(3)}(l) \\ &+ z^{(1)}z^{(2)}(1 - z^{(3)})B^{(4)}(l) \\ &+ (1 - z^{(1)})z^{(2)}z^{(3)}B^{(5)}(l)\}y \quad l = 1, \ldots, s \end{aligned}$$

$$
\begin{aligned}
z^{(1)}+z^{(2)}+z^{(3)} &\geq 1\\
z^{(1)}+z^{(2)}+z^{(3)} &\leq 2\\
z^{(1)}+z^{(3)} &\leq 1\\
z^{(1)}, z^{(2)}, z^{(3)} &= 0/1\\
x &\in X\\
y &\in Y
\end{aligned}
$$

Model 6.b

$$
\begin{aligned}
\max \quad &: \quad v_1(l) \;\; l=1,\ldots,s\\
\max \quad &: \quad v_2(l) \;\; l=1,\ldots,s\\
\text{subject to} \quad v_1(l) &= x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})A^{(1)}(l)\\
&+ (1-z^{(1)})z^{(2)}(1-z^{(3)})A^{(2)}(l)\\
&+ (1-z^{(1)})(1-z^{(2)})z^{(3)}A^{(3)}(l)\\
&+ z^{(1)}z^{(2)}(1-z^{(3)})A^{(4)}(l)\\
&+ z^{(1)}(1-z^{(2)})z^{(3)}A^{(5)}(l)\}y \;\; l=1,\ldots,s\\
v_2(l) &= x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})B^{(1)}(l)\\
&+ (1-z^{(1)})z^{(2)}(1-z^{(3)})B^{(2)}(l)\\
&+ (1-z^{(1)})(1-z^{(2)})z^{(3)}B^{(3)}(l)\\
&+ z^{(1)}z^{(2)}(1-z^{(3)})B^{(4)}(l)\\
&+ z^{(1)}(1-z^{(2)})z^{(3)}B^{(5)}(l)\}y \;\; l=1,\ldots,s\\
z^{(1)}+z^{(2)}+z^{(3)} &\geq 1\\
z^{(1)}+z^{(2)}+z^{(3)} &\leq 2\\
z^{(2)}+z^{(3)} &\leq 1\\
z^{(1)}, z^{(2)}, z^{(3)} &= 0/1\\
x &\in X, \;\; y \in Y
\end{aligned}
$$

Model 6.c

$$
\begin{array}{rcl}
\max & : & v_1(l) \;\; l=1,\dots,s \\
\max & : & v_2(l) \;\; l=1,\dots,s \\
\text{subject to} \;\; v_1(l) & = & x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})A^{(1)}(l) \\
 & + & (1-z^{(1)})z^{(2)}(1-z^{(3)})A^{(2)}(l) \\
 & + & (1-z^{(1)})(1-z^{(2)})z^{(3)}A^{(3)}(l) \\
 & + & (1-z^{(1)})z^{(2)}z^{(3)}A^{(4)}(l) \\
 & + & z^{(1)}(1-z^{(2)})z^{(3)}A^{(5)}(l)\}y \;\; l=1,\dots,s \\
v_2(l) & = & x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})B^{(1)}(l) \\
 & + & (1-z^{(1)})z^{(2)}(1-z^{(3)})B^{(2)}(l) \\
 & + & (1-z^{(1)})(1-z^{(2)})z^{(3)}B^{(3)}(l) \\
 & + & (1-z^{(1)})z^{(2)}z^{(3)}B^{(4)}(l) \\
 & + & z^{(1)}(1-z^{(2)})z^{(3)}B^{(5)}(l)\}y \;\; l=1,\dots,s \\
z^{(1)}+z^{(2)}+z^{(3)} & \geq & 1 \\
z^{(1)}+z^{(2)}+z^{(3)} & \leq & 2 \\
z^{(1)}+z^{(2)} & \leq & 1 \\
z^{(1)}, z^{(2)}, z^{(3)} & = & 0/1 \\
x & \in & X \\
y & \in & Y
\end{array}
$$

Case(v) $k=6$

Setting $k=6$, we have (9) and (10) as:
$v_1(l) = x^T(A^{(1)}(l), A^{(2)}(l), A^{(3)}(l),, A^{(4)}(l), A^{(5)}(l), A^{(6)}(l)), \quad l=1,\dots,s$
$v_2(l) = x^T(B^{(1)}(l), B^{(2)}(l), B^{(3)}(l),, B^{(4)}(l), B^{(5)}(l), B^{(6)}(l)), \quad l=1,\dots,s$
For each $l=1,\dots,s$ RHS of the constraint has six(pair-wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$, $(A^{(2)}(l), B^{(2)}(l))$, $(A^{(3)}(l), B^{(3)}(l))$, $(A^{(4)}(l), B^{(4)}(l))$, $(A^{(5)}(l), B^{(5)}(l))$, $(A^{(6)}, B^{(6)})$, out of which one(pair) is to be selected. Since total number of elements of

the set is 5 and $2^2 < 6 < 2^3$, two binary variable $z^{(1)}, z^{(2)}, z^{(3)}$ is required. Express 6 as $(3C_1 + 3C_2)$. Hence we have to give restriction to remaining two(8-6) term by introducing additional constraint. Taking the binary variables and introducing additional constraints, three models are formulated as:

Model 7

$$
\begin{aligned}
\max &: \quad v_1(l), \quad l = 1, \ldots, s \\
\max &: \quad v_2(l), \quad l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) &= x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})A^{(1)}(l) \\
&+ (1-z^{(1)})z^{(2)}(1-z^{(3)})A^{(2)}(l) \\
&+ (1-z^{(1)})(1-z^{(2)})z^{(3)}A^{(3)}(l) \\
&+ z^{(1)}z^{(2)}(1-z^{(3)})A^{(4)}(l) \\
&+ z^{(1)}(1-z^{(2)})z^{(3)}A^{(5)}(l) \\
&+ (1-z^{(1)})z^{(2)}z^{(3)}A^{(6)}(l)\}y \quad l = 1, \ldots, s \\
v_2(l) &= x^T\{z^{(1)}(1-z^{(2)})(1-z^{(3)})B^{(1)}(l) \\
&+ (1-z^{(1)})z^{(2)}(1-z^{(3)})B^{(2)}(l) \\
&+ (1-z^{(1)})(1-z^{(2)})z^{(3)}B^{(3)}(l) \\
&+ z^{(1)}z^{(2)}(1-z^{(3)})B^{(4)}(l) \\
&+ z^{(1)}(1-z^{(2)})z^{(3)}B^{(5)}(l) \\
&+ (1-z^{(1)})z^{(2)}z^{(3)}B^{(6)}(l)\}y \quad l = 1, \ldots, s \\
z^{(1)} + z^{(2)} + z^{(3)} &\geq 1 \\
z^{(1)} + z^{(2)} + z^{(3)} &\leq 2 \\
z^{(1)}, z^{(2)}, z^{(3)} &= 0/1 \\
x &\in X \\
y &\in Y
\end{aligned}
$$

Case(vi) $k = 7$

Setting $k = 7$, we have (9) and (10) as:

$v_1(l) = x^T(A^{(1)}(l), A^{(2)}(l), A^{(3)}(l), A^{(4)}(l), A^{(5)}(l), A^{(6)}(l), A^{(7)}(l))\ \ l = 1, \ldots, s$

$v_2(l) = x^T(B^{(1)}(l), B^{(2)}(l), B^{(3)}(l), B^{(4)}(l), B^{(5)}(l),, B^{(6)}(l), B^{(7)}(l))\ \ l = 1, \ldots, s$

For each $l = 1, \ldots, s$ RHS of the constraint has seven(pair-wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$, $(A^{(2)}(l), B^{(2)}(l))$, $(A^{(3)}(l), B^{(3)}(l))$, $(A^{(4)}(l), B^{(4)}(l))$, $(A^{(5)}(l), B^{(5)}(l))$, $(A^{(6)}(l), B^{(6)}(l))$, $(A^{(7)}(l), B^{(7)}(l))$, out of which one(pair) is to be selected. Since total number of elements of the set is 7 and $2^2 < 7 < 2^3$, two binary variable $z^{(1)}, z^{(2)}, z^{(3)}$ is required. Express 7 as $(3C_1 + 3C_2 + 3C3)$. Hence we have to give restriction to remaining one(8-7) term by introducing additional constraint. Taking the binary variables and introducing additional constraints, different models are formulated as:

Model 8a

$$\begin{aligned}
\max &: \quad v_1(l), \quad l = 1, \ldots, s \\
\max &: \quad v_2(l), \quad l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) &= x^T\{(1 - z^{(1)})(1 - z^{(2)})(1 - z^{(3)})A^{(1)}(l) \\
&+ z^{(1)}(1 - z^{(2)})(1 - z^{(3)})A^{(2)}(l) \\
&+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})A^{(3)}(l) \\
&+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}A^{(4)}(l) \\
&+ z^{(1)}z^{(2)}(1 - z^{(3)})A^{(5)}(l) \\
&+ z^{(1)}(1 - z^{(2)})z^{(3)}A^{(6)}(l) \\
&+ (1 - z^{(1)})z^{(2)}z^{(3)}A^{(7)}(l)\}y \quad l = 1, \ldots, s \\
v_2(l) &= x^T\{(1 - z^{(1)})(1 - z^{(2)})(1 - z^{(3)})B^{(1)}(l) \\
&+ z^{(1)}(1 - z^{(2)})(1 - z^{(3)})B^{(2)}(l) \\
&+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})B^{(3)}(l) \\
&+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}B^{(4)}(l) \\
&+ z^{(1)}z^{(2)}(1 - z^{(3)})B^{(5)}(l) \\
&+ z^{(1)}(1 - z^{(2)})z^{(3)}B^{(6)}(l) \\
&+ (1 - z^{(1)})z^{(2)}z^{(3)}B^{(7)}(l)\}y \quad l = 1, \ldots, s \\
z^{(1)} + z^{(2)} + z^{(3)} &\leq 2
\end{aligned}$$

$$
\begin{aligned}
z^{(1)}, z^{(2)}, z^{(3)} &= 0/1 \\
x &\in X \\
y &\in Y
\end{aligned}
$$

Model 8b

$$
\begin{aligned}
\max &: \quad v_1(l), \quad l = 1, \ldots, s \\
\max &: \quad v_2(l), \quad l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) &= x^T \{ z^{(1)}(1 - z^{(2)})(1 - z^{(3)})A^{(1)}(l) \\
&+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})A^{(2)}(l) \\
&+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}A^{(3)}(l) \\
&+ z^{(1)}z^{(2)}(1 - z^{(3)})A^{(4)}(l) \\
&+ z^{(1)}(1 - z^{(2)})z^{(3)}A^{(5)}(l) \\
&+ (1 - z^{(1)})z^{(2)}z^{(3)}A^{(6)}(l) \\
&+ z^{(1)}z^{(2)}z^{(3)}A^{(7)}(l)\}y \quad l = 1, \ldots, s \\
v_2(l) &= x^T \{ z^{(1)}(1 - z^{(2)})(1 - z^{(3)})B^{(1)}(l) \\
&+ (1 - z^{(1)})z^{(2)}(1 - z^{(3)})B^{(2)}(l) \\
&+ (1 - z^{(1)})(1 - z^{(2)})z^{(3)}B^{(3)}(l) \\
&+ z^{(1)}z^{(2)}(1 - z^{(3)})B^{(4)}(l) \\
&+ z^{(1)}(1 - z^{(2)})z^{(3)}B^{(5)}(l) \\
&+ (1 - z^{(1)})z^{(2)}z^{(3)}B^{(6)}(l) \\
&+ z^{(1)}z^{(2)}z^{(3)}B^{(7)}(l)\}y \quad l = 1, \ldots, s \\
z^{(1)} + z^{(2)} + z^{(3)} &\geq 1 \\
z^{(1)}, z^{(2)}, z^{(3)} &= 0/1 \\
x &\in X \\
y &\in Y
\end{aligned}
$$

Case(vii)k=8 Setting $k = 8$, we have (9) and (10) as:

$v_1(l) = x^T(A^{(1)}(l), A^{(2)}(l), A^{(3)}(l), A^{(4)}(l), A^{(5)}(l), A^{(6)}(l), A^{(7)}(l), A^{(8)}(l)) \;\; l = 1, \ldots, s$

$v_2(l) = x^T(B^{(1)}(l), B^{(2)}(l), B^{(3)}(l), B^{(4)}(l), B^{(5)}(l), B^{(6)}(l), B^{(7)}(l), B^{(8)}(l)) \;\; l = 1, \ldots, s$

For each $l = 1, \ldots, s$ RHS of the constraint has eight(pair-wise) alternatives, namely, $(A^{(1)}(l), B^{(1)}(l))$, $(A^{(2)}(l), B^{(2)}(l))$, $(A^{(3)}(l), B^{(3)}(l))$, $(A^{(4)}(l), B^{(4)}(l))$, $(A^{(5)}(l), B^{(5)}(l))$, $(A^{(6)}(l), B^{(6)}(l))$, $(A^{(7)}(l), B^{(7)}(l))$, $(A^{(8)}(l), B^{(8)}(l))$, out of which one(pair) is to be selected. Since total number of elements of the set is 7 and $8 = 2^3$, two binary variable $z^{(1)}, z^{(2)}, z^{(3)}$ is required. Taking the binary variables models is formulated as:

Model 9

$$\begin{aligned}
\max \quad &: \quad v_1(l) \;\; l = 1, \ldots, s \\
\max \quad &: \quad v_2(l) \;\; l = 1, \ldots, s \\
\text{subject to} \quad v_1(l) \quad &= \quad x^T\{z^{(1)}z^{(2)}z^{(3)}A^{(1)}(l) \\
&+ \quad (1 - z^{(1)})z^{(2)}z^{(3)}A^{(2)}(l) \\
&+ \quad z^{(1)}(1 - z^{(2)})z^{(3)}A^{(3)}(l) \\
&+ \quad z^{(1)}z^{(2)}(1 - z^{(3)})A^{(4)}(l) \\
&+ \quad (1 - z^{(1)})(1 - z^{(2)})z^{(3)}A^{(5)}(l) \\
&+ \quad (1 - z^{(1)})z^{(2)}(1 - z^{(3)})A^{(6)}(l) \\
&+ \quad z^{(1)}(1 - z^{(2)})(1 - z^{(3)})A^{(7)}(l) \\
&+ \quad (1 - z^{(1)})(1 - z^{(2)})(1 - z^{(3)})A^{(8)}(l)\}y \;\; l = 1, \ldots, s \\
v_2(l) \quad &= \quad x^T\{z^{(1)}z^{(2)}z^{(3)}B^{(1)}(l) \\
&+ \quad (1 - z^{(1)})z^{(2)}z^{(3)}B^{(2)}(l) \\
&+ \quad z^{(1)}(1 - z^{(2)})z^{(3)}B^{(3)}(l) \\
&+ \quad z^{(1)}z^{(2)}(1 - z^{(3)})B^{(4)}(l) \\
&+ \quad (1 - z^{(1)})(1 - z^{(2)})z^{(3)}B^{(5)}(l) \\
&+ \quad (1 - z^{(1)})z^{(2)}(1 - z^{(3)})B^{(6)}(l) \\
&+ \quad z^{(1)}(1 - z^{(2)})(1 - z^{(3)})B^{(7)}(l) \\
&+ \quad (1 - z^{(1)})(1 - z^{(2)})(1 - z^{(3)})B^{(8)}(l)\}y \;\; l = 1, \ldots, s
\end{aligned}$$

$$\begin{aligned} z^{(1)}, z^{(2)}, z^{(3)} &= 0/1 \\ x &\in X \\ y &\in Y \end{aligned}$$

Generalized this transformation techniques is modified for bimatrix game in the following (3.1)

3.1 Transformation Techniques For Bimatrix Game

We present two transformation techniques of MCLPP to formulate an equivalent mathematical model.

3.1.1 Transformation techniques 1

Restrictions are given on the upper bound of binary variables.
Step 1: Find the total set of choices for set of payoff bimatrices.

$$\begin{aligned} v_1(d) &= x^T(A^{(1)}(d), A^{(2)}(d) \ldots, A^{(k)}(d))y, \quad d = 1, \ldots, s \\ v_2(d) &= x^T(B^{(1)}(d), B^{(2)}(d) \ldots, B^{(k)}(d))y, \quad d = 1, \ldots, s \end{aligned}$$

Step 2: Find the number of binary variables, which is required to handle the multi-choice parameters in LHS of the constraint in following manner.
Find l, for which $2^{l-1} < k < 2^l$. Here l number of binary variables are needed. Let the binary variables are $z^1, z^2, z^3, \ldots, z^l$.

Step 3: Expand 2^l as $\binom{l}{0} + \binom{l}{1} + \binom{l}{2} + \ldots + \binom{l}{r_1} + \ldots + \binom{l}{r_2} + \ldots + \binom{l}{l}$ and select the smallest number of consecutive terms whose sum is equal to or just greater than k_i from the expansion. Let the terms be $\binom{l}{r_1}, \binom{l}{r_1+1}, \binom{l}{r_1+2}, \ldots, \binom{l}{l}$.

Step 4**:** Assign k binary codes to k number of choices follows:

$$
\begin{aligned}
& X^T\{\sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} A^{(j)}(d) + \\
& \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} A^{(\binom{l}{r_1}+j)}(d) + \ldots + \\
& \sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} A^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d) \\
& + \sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} A^{(L_1+j)}(d)\}Y = v_1(d), \quad d = 1, \ldots, s \\
& X^T\{\sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} B^{(j)}(d) + \\
& \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} B^{(\binom{l}{r_1}+j)}(d) + \ldots + \\
& \sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} B^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d) \\
& + \sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} B^{(L_1+j)}(d)\}Y = v_2(d), \quad d = 1, \ldots, s
\end{aligned}
$$

$$\text{Where} \quad L^1 = \binom{l}{r_1} + \binom{l}{r_1+1} + \ldots + \binom{l}{r_2-1}$$

$$
\begin{aligned}
j_1 &\in (1, 2, 3, \ldots, (l-s)+1), \\
j_2 &\in (2, 3, \ldots, (l-s)+2), \ldots, j_s \in (s, s+1, \ldots, l) \\
I_s^{(j)} &= \{\{j_1, j_2, \ldots, j_s\} | j_1 < j_2 < \ldots < j_s, s = r_1, r_1+1, \ldots, r_2\} \\
P_j^{(s)} &= \{z^{j_1} z^{j_2} z^{j_3} \ldots z^{j_s} | \{j_1, j_2, \ldots, j_s\} \in I_s^{(j)}, s = r_1, r_1+1, \ldots, r_2\} \\
Q_j^s &= \{\prod_{j=1}^{l} (1 - z^{(j)}) | j \notin \{j_1, j_2, \ldots, j_s\}
\end{aligned}
$$

Step 5: Restrict $(2^l - k)$ number of binary codes to overcome repetitions as follows:

$$\begin{aligned} z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l)} &\geq r_1 \\ z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l_i)} &\leq r_2 \\ z^{(j_1)} + z^{(j_2)} + z^{(j_3)} + \ldots + z^{(j_{r_2})} &\leq r_2 - 1, \\ j = (k - L^1) + 1, (k - L^1) + 2, \ldots, &\binom{l}{r_2} \end{aligned}$$

Restrictions should be imposed on $z^{(j_1)} + z^{(j_2)} + z^{(j_3)} + \ldots + z^{(j_{r_2})} \in P_j^{r_2}$ but not included in T_{r_2}. T_{r_2} contains the terms $P_j^{r_2}$ in transformed constraint.

Step 6:

$$\begin{aligned} \max &: \quad v_1(d) \quad d = 1, \ldots, s \\ \max &: \quad v_2(d), \quad d = 1, \ldots, s \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & X^T\{ \sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} A^{(j)}(d) + \\ & \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} A^{(\binom{l}{r_1}+j)}(d) + \ldots + \\ & \sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} A^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d) \\ & + \sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} A^{(L_1+j)}(d)\} Y = v_1(d), \quad d = 1, \ldots, s \\ & X^T\{ \sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} B^{(j)}(d) + \\ & \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} B^{(\binom{l}{r_1}+j)}(d) + \ldots + \\ & \sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} B^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d) \\ & + \sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} B^{(L_1+j)}(d)\} Y = v_2(d), \quad d = 1, \ldots, s \end{aligned}$$

$$
\begin{aligned}
z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l)} &\geq r_1\\
z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l_i)} &\leq r_2\\
z^{(j_1)}+z^{(j_2)}+z^{(j_3)}+\ldots+z^{(j_{r_2})} &\leq r_2-1,\\
j=(k-L^1)+1,(k-L^1)+2,\ldots,&\binom{l}{r_2}\\
x &\in X\\
Y &\in Y\\
z^{(l)}=0/1, l=1,2,\ldots,&[\frac{\ln(k)}{\ln(2)}]\\
\text{Where} \quad L^1=\binom{l}{r_1}+\binom{l}{r_1+1}&+\ldots+\binom{l}{r_2-1}
\end{aligned}
$$

Step 7: Above mathematical model is a mixed integer non-linear programming problem. Solve the model with the help of existing methodology.

3.1.2 Transformation techniques 2

Restrictions are given on the upper bound of binary variables.

Step 1: Find the total number of choices for payoff bimatrix.

$$
\begin{aligned}
v_1(d) &= x^T(A^{(1)}(d),A^{(2)}(d)\ldots,A^{(k)}(d))y, \quad d=1,\ldots,s\\
v_2(d) &= x^T(B^{(1)}(d),B^{(2)}(d)\ldots,B^{(k)}(d))y, \quad d=1,\ldots,s
\end{aligned}
$$

Step 2: Find the number of binary variables, which is required to handle the multi-choice parameters in LHS of the constraint in following manner.

Find l, for which $2^{l-1}<k<2^l$. Here l number of binary variables are needed. Let the binary variables are $z^1,z^2,z^3,\ldots,z^l$.

Step 3: Expand 2^l as $\binom{l}{0}+\binom{l}{1}+\binom{l}{2}+\ldots+\binom{l}{r_1}+\ldots+\binom{l}{r_2}+\ldots+\binom{l}{l}$ and select the smallest number of consecutive terms whose sum is equal to or just greater than k_i from the expansion. Let the terms be $\binom{l}{r_1},\binom{l}{r_1+1},\binom{l}{r_1+2},\ldots,\binom{l}{l}$.

Step 4**:** Assign k binary codes to k number of choices for payoff bimatrices:

$$X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)}Q_j^{(r_2)}A^{(j)}(d) +$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)}Q_j^{(r_2-1)}A^{(\binom{l}{r_2}+j)}(d) + \ldots +$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)}Q_j^{(r_1+1)}A^{(\binom{l}{r_2}+\ldots+\binom{l}{r_1+2}+j)}(d)$$

$$+\sum_{j=1}^{k-L_2} P_j^{(r_1)}Q_j^{(r_1)}A^{(L_2+j)}(d)\}Y \quad = \quad v_1(d), \text{ `}d = 1, \ldots, s$$

$$X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)}Q_j^{(r_2)}B^{(j)}(d) +$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)}Q_j^{(r_2-1)}B^{(\binom{l}{r_2}+j)}(d) + \ldots +$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)}Q_j^{(r_1+1)}B^{(\binom{l}{r_2}(d)+\ldots+\binom{l}{r_1+2}+j)}$$

$$+\sum_{j=1}^{k-L_2} P_j^{(r_1)}Q_j^{(r_1)}B^{(L_2+j)}(d)\}Y \quad = \quad v_2(d), \quad d = 1, \ldots, s$$

$$\text{Where} \quad L_2 = \binom{l}{r_2} + \binom{l}{r_2-1} + \ldots + \binom{l}{r_1+1}$$

$$j_1 \in (1, 2, 3, \ldots, (l-s)+1),$$

$$j_2 \in (2, 3, \ldots, (l-s)+2), \ldots, j_s \in (s, s+1, \ldots, l)$$

$$I_s^{(j)} = \{\{j_1, j_2, \ldots, j_s\} | j_1 < j_2 < \ldots < j_s, s = r_1, r_1+1, \ldots, r_2\}$$

$$P_j^{(s)} = \{z^{j_1}z^{j_2}z^{j_3}\ldots z^{j_s} | \{j_1, j_2, \ldots, j_s\} \in I_s^{(j)}, s = r_1, r_1+1, \ldots, r_2\}$$

$$Q_j^s = \{\prod_{j=1}^{l}(1-z^{(j)}) | j \notin \{j_1, j_2, \ldots, j_s\}$$

Step 5**:** Restrict $(2^l - k)$ number of binary codes to overcome repetitions as follows:

$$z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l)} \geq r_1$$

$$z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l_i)} \leq r_2$$

$$\sum_{t=1}^{l} z^{(t)} \geq 1, t \notin \{j_1, \ldots, j^l_{r_1}\}$$

$$j = (k - L^2) + 1, (k - L^2) + 2, \ldots, \binom{l}{r_1}$$

Restrictions should be imposed on $z^{(j_1)} + z^{(j_2)} + z^{(j_3)} + \ldots + z^{(j_{r_1})} \in P_j^{r_2}$ but not included in T_{r_1}. T_{r_1} contains the terms $P_j^{r_1}$ in transformed constraint.

Step 6**:**

$$\max : v_1$$

$$\max : v_2$$

$$\text{subject to} \quad X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)} Q_j^{(r_2)} A^{(j)}(d) +$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} A^{(\binom{l}{r_2}+j)}(d) + \ldots +$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} A^{(\binom{l}{r_2}+\ldots+\binom{l}{r_1+2}+j)}(d)$$

$$+ \sum_{j=1}^{k-L_2} P_j^{(r_1)} Q_j^{(r_1)} A^{(L_2+j)}(d)\}Y = v_1(d), \text{ `}d = 1, \ldots, s$$

$$X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)} Q_j^{(r_2)} B^{(j)}(d) +$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} B^{(\binom{l}{r_2}+j)}(d) + \ldots +$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} B^{(\binom{l}{r_2}(d)+\ldots+\binom{l}{r_1+2}+j)}$$

$$+ \sum_{j=1}^{k-L_2} P_j^{(r_1)} Q_j^{(r_1)} B^{(L_2+j)}(d)\}Y = v_2(d), \quad d = 1, \ldots, s$$

$$z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l)} \geq r_1$$
$$z^{(1)} + z^{(2)} + z^{(3)} + \ldots + z^{(l_i)} \leq r_2$$
$$\sum_{t=1,\quad t \notin I^j_{r_1}}^{l} z^{(t)} \geq 1$$
$$j = (k - L^2) + 1, (k - L^2) + 2, \ldots, \binom{l}{r_1}$$
$$x \in X$$
$$Y \in Y$$
$$z^{(l)} = 0/1, l = 1, 2, \ldots, [\frac{\ln(k)}{\ln(2)}]$$
$$\text{Where} \quad L^2 = \binom{l}{r_2} + \binom{l}{r_2 - 1} + \ldots + \binom{l}{r_1 + 1}$$

Step 7: Above mathematical model is a mixed integer non-linear programming problem. Solve the model with the help of existing methodology.

4 Solution Procedure

In previous section, we have seen that, **Model A** or **Model B**is a multi-objective non-linear programming (MONLP) problem. To get a satisfactory solution of the above model, we have introduced Zimmerman's[] the fuzzy programming which is defined in the following subsection.

4.1 Fuzzy Programming:

In fuzzy programming, we first construct the membership function for each objective function in **Model 6.3**. Let $\mu_{1d}(v_1(d)), \mu_{2d}(v_2(d)),\ d = 1, \ldots, s$ be the membership functions for objectives respectively and these are defined as follows:

$$\mu_{1d}(v_1(d)) = \begin{cases} 0 & \text{if } \ v_1(d) \leq v_1^-(d) \\ \frac{v_1(d) - v_1^-(d)}{v_1^+(d) - v_1^-(d)} & \text{if } \ v_1^-(d) \leq v_1(d) \leq v_1^+(d), \\ 1 & \text{if } \ v_1(d) \geq v_1^+(d) \end{cases} \tag{14}$$

$$\mu_{2d}(v_2(d)) = \begin{cases} 0 & \text{if } v_2(d) \leq v_2^-(d) \\ \frac{v_2(d)-v_2^-(d)}{v_2^+(d)-v_2^-(d)} & \text{if } v_2^-(d) \leq v_2(d) \leq v_2^+(d), \\ 1 & \text{if } v_2(d) \geq v_2^+(d) \end{cases} \tag{15}$$

where $v_1^+(d), v_1^-(d),\ d = 1, \ldots, s$ respectively, represent maximum and minimum values of v_1; $v_2^+(d), v_2^-(d),\ d = 1, \ldots, s$ respectively, represent maximum and minimum values of v_2.

To conversion in a single objective non-linear model from multi-objective non-linear model, we have introduced the concept of fuzzy programming technique with the help of membership functions and the **Model A** then we have formulated the following single objective non-linear **Model C** as follows.

Model C

$$\begin{aligned} \max &: \lambda \\ \text{subject to } \lambda &\leq \frac{v_1(d) - v_1^-(d)}{v_1^+(d) - v_1^-(d)},\ d = 1, \ldots, s \\ \lambda &\leq \frac{v_2(d) - v_2^-(d)}{v_2^+(d) - v_2^-(d)},\ d = 1, \ldots, s \end{aligned}$$

$$\begin{aligned} & X^T\{ \sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} A^{(j)}(d) + \\ & \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} A^{(\binom{l}{r_1}+j)}(d) + \ldots + \\ & \sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} A^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d) \\ & + \sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} A^{(L_1+j)}(d)\}Y = v_1(d), \quad d = 1, \ldots, s \end{aligned} \tag{16}$$

$$\begin{aligned} & X^T\{ \sum_{j=1}^{\binom{l}{r_1}} P_j^{(r_1)} Q_j^{(r_1)} B^{(j)}(d) + \\ & \sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} B^{(\binom{l}{r_1}+j)}(d) + \ldots + \end{aligned}$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} B^{(\binom{l}{r_1}+\ldots+\binom{l}{r_2-2}+j)}(d)$$

$$+\sum_{j=1}^{k-L_1} P_j^{(r_2)} Q_j^{(r_2)} B^{(L_1+j)}(d)\}Y \quad = \quad v_2(d), \quad d=1,\ldots,s \tag{17}$$

$$z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l)} \geq r_1 \tag{18}$$

$$z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l_i)} \leq r_2 \tag{19}$$

$$z^{(j_1)}+z^{(j_2)}+z^{(j_3)}+\ldots+z^{(j_{r_2})} \leq r_2-1, \tag{20}$$

$$j=(k-L^1)+1,(k-L^1)+2,\ldots,\binom{l}{r_2} \tag{21}$$

$$x \in X$$

$$Y \in Y$$

$$z^{(l)}=0/1, l=1,2,\ldots,[\frac{\ln(k)}{\ln(2)}] \tag{22}$$

Where $L^1=\binom{l}{r_1}+\binom{l}{r_1+1}+\ldots+\binom{l}{r_2-1}$

OR **Model D**

$$\max \quad : \quad \lambda$$

$$\text{subject to} \quad \lambda \leq \frac{v_1(d)-v_1^-(d)}{v_1^+(d)-v_1^-(d)}, \ d=1,\ldots,s$$

$$\lambda \leq \frac{v_2(d)-v_2^-(d)}{v_2^+(d)-v_2^-(d)}, \ d=1,\ldots,s$$

$$X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)} Q_j^{(r_2)} A^{(j)}(d)+$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)} Q_j^{(r_2-1)} A^{(\binom{l}{r_2}+j)}(d)+\ldots+$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)} Q_j^{(r_1+1)} A^{(\binom{l}{r_2}+\ldots+\binom{l}{r_1+2}+j)}(d)$$

$$+\sum_{j=1}^{k-L_2} P_j^{(r_1)} Q_j^{(r_1)} A^{(L_2+j)}(d)\}Y \quad = \quad v_1(d), \ `d=1,\ldots,s$$

$$X^T\{\sum_{j=1}^{\binom{l}{r_2}} P_j^{(r_2)}Q_j^{(r_2)}B^{(j)}(d)+$$

$$\sum_{j=1}^{\binom{l}{r_2-1}} P_j^{(r_2-1)}Q_j^{(r_2-1)}B^{(\binom{l}{r_2}+j)}(d)+\ldots+$$

$$\sum_{j=1}^{\binom{l}{r_1+1}} P_j^{(r_1+1)}Q_j^{(r_1+1)}B^{(\binom{l}{r_2}(d)+\ldots+\binom{l}{r_1+2}+j)}$$

$$+\sum_{j=1}^{k-L_2} P_j^{(r_1)}Q_j^{(r_1)}B^{(L_2+j)}(d)\}Y \quad = \quad v_2(d), \quad d=1,\ldots,s$$

$$z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l)}\geq r_1$$

$$z^{(1)}+z^{(2)}+z^{(3)}+\ldots+z^{(l_i)}\leq r_2$$

$$\sum_{t=1,\quad t\notin I_{r_1}^j}^{l} z^{(t)}\geq 1$$

$$j=(k-L^2)+1,(k-L^2)+2,\ldots,\binom{l}{r_1}$$

$$x\in X$$

$$Y\in Y$$

$$z^{(l)}=0/1, l=1,2,\ldots,[\frac{\ln(k)}{\ln(2)}]$$

$$\text{Where}\quad L^2=\binom{l}{r_2}+\binom{l}{r_2-1}+\ldots+\binom{l}{r_1+1}$$

Now v_l^+, v_l^-, and v_2^+, v_2^-, which are determined by Genetic Algorithm[]. Then solve it by LINGO software.

5 Numerical example

Example : Let us consider a multi choice game problem having three alternatives the pay-off matrices are as follows:

$$A^1(1) = \begin{bmatrix} 6 & 3 & 5 \\ 7 & 9 & 3 \\ 8 & 7 & 8 \end{bmatrix} \quad A^2(1) = \begin{bmatrix} 0 & 5 & 3 \\ 6 & 0 & 8 \\ 7 & 9 & 0 \end{bmatrix} \quad A^3(1) = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$A^1(2) = \begin{bmatrix} 4 & 3 & 6 \\ 2 & 5 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad A^2(2) = \begin{bmatrix} 3 & 5 & 6 \\ 4 & 2 & 3 \\ 3 & 7 & 3 \end{bmatrix} \quad A^3(2) = \begin{bmatrix} 8 & 3 & 4 \\ 6 & 7 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$

$$A^1(3) = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \quad A^2(3) = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 2 & 4 \end{bmatrix} \quad A^3(3) = \begin{bmatrix} 4 & 0 & 6 \\ 6 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix}$$

$$B^1(1) = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 6 & 5 \\ 3 & 1 & 3 \end{bmatrix} \quad B^2(1) = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 2 & 3 \\ 6 & 8 & 4 \end{bmatrix} \quad B^3(1) = \begin{bmatrix} 3 & 7 & 6 \\ 4 & 5 & 3 \\ 8 & 10 & 12 \end{bmatrix}$$

$$B^1(2) = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 5 \end{bmatrix} \quad B^2(2) = \begin{bmatrix} 4 & 0 & 4 \\ 3 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad B^3(2) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$B^1(3) = \begin{bmatrix} 6 & 3 & 5 \\ 7 & 9 & 3 \\ 8 & 7 & 8 \end{bmatrix} \quad B^2(3) = \begin{bmatrix} 3 & 5 & 6 \\ 4 & 2 & 3 \\ 3 & 7 & 3 \end{bmatrix} \quad B^3(3) = \begin{bmatrix} 4 & 0 & 6 \\ 6 & 4 & 2 \\ 0 & 5 & 2 \end{bmatrix}$$

Using above pay-off matrices we develop following **Model 10** from **Model 4.a**

Model 10

$$\begin{aligned}
\max &: \quad v_1(l) \;\; l=1,2,3\\
\max &: \quad v_2(l) \;\; l=1,2,3\\
\text{subject to} \quad v_1(l) &= x^T\{(1-z^{(1)})(1-z^{(2)})A^{(1)}(l)\\
&+ (1-z^{(1)})z^{(2)}A^{(2)}(l)+z^{(1)}(1-z^{(2)})A^{(3)}(l)\}y \;\; l=1,2,3\\
v_2(l) &= x^T\{(1-z^{(1)})(1-z^{(2)})B^{(1)}(l)\\
&+ (1-z^{(1)})z^{(2)}B^{(2)}(l)+z^{(1)}(1-z^{(2)})B^{(3)}(l)\}y \;\; l=1,2,3\\
z^{(1)}+z^{(2)} &\le 1\\
z^{(1)},z^{(2)} &= 0/1
\end{aligned}$$

$x_1+x_2+x_3=1$

$y_1+y_2+y_3=1$

$x_1,x_2,x_3\ge 0$

$y_1,y_2,y_3\ge 0$

The values of $v_l^+(d), v_l^-(d)$, and $v_2^+(d), v_2^-(d), d=1,2,3$ are presented in the following table **T1**
which are computed by Genetic Algorithm.

	maximum value	*minimum value*
$v_1(1)$	$v_1^+(1)=9$	$v_1^-(1)=2$
$v_1(2)$	$v_1^+(2)=8$	$v_1^-(2)=1$
$v_1(3)$	$v_1^+(3)=6$	$v_1^-(3)=1$
$v_2(1)$	$v_2^+(1)=8$	$v_2^-(1)=2$
$v_2(2)$	$v_2^+(2)=6$	$v_2^-(2)=2$
$v_2(3)$	$v_2^+(3)=7$	$v_2^-(3)=4$

T1 : Max and Min values of objectives.

The single objective model is

Model 11

$$\begin{aligned}
\max \quad &:& \lambda \\
\text{subject to} \quad \lambda &\leq& \frac{v_1(1)-2}{7}, \\
\lambda &\leq& \frac{v_1(2)-1}{7}, \\
\lambda &\leq& \frac{v_1(3)-1}{5}, \\
\lambda &\leq& \frac{v_2(1)-2}{6}, \\
\lambda &\leq& \frac{v_2(2)-2}{4}, \\
\lambda &\leq& \frac{v_2(3)-4}{3}, \\
v_1(l) &=& x^T\{(1-z^{(1)})(1-z^{(2)})A^{(1)}(l) \\
&+& (1-z^{(1)})z^{(2)}A^{(2)}(l)+z^{(1)}(1-z^{(2)})A^{(3)}(l)\}y \quad l=1,2,3 \\
v_2(l) &=& x^T\{(1-z^{(1)})(1-z^{(2)})B^{(1)}(l) \\
&+& (1-z^{(1)})z^{(2)}B^{(2)}(l)+z^{(1)}(1-z^{(2)})B^{(3)}(l)\}y \quad l=1,2,3 \\
z^{(1)}+z^{(2)} &\leq& 1 \\
z^{(1)}, z^{(2)} &=& 0/1
\end{aligned}$$

$$x_1+x_2+x_3=1$$

$$y_1+y_2+y_3=1$$

$$x_1, x_2, x_3 \geq 0$$

$$y_1, y_2, y_3 \geq 0$$

The **Model 11** is a mixed integer programming model. Solved it by Lingo(9)package the value of the multi objective bimatrix game is $v_1^*(1)=3.928440$, $v_1^*(2)=5.055310$,$v_1^*(3)=2.377457$,$v_2^*(1)=4.219399$, $v_2^*(2)=3.101966$, $v_2^*(3)=5.055310$ for $x*=(0.756, 0.244, 0.0)$, $y*=(0.105, 0.0, 0.895)$.

6 Conclusions

The proposed method provides a solution for multiple alternatives of a matrix game model. If we have k alternatives then $[\frac{lnk}{ln2}]$ number of binary variables are needed. In present example if we solve any alternative model(i.e. $4a$ or $4b$) we get same results. So we need not solve all the alternative models. The present method also can be extended for more than eight alternatives, but for simplicity we give only eight alternative.

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